Fixed Point and Contraction Theorems in Metric Spaces

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1. Let us consider a complete metric space M and a function $f: M \to M$. The following theorem is well-known.

PROPOSITION 1 (Banach). *Suppose* $\alpha < 1$ *and for each couple of points* $x, y \in M$,

$$d(f(x), f(y)) \leqslant \alpha d(x, y).$$

Then f has a unique fixed point.

The following interesting result is of a similar nature.

PROPOSITION 2 (Kannan [2]). *Suppose* $\beta < \frac{1}{2}$ *and for each couple of points* $x, y \in M$,

$$d(f(x), f(y)) \leqslant \beta (d(x, f(x)) + d(y, f(y))).$$

Then f has a unique fixed point.

Both of these propositions are generalized by the following recent result, proved independently by S. Reich and I. A. Rus.

PROPOSITION 3 (Reich [4], Rus [5]). *Suppose* $a, b, c \in R_+$ [1]), $a+b+c<1$, *and for each couple of points* $x, y \in M$,

$$d(f(x), f(y)) \leqslant ad(x, f(x)) + bd(y, f(y)) + cd(x, y).$$

Then f has a unique fixed point.

Obviously, the next theorem also generalizes Propositions 1 and 2.

PROPOSITION 4. *Suppose* $\alpha < 1$, $\beta < \frac{1}{2}$ *and for each couple of points* $x, y \in M$, *at least one of the following conditions is satisfied:*

$$d(f(x), f(y)) \leqslant \alpha d(x, y),$$
$$d(f(x), f(y)) \leqslant \beta (d(x, f(x)) + d(y, f(y))).$$

Then f has a unique fixed point.

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1) R_+ is the set of all nonnegative real numbers.

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A proof is here unnecessary, because Proposition 4 is included in Theorem 1 of [8]. Let us show that Proposition 3 is implied by Proposition 4. Indeed, suppose Proposition 4 is true and let $x, y \in M$. We have

$$d(f(x), f(y)) \leqslant ad(x, f(x)) + bd(y, f(y)) + cd(x, y),$$
$$d(f(x), f(y)) \leqslant ad(y, f(y)) + bd(x, f(x)) + cd(y, x),$$

hence

$$d(f(x), f(y)) \leqslant \frac{a+b}{2}(d(x, f(x)) + d(y, f(y))) + cd(x, y).$$

Therefore,

$$d(f(x), f(y)) \leqslant (a+b+c)\cdot \max\{\tfrac{1}{2}(d(x, f(x)) + d(y, f(y))), d(x, y)\}$$

and by Proposition 4 the function f has a unique fixed point in M. It is easily seen that Proposition 3 is not equivalent to Proposition 4. However, it suggests the following improvement.

PROPOSITION 5. *Let us consider the number* $\delta < 1$ *and the functions* $a, b, c: M \times \times M \to R_+$. *If for each couple of points* $x, y \in M$,

$$a(x, y) + b(x, y) + c(x, y) = \delta$$

and

$$d(f(x), f(y)) \leqslant a(x, y)d(x, f(x)) + b(x, y)\, d(y, f(y)) + c(x, y)\, d(x, y),$$

then f has a unique fixed point.

Proposition 5 clearly improves not only Proposition 3, but also Proposition 4. We will not prove Proposition 5, because this will follow from the next theorem.

THEOREM 1. *Suppose* $\alpha < 1$ *and for each couple of points* $x, y \in M$, *at least one of the following conditions is satisfied:*

1) $d(f(x), f(y)) \leqslant \alpha d(x, y)$
2) $d(f(x), f(y)) \leqslant \alpha d(x, f(x))$
3) $d(f(x), f(y)) \leqslant \alpha d(y, f(y))$.

Then f has a fixed point.

Proof [2]). Suppose f has no fixed points. Choose arbitrarily the point $x_0 \in M$ and

[2]) Let us notice that there is no essential difference between the proof technique used for Theorem 1 and that used for other similar known results (for instance for Theorem 1 of [8]). The only new ideas are in the statement of Theorem 1.

the integer $n \geqslant 0$. Take $x = f^n(x_0)$, $y = f^{n+1}(x_0)$; the hypothesis of the theorem implies

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) \leqslant \alpha d(f^n(x_0), f^{n+1}(x_0))$$

or

$$d(f^{n+1}(x_0), f^{n+2}(x_0)) \leqslant \alpha d(f^{n+1}(x_0), f^{n+2}(x_0)).$$

