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On the Line-connectivity of Line-graphs

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Introduction

Throughout the paper, G will denote a finite undirected graph without loops or multiple lines. The *line-graph* $L(G)$ of G is that graph whose point set can be put in one-to-one correspondence with the line set of G, such that two points of $L(G)$ are adjacent if and only if the corresponding lines of G are adjacent. The *line-connectivity* $\lambda(G)$ of G is defined to be the smallest number of lines whose removal results in a disconnected graph or the trivial graph. Thus, a nontrivial graph is connected if and only if it has positive line-connectivity. If $m \leqq \lambda(G)$, then the graph G is said to be *m-line-connected.*

We shall make use of the following simple known propositions:

Proposition 1. *The graph G is m-line-connected if and only if for every nonempty subset A of the point set X of G, there exist m lines joining points in A with points in $X - A$* [2].

Proposition 2.

$$\lambda(G) \leqq \min \deg G \quad [3].$$

Our terminology also includes the following:

The *order* of a graph is the cardinality of its point set. If G' is a subgraph of G and X', X are the point sets of G', G (respectively), then the *degree of G' in G* is the number of all lines of G joining points in X' with points in $X - X'$.

The aim of this note is to estimate the line-connectivity of the line-graph in connection with the degree of the vertices of the line-graph and with the line-connectivity of the original graph. This will complete the description given in [1].

A Lemma

Lemma. *If*

$$\lambda(L(G)) < \lambda(G)\left[\frac{\lambda(G)+1}{2}\right],$$

then there exists a connected subgraph of G, of order 2 and degree $\lambda(L(G))$ in G. (Also, by the following Corollary 1, $\lambda(G) \neq 2$.)

Proof. We use the same notation as in the proof of Theorem 2 in [1], namely let Y' denote an arbitrary proper subset of the point set X' of $L(G)$; put Y the subset of the line set X of G induced by Y'; denote by $\delta(u)$ the number of lines of Y incident with the vertex u of G and by $\bar{\delta}(u)$ the number of lines of $X - Y$ incident with u; and set

$$W = \{u\colon \delta(u)\,\bar{\delta}(u) > 0\}\,.$$

Suppose that each connected subgraph of G with 2 vertices has degree at least $\lambda(L(G)) + 1$ in G. We shall show that

$$\sum_{u \in W} \delta(u)\,\bar{\delta}(u) \geqq \lambda(L(G)) + 1\,.$$

First, suppose that no 2 points in W are adjacent.

Following Proposition 2, $\deg u \geqq \lambda(G)$ for every point $u \in W$. Thus, at least one of the numbers $\delta(u)$ and $\bar{\delta}(u)$ must be at least $\left\lceil \frac{\lambda(G)+1}{2} \right\rceil$. Consequently,

$$\sum_{u \in W} \delta(u)\,\bar{\delta}(u) \geqq \left\lceil \frac{\lambda(G)+1}{2} \right\rceil \sum_{u \in W} \delta_u(u)\,,$$

where δ_u means δ or $\bar{\delta}$. From the $\lambda(G)$-line-connectivity of G it follows

$$\sum_{u \in W} \delta_u(u) \geqq \lambda(G)\,,$$

and therefore

$$\sum_{u \in W} \delta(u)\,\bar{\delta}(u) \geqq \lambda(G) \left\lceil \frac{\lambda(G)+1}{2} \right\rceil > \lambda(L(G))\,.$$

Suppose now that 2 adjacent points, say v and w, belong to W.

We assumed that the degree of the subgraph generated by v and w is at least $\lambda(L(G)) + 1$ in G, i.e.

$$\delta(v) + \bar{\delta}(v) + \delta(w) + \bar{\delta}(w) \geqq \lambda(L(G)) + 3\,.$$

Since for any natural numbers N_1 and N_2, $N_1 N_2 \geqq N_1 + N_2 - 1$, we may write

$$\begin{aligned}\sum_{u \in W} \delta(u)\,\bar{\delta}(u) &\geqq \delta(v)\,\bar{\delta}(v) + \delta(w)\,\bar{\delta}(w)\\ &\geqq \delta(v) + \bar{\delta}(v) - 1 + \delta(w) + \bar{\delta}(w) - 1\\ &\geqq \lambda(L(G)) + 1\,.\end{aligned}$$

The inequality

$$\sum_{u \in W} \delta(u)\,\bar{\delta}(u) \geqq \lambda(L(G)) + 1$$

proved above for a set W derived from an arbitrary proper subset Y' of X' would show, by Proposition 1, that $L(G)$ is $(\lambda(L(G)) + 1)$-line-connected, which is by definition impossible.

