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A Two-Connected Planar Graph without Concurrent Longest Paths

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An example of a two-connected planar graph without concurrent longest paths is provided.

INTRODUCTION

Does there exist any connected graph without concurrent longest paths?

This non-trivial question was raised by T. Gallai at the Colloquim on Graph Theory held at Tihany, Hungary, in 1966. The answer is already known, H. Walther producing an example in 1969. Now, in connection with this problem a number of related questions may be put, and some of them seem to be at least as non-trivial as the original question of T. Gallai.

Let $P_i^j = \infty$ ($\bar{P}_i^j = \infty$) if there is no i-connected graph (planar graph) such that each set of j points is disjoint from some longest path. If $P_i^j \neq \infty$ ($\bar{P}_i^j \neq \infty$), let P_i^j ($\bar{P}_i^j$) denote the minimum number of points of an i-connected graph (planar graph), such that each set of j points is disjoint from some longest path. Analogously are defined C_i^j and $\bar{C}_i^j$, for longest circuits instead of longest paths.

The collection of questions we mentioned above may be concisely formulated as follows: Determine all P_i^j, $\bar{P}_i^j$, C_i^j, and $\bar{C}_i^j$! Until now we know

1. $\bar{P}_1^1 \leqslant 25$ (Walther [3]),
2. $C_3^1 \leqslant 10$ (Petersen's graph),
3. $\bar{C}_2^1 \leqslant 105$ (Walther [3]),
4. $C_2^2 \leqslant 220$ (Walther [4]).

(Besides the (answered) question "Is $P_1^1 = \infty$?" of Gallai [1, p. 362], the questions "Is $C_3^2 = \infty$?" (H. Sachs [1, p. 368]), "$P_1^1 = ?$" (Walther

[3, p. 6]), and "Is $P_1^n = \infty$ for some n?" (Walther [3, p. 6]) have also been explicitly raised.)

The purpose of this note is to show that

$$\bar{P}_2^1 \leqslant 82.$$

Lemmas

We use some notations from [3]: Let the part S of a graph G be defined by Fig. 1, where its twenty points and the lines joining points in S with

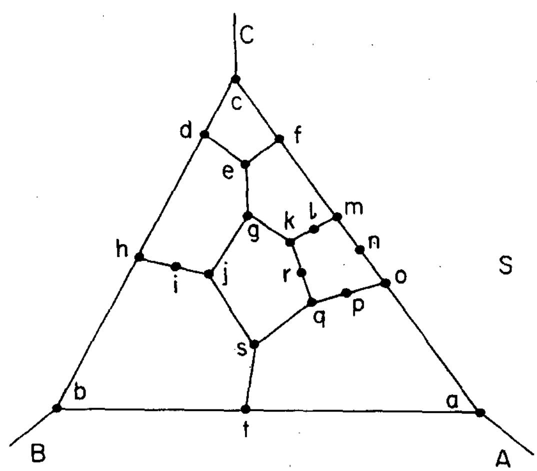

FIGURE 1

points in $G - S$ are denoted. (Figure 1 is derived from part of Fig. 2 of W. T. Tutte [2].) Let W be a longest path in G.

LEMMA 1 (Walther [3]). *If W has no end-point in S and contains the lines A and B (B and C, C and A), then $W \cap S$ has* 17 (18, 18) *points.*

LEMMA 2 (Walther [3]). *For each point u in S, there exists a path joining A with B (or B with C, or C with A), containing* 17 (18, 18) *points, and not passing through u.*

LEMMA 3. *If W has exactly one end-point in S, and contains exactly one of the lines A, B, C, then $W \cap S$ has less than* 20 *points.*

Proof. In case A or B lies in W the proof is provided by a Hilfssatz in [3].

Consider now the case in which C lies in W. Suppose $W \cap S$ has 20 points. Then the end-point (considered the first point) of W in S must be one of the points p, r, l, n; otherwise at least one of these points will never be met by W. For the same reason the first 8 points in W must be p, q, r, k, l, m, n, o (possibly in some other order). Then must come a and t; but, after t, at least one of the points b and s will irremediably be lost, which provides a contradiction.

LEMMA 4. *If W has exactly one end-point in S, and contains all lines A, B, and C, then $W \cap S$ has exactly* 19 *points.*

Proof. Suppose $W \cap S$ has 20 points. Obviously $W \cap S$ has two connected components α, β. Let α be the component containing the end-point w of W in S (first point of W). Then w is one of the points k, l, m, n, o, p, q, r, which are the first 8 points in α.

1. Suppose A is in α. If mf or kg is in α, then $fegjs$ or gjs continues the path, and i will not be reached by β. If qs is in α, then $bhijge$ lies on β, but then one of the points d and f will not be contained by β. If oa is in α, then $btsjihde$ is in β, but then either g is forgotten or C will not be reached.

2. Suppose now B is in α. Then $S - \alpha$ is disconnected, which contradicts the existence of the path β joining C with A.

3. Suppose C is in α. If mfc or $mfedc$ is in α, then $atsj$ lies on β, but then either β continues through ihb and does not meet g, or meets g and never reaches i. If $mfegjih$ or $kgjih$ is in α, then β loses either s or b.

Hence $W \cap S$ has at most 19 points. The proof will be completed by giving for every case an example of possible shape for $W \cap S$, such that W always omits from S exactly one point:

1.	A is in α :	$\alpha = pqrklmnoaA$,	$\beta = BbtsjihdefcC$.
2.	B is in α :	$\alpha = pqrkgedhijstbB$,	$\beta = CcfmnoaA$.
3.	C is in α :	$\alpha = lkrqponmfcC$,	$\beta = AatsjgedhbB$.