Because y is supposed not to be a fixed point of f, the first one of the above inequalities holds. Since we have chosen n arbitrarily, it follows that $\{f^n(x_0)\}_{n=0}^{\infty}$ is a Cauchy sequence and therefore converges to some point $z \in M$. Again taking an arbitrary positive integer n and putting $x = f^n(x_0)$, $y = z$, the hypothesis of the theorem yields that at least one of the following inequalities is valid:

$$d(f^{n+1}(x_0), f(z)) \leqslant \alpha d(f^n(x_0), z),$$
$$d(f^{n+1}(x_0), f(z)) \leqslant \alpha d(f^n(x_0), f^{n+1}(x_0)),$$
$$d(f^{n+1}(x_0), f(z)) \leqslant \alpha d(z, f(z)).$$

Since $\{f^n(x_0)\}_{n=0}^{\infty}$ converges to z, we have $z = f(z)$, which contradicts the assumption that f has no fixed points.

Now let us show that f does not have more than one fixed point. Suppose, on the contrary, that $z \neq z'$, $f(z) = z$ and $f(z') = z'$. Then

$$d(f(z), f(z')) = d(z, z')$$

and

$$d(f(z), f(z')) > d(z, f(z)) = d(z', f(z'));$$

thus none of the three conditions of the statement is satisfied at the points z, z'. The proof is complete.

Proposition 5 is a consequence of Theorem 1; indeed, if for each couple of different points $x, y \in M$,

$$d(f(x), f(y)) \leqslant a(x, y)\, d(x, f(x)) + b(x, y)\, d(y, f(y)) + c(x, y)\, d(x, y),$$

then

$$d(f(x), f(y)) \leqslant \delta \cdot \max\{d(x, f(x)), d(y, f(y)), d(x, y)\}$$

and by Theorem 1 the function f has a unique fixed point. Also, a very recent result of R. M. Tiberio Bianchini (Theorem 2 in [7]) is similar to and a consequence of our Theorem 1.

2. The following result clearly improves both the preceding theorem and Theorem 1 from [8]. We do not prove it here since the argument is an obvious combination of the proofs of both mentioned theorems.

THEOREM 2. *Suppose M is a complete metric space, $\alpha < 1$ and for each couple of points $x, y \in M$, at least one of the following conditions is satisfied:*

1) $d(f(x), f(y)) \leqslant \alpha d(x,y)$
2) $d(f(x), f(y)) \leqslant \alpha d(x, f(x))$
3) $d(f(x), f(y)) \leqslant \alpha d(y, f(y))$
4) $d(f(x), f(y)) \leqslant (\alpha/2)\,(d(x, f(y)) + d(y, f(x)))$.

Then f has a unique fixed point[3]).

Similarly, all the other theorems in [8] except the last one may be improved. Since the proofs of these theorems are available, it is only routine to adapt them to the improved variants. Consequently, we give here only the statements.

THEOREM 3. *Suppose M is a metric space, $\alpha < 1$, for each couple of points $x, y \in M$, at least one of the conditions* 1)–4) *of Theorem* 2 *is satisfied, and for some $x_0 \in M$, the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ has a limit point z in M. Then z is a unique fixed point of f.*

Theorem 3 improves Theorem 2 from [8] and thus also generalizes Theorem 1 of R. Kannan [3].

THEOREM 4. *Suppose M is a set in a complete metric space X, $\alpha < 1$ and for each couple of points $x, y \in M$, at least one of the conditions* 1)–4) *of Theorem* 2 *is satisfied. Then, for $x_0 \in M$, the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ converges to a point in X independent of the choice of x_0.*

Theorem 4 improves Theorem 3 from [8].

THEOREM 5. *Suppose M is a metric space, for each couple of different points $x, y \in M$, at least one of the following conditions is satisfied:*

1° $d(f(x), f(y)) < d(x, y)$
2° $d(f(x), f(y)) < d(x, f(x))$
3° $d(f(x), f(y)) < d(y, f(y))$
4° $d(f(x), f(y)) < \frac{1}{2}(d(x, f(y)) + d(y, f(x)))$,

for some $x_0 \in M$, the sequence $\{f^n(x_0)\}_{n=0}^{\infty}$ has a limit point z in M, and the functions f and f^2 are continuous at z. Then z is a unique fixed point of f.

The above theorem improves Theorem 5 from [8]. Consequently, it also generalizes Theorem 4 from [8], Theorem 1 of M. Edelstein [1], and Theorem 1 of S. P. Singh [6].

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3) Examples illustrating the independence of the four conditions may be easily constructed (see also [2], [8]).

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