Therefore there exists a connected subgraph G' of G, of order 2 and degree at most $\lambda(L(G))$; if this degree were smaller than $\lambda(L(G))$, then the corresponding vertex of $L(G)$ would also have degree smaller than $\lambda(L(G))$, violating the Proposition 2. Hence G' has precisely the degree $\lambda(L(G))$ in G.

Corollaries and the Theorem

Immediate proofs will be provided for the next corollaries (two of them first stated in [1]), with the direct help of our Lemma.

Corollary 1 (Chartrand-Stewart).

$$\lambda(L(G)) \geqq 2\,\lambda(G) - 2\,.$$

Proof. Suppose, on the contrary, that the inverse strict inequality holds. Since

$$2\,\lambda(G) - 2 \leqq \lambda(G)\left[\frac{\lambda(G)+1}{2}\right],$$

the Lemma implies the existence of a connected subgraph G' of G with 2 vertices, of degree $\lambda(L(G))$ in G; since this degree is smaller than $2\,\lambda(G)-2$, the degree of at least one of the vertices of G' is at most $\lambda(G)-1$, violating Proposition 2.

Corollary 2 (Chartrand-Stewart). *If* $\lambda(G) \neq 2$, *then*

$$\lambda(L(G)) = 2\,\lambda(G) - 2$$

if and only if there exist two adjacent points in G with degree $\lambda(G)$.

Proof. For $\lambda(G) \neq 2$,

$$2\,\lambda(G) - 2 < \lambda(G)\left[\frac{\lambda(G)+1}{2}\right].$$

Hence, following our Lemma, if $\lambda(L(G)) = 2\,\lambda(G) - 2$, then there exist two adjacent vertices v, w in G so that

$$\deg v + \deg w = \lambda(L(\mathrm{G})) + 2\,.$$

Since both v and w have degree at least $\lambda(G)$ (see Proposition 2) and $\deg v + \deg w = 2\,\lambda(G)$, it follows immediately that

$$\deg v = \deg w = \lambda(G)\,.$$

Conversely, if v, w are adjacent vertices of G and $\deg v = \deg w = \lambda(G)$, then the point in $L(G)$ corresponding to the line joining v and w has degree $2\,\lambda(G)-2$, whence, by Proposition 2,

$$\lambda(L(G)) \leqq 2\,\lambda(G) - 2\,.$$

Now, by Corollary 1, it follows

$$\lambda(L(G)) = 2\,\lambda(G) - 2\,.$$

Corollary 3. *If* $\lambda(G) \geqq 3$, *then*

$$\lambda(L(G)) = 2\,\lambda(G) - 1$$

only if there exist two adjacent points in G, one of degree $\lambda(G)$ *and the other of degree* $\lambda(G)+1$.

The proof is similar to that of Corollary 2.

The sequence of such examples can be continued, but they may all be essentially condensed into the more significant following theorem.

Theorem. *If*

$$\min\deg L(G) \leqq \lambda(G)\left[\frac{\lambda(G)+1}{2}\right],$$

then

$$\lambda(L(G)) = \min\deg L(G)\,.$$

If

$$\min\deg L(G) \geqq \lambda(G)\left[\frac{\lambda(G)+1}{2}\right],$$

then

$$\lambda(G)\left[\frac{\lambda(G)+1}{2}\right] \leqq \lambda(L(G)) \leqq \min\deg L(G)\,.$$

Proof. The Proposition 2 in the Introduction implies

$$\lambda(L(G)) \leqq \min\deg L(G)\,.$$

Now, for the case

$$\min\deg L(G) \leqq \lambda(G)\left[\frac{\lambda(G)+1}{2}\right],$$

suppose

$$\lambda(L(G)) < \min\deg L(G)\,.$$

Then, the Lemma asserts that there exists a connected subgraph of order 2 and degree $\lambda(L(G))$ in G; this means that there is a vertex in $L(G)$ of degree $\lambda(L(G))$, violating the supposed inequality. Consequently,

$$\lambda(L(G)) = \min\deg L(G)\,.$$

For the case

$$\min\deg L(G) \geqq \lambda(G)\left[\frac{\lambda(G)+1}{2}\right],$$

it remains to be shown that

$$\lambda(G)\left[\frac{\lambda(G)+1}{2}\right] \leqq \lambda(L(G))\,.$$

Suppose, on the contrary, that

$$\lambda(G)\left[\frac{\lambda(G)+1}{2}\right] > \lambda(L(G)).$$

Then, by our Lemma, some vertex in $L(G)$ has degree $\lambda(L(G))$, whence

$$\min \deg L(G) \leqq \lambda(L(G));$$

it follows

$$\lambda(G)\left[\frac{\lambda(G)+1}{2}\right] \leqq \lambda(L(G)),$$

contradicting the inequality assumed above.

Thus, the proof is complete.

References

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