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ON PENCILS OF DIAMETERS IN CONVEX BODIES

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This note gives the solution to a problem of P. C. Hammer on pairs of convex bodies, by proving a result on pencils of diameters in a convex body.

A chord of a convex body is called diameter if its extremities lie on two parallel lines which do not intersect the interior of the body.

It is established in this note, that the set of the ratios in which a point divides diameters of a plane convex body is not necessarily countable, which answers the following question raised by P. C. Hammer.

Let the origin be an interior point of each of two bounded convex bodies B_1, B_2 in the Euclidean n-dimensional space E_n, $n \geqslant 2$. Let R be the set of positive numbers such that $t \in R$ if and only if tB_1 and B_2 have a common hyperplane of support at a point in the boundary of both. Is R always countable?

First, we establish the following simple

LEMMA. *Given in E_2 a circular arc (r, φ), $\varphi' < \varphi < \varphi''$, there exists a convex arc $\rho = A(\varphi)$ such that $A(\varphi') = r$, $A(\varphi'') > r$, and the tangents at the end points of the two arcs have the same directions* (fig. 1).

Proof. We shall call the 2nd arc, whose origin coincides with that of the circular arc and the end point is beyond it, a *sticking out arc*. The proof comes from inspecting the diagram. $OA = OB$, $A'A \perp OA$, B' between B and A', $B'B'' \perp OB$. Any convex arc AB' inscribed into the angle $AB''B'$, touching the sides of the angle in A and B', is a required sticking out arc. We shall always assume it having a continuously varying tangent.

We prove now the principal result of the paper.

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THEOREM. *There exist a plane convex body and an interior point such that the set of ratios in which this point divides diameters through it is uncountable.*

Proof. Let $\sum(\varphi_i', \varphi_i'')$, $i = 1, 2, \ldots$, be a union of disjoint intervals, everywhere dense on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and let

$$\sum(\varphi_i'' - \varphi_i') = \frac{\pi}{4}.$$

Write

$$\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) - \sum(\varphi_i', \varphi_i'') = \mathcal{E},$$

so that

$$\mathrm{m}\,\mathcal{E} = \frac{\pi}{4}.$$

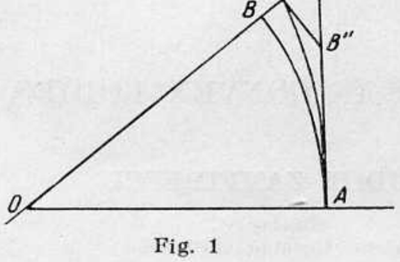

Fig. 1

With every interval $(\varphi_i', \varphi_i'')$ associate a sticking out arc $r(\varphi)$, $\varphi_i' \leqslant \varphi \leqslant \varphi_i''$, and denote by $\theta(\varphi)$ the angle in the positive direction of the tangent at the point $(r(\varphi), \varphi)$ of the arc. Assume further

$$\theta(\varphi) = \varphi + \frac{\pi}{2}$$

for all $\varphi \in \mathcal{E}$. Defined in this way, $\theta(\varphi)$ is continuous in $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$.

Let $r = C(\varphi)$ be the arc passing through $\left(1, \frac{\pi}{4}\right)$, such that the tangent at every point φ, $\left(\frac{\pi}{4} \leqslant \varphi \leqslant \frac{3\pi}{4}\right)$, makes an angle $\theta(\varphi)$ with the axis. ~~As~~ Since at all points of $\mathcal{E}$ the tangent is perpendicular to the radius vector, we have

$$\frac{dr}{d\varphi} = \frac{dC(\varphi)}{d\varphi} = 0 \tag{1}$$

everywhere on $\mathcal{E}$.

Also, for $\varphi_i' \leqslant \varphi \leqslant \varphi_i''$, $r = C(\varphi)$ is similar to the sticking out arc, so that

$$C(\varphi_i'') > C(\varphi_i'). \tag{2}$$

Let $\varphi' < \varphi''$ and $\varphi' \in \mathcal{E}$, $\varphi'' \in \mathcal{E}$; we have

$$C(\varphi'') - C(\varphi') = \int_{\varphi'}^{\varphi''} \frac{dC(\varphi)}{d\varphi}\, d\varphi =$$

$$= \int_{\mathcal{E} \cap (\varphi', \varphi'')} \frac{dC(\varphi)}{d\varphi}\, d\varphi + \sum_{(\varphi_i', \varphi_i'') \subset (\varphi', \varphi'')} (C(\varphi_i'') - C(\varphi_i')) > 0$$

by (1), (2) (since there are sticking out arcs between φ' and φ''). Hence all values of $C(\varphi)$ on $\mathcal{E}$ are distinct.

Expand now $C(\varphi)$ for φ in $\left(\frac{3\pi}{4}, \frac{9\pi}{4}\right)$ to a convex curve so that, in $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$, $C(\varphi)$ be the circular arc of radius 1.

For every $\varphi \in \mathcal{E}$ the line through 0 of the director φ is a diametral chord and is divided by 0 into segments of lengths $C(\varphi)$, 1, that is in the ratio $C(\varphi) : 1$ which has different values for all $\varphi \in \mathcal{E}$. The proof is complete.

Evidently, the theorem is true in arbitrary E_n $(n \geqslant 2)$.

Observe also that all values of $[0, \pi)$ are possible for the measure of the set of directors of a pencil of diameters, but the measure of the set of ratios is always zero.

As one easily sees, the theorem provides a negative answer to Hammer's question.

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