THE EXAMPLE

Consider the part F of a graph, shown in Fig. 2, the graph G' of Fig. 3, where each of its parts H, I, J, K (called F-parts) is isomorphic to F[1], and the graph G obtained from G' by contracting the lines appearing in Fig. 3.

[1] Since F is not symmetric, more than one graph may be imagined for Fig. 3. Take G' to be one of them.

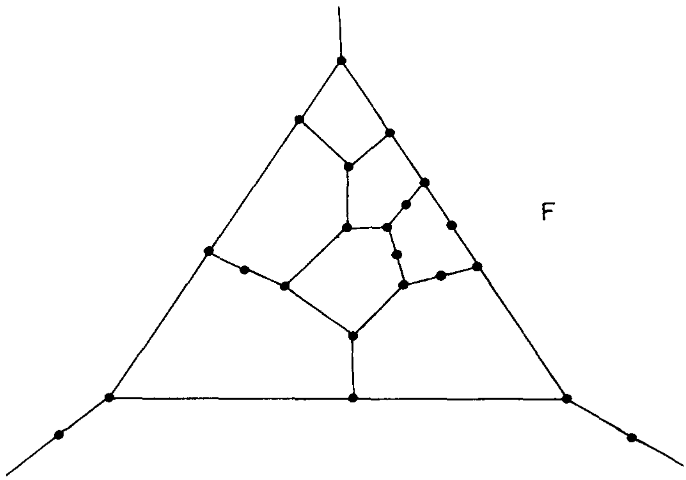

FIGURE 2

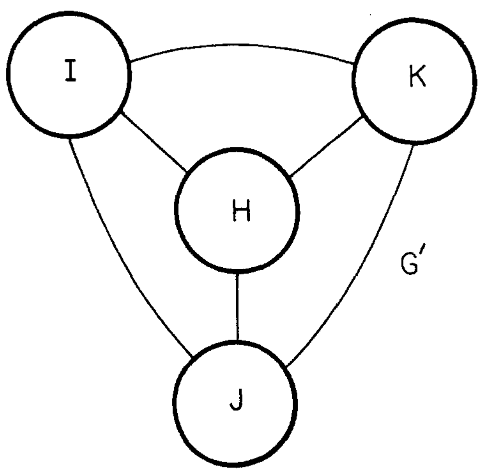

FIGURE 3

We construct a path W_0 originating in H, passing successively through H, I, J, H, K, J, and having 21 points in H, 19 points in I, 21 points in J, and 19 points in K. For the construction (which obviously does not lead to a unique path), we use Lemmas 2 and 4. Because some points from different F-parts are identified when forming G, this path W_0 has 75 points.

Now let W be an arbitrary longest path in G.

LEMMA 5. *If there is no F-part of G containing both end-points of W, then W has at most* 75 *points.*

Proof. Let us say the end-points of W lie in H and I. If W lies in the union of only three F-parts, then it has at most 63 points. Suppose in the sequel that there do not exist three F-parts of G such that W lies in their union.

Let ν be the number of contracted lines of G', whose identified end-points are now on W.

If W passes successively only four times through consecutively different F-parts, then $\nu = 3$; since all these F-parts are different, $W \cap H$ and $W \cap I$ are connected, and Lemma 3 asserts that they have at most 20 points each (pay attention, now and in the following, to the difference between Figs. 1 and 2). Furthermore, the rest of W has, by Lemma 1, exactly 19 points in each of the F-parts J and K. Hence W has at most 75 points.

If W passes successively five times through consecutively different F-parts, then $\nu = 4$, and one of the parts $W \cap H$ and $W \cap I$, say $W \cap H$, is connected and the other, $W \cap I$, disconnected. Following Lemma 3, $W \cap H$ has at most 20 points; following Lemma 4, $W \cap I$ has 21 points. Hence, analogously, W has at most 75 points.

If W passes successively six times through consecutively different F-parts, then $\nu = 5$; $W \cap H$ and $W \cap I$ are disconnected and have, by Lemma 4, 21 points each. Hence, again, W has at most 75 points.

Supposing W passes successively more than six times through consecutively different F-parts, W would have disconnected intersection with more than two F-parts. Thus W would include 4 of the lines entering some F-part. Since each F-part has only 3 entry lines, this is absurd.

LEMMA 6. *If both end-points of W are in some F-part of G, then W has at most* 75 *points.*

Proof. Let us say the end-points of W lie in H. If W lies in the union of only three F-parts, then the statement is known to be verified. If not, then the number ν introduced in the proof of Lemma 5, clearly equals 4, since W passes successively five times through consecutively different F-parts. Thus, W has at most 22 points in H, and the rest of W has, by Lemma 1, exactly 19 points in each of the other F-parts. Hence W has at most 75 points.

THEOREM. *The graph G has* 82 *points and is a two-connected planar graph without concurrent longest paths.*

Proof. One has only to prove that, for each point u in G, there exists a longest path avoiding u. The existence of the path W_0 and Lemmas 5 and 6 show every longest path has 75 points. Suppose, for instance, u is in I, but not in H, J, or K. Return to the construction of the path W_0.

Lemma 2 enables us to reconstruct W_0 in such a way that $W_0 \cap I$ avoids u. To complete the proof, consider now the case in which u is precisely one of the points resulting from the contraction of lines in G'. Then we may say without loss of generality that u lies simultaneously in I and K; by applying again Lemma 2, we can reconstruct W_0 so that both $W_0 \cap I$ and $W_0 \cap K$ avoid u.

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