R. BLIND, G. VALETTE AND T. ZAMFIRESCU

RECTANGULAR CONVEXITY

1. Introduction

Among the problems asked by participants at the 1974 meeting in Oberwolfach, about convexity, the following has attracted our attention:

Let $\mathscr{F}$ be a class of (convex) sets in $\mathbb{R}^n$. We say that a set $M \subset \mathbb{R}^n$ is $\mathscr{F}$-convex if, for each two distinct points $x, y \in M$, there exists $F \in \mathscr{F}$ such that $x, y \in F$ and $F \subset M$. Study the $\mathscr{F}$-convexity for remarkable classes $\mathscr{F}$ (Zamfirescu).

For example, the members of $\mathscr{F}$ may be the usual closed segments, and in this case the $\mathscr{F}$-convexity is nothing else but the classical convexity; the members of $\mathscr{F}$ may be the lines in a vector space and then the $\mathscr{F}$-convex sets are exactly its linear manifolds (affine subspaces); or the members of $\mathscr{F}$ may be arcs and $\mathscr{F}$-convexity becomes the usual arcwise connectedness.

The problem of describing the $\mathscr{F}$-convex sets may be difficult for easily defined classes $\mathscr{F}$. It is so—in the opinion of the authors—when $\mathscr{F}$ is the class of all 2-dimensional rectangles in the Euclidean n-space; this particular $\mathscr{F}$-convexity will be called *rectangular convexity* or, shorter, *r-convexity*. The present paper deals with r-convexity for $n = 2$ and $n = 3$.

Noting first that an open set in $\mathbb{R}^n$ is r-convex if and only if it is convex, we immediately pass on to the study of closed r-convex sets. We begin with the case $n = 2$; in the following statements, we shall say that a subset of $\mathbb{R}^2$ is: a *strip* if it is similar to $\{(x, y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant 1\}$; a *half-strip* if it is similar to $\{(x, y) \in \mathbb{R}^2 : 0 \leqslant x, 0 \leqslant y \leqslant 1\}$; *extremely circular* if all its extreme points lie on a circle.

THEOREM 1. *The following sets are r-convex:*

(A) *every closed unbounded convex set whose asymptotic cone has its angular measure in $[\pi/2, \pi] \cup \{2\pi\}$;*

(B) *the strips and the half-strips;*

(C) *the compact 2-dimensional convex sets which are centrally symmetric and extremely circular.*

We conjecture that there are no other closed r-convex sets in the Euclidean plane; this is supported by the following results:

THEOREM 2. *The only non bounded closed r-convex sets in the Euclidean plane are those described in* (A) *and* (B) *of Theorem* 1.

THEOREM 3. *If P is an r-convex polygon, then P is centrally symmetric and extremely circular.*

THEOREM 4. *If M is a compact r-convex set which is extremely circular, then M is also centrally symmetric.*

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THEOREM 5. *If S is a compact r-convex set which is centrally symmetric, then S is also extremely circular.*

The description of all closed r-convex sets in $\mathbb{R}^n$ seems to be an even more difficult task. In the bounded case, we can only give several examples: a centrally symmetric extremely spherical (analogue to extremely circular) convex body without $(n-2)$-dimensional faces, a cylinder $K \times [0, 1]$ with an $(n-1)$-dimensional compact convex set K as basis, the intersection of two n-dimensional balls. So, one sees that there exist in $\mathbb{R}^n$ $(n \geqslant 3)$ r-convex sets which are compact but neither centrally symmetric nor extremely spherical.

In the non-bounded case, we have obtained a result concerning the closed r-convex sets in $\mathbb{R}^3$. Its formulation needs two definitions: Let S_2 be the unit sphere; a closed spherically convex set $A \subset S_2$ will be called *q-large* if there is no open quarter of S_2 (a component of the complement on S_2 of the union of two orthogonal great circles) which includes A. The intersection of the asymptotic cone of a non-bounded convex set B with S_2 will be called *asymptotic* set of B.

THEOREM 6. *Let B be a non-bounded closed strictly convex set in $\mathbb{R}^3$ having a strictly convex asymptotic set $A \neq S_2$. Then B is r-convex if and only if A is q-large.*

It is clear that the strict convexity conditions in the last theorem do not allow us to consider the non-bounded case as solved. However, we are optimistic and believe that Theorem 6 is true without supposing the strict convexity of A; the detailed investigation remains to be done.

We shall use the following notations: d for the Euclidean metric; ab for the segment joining the points a, b; $\langle a, b\rangle$ for the line through the points a, b.

The following sections present proofs of the above theorems.

2. Rectangular convexity in the plane

Proof of Theorem 1. Let M be one of the sets described in the statement. It is sufficient to show that any two points of the boundary ∂M are contained in a rectangle included in M. This is clear if M is of type (A) or (B). When M is of type (C), let K be its circumscribed circle. If no supporting line of M through a or b is orthogonal to ab, then it is easy to find a rectangle having a, b as vertices and contained in M. If there is a supporting line through a or b (say a) which is orthogonal to ab, three cases are possible:

(1) a is not on K. Then a lies on a chord of K contained in ∂M, and the symmetry of M implies that ab is the side of a rectangle included in M.

(2) a is on K and is a regular point of ∂M. Then a and b are diametral points of K and, because M has other extremal points of K (symmetrically disposed), ab is the diagonal of a rectangle contained in M.

(3) a is on K and is not a regular point of ∂M. Let L_1 and L_2 be the extremal

supporting lines of M through a and let R_1 (resp. R_2) be the ray with endpoint a, orthogonal to L_1 (resp. L_2) and meeting $K \setminus \{a\}$. As b lies between R_1 and R_2 on the boundary of M, which is centrally symmetric, it must belong to the image of L_1 or L_2 under the central symmetry which preserves M. Hence ab is contained in a rectangle included in M.

Proof of Theorem 2. Let M be a closed and non-bounded r-convex set which contains no line. It is sufficient to show that if the asymptotic cone of M has its angular measure less than $\pi/2$, then M is a half-strip. We do this using the following notations: B is the boundary of M; d_1 and d_2 are the extremal directions of infinity of M; and L_1 is the unique supporting line of M which is orthogonal to d_1. Then we choose a Cartesian coordinate system as follows: the x-axis is L_1 and the upper half-plane contains M; the angle between d_2 and the positive x-axis is at most $\pi/2$; the origin O belongs to $M \cap L_1$ which is contained in the negative x-axis. Now we distinguish two cases.

(1) $B \cap \{x \geqslant 0\}$ and $L_1 \cap \{x \geqslant 0\}$ are not tangent. Let T be the ray tangent to $B \cap \{x \geqslant 0\}$ at O and $\{O, p\}$ be the intersection of B with the bissectrice of T and the negative x-axis. It is clear that the segment Op cannot be the side of a rectangle contained in M. As M is r-convex, Op is the diagonal of a rectangle R included in M. But M does not meet the sets $\{y < 0\}$ and $\{x < x(p), y < y(p)\}$. Hence R does not intersect these sets and there remains just one position for R, namely the rectangle $\{x(p) \leqslant x \leqslant 0, 0 \leqslant y \leqslant y(p)\}$. This implies first that the projection $(k, 0)$ of p on L_1 belongs to M, and further that

$$M \cap \{x \leqslant 0\} = \{k \leqslant x \leqslant 0, y \geqslant 0\}.$$

(2) $B \cap \{x \geqslant 0\}$ and $L_1 \cap \{x \geqslant 0\}$ are tangent. Let C be the part of $B \cap \{x \leqslant 0\}$ which is above the line through O, orthogonal to d_2. We define a map $f\colon C \to B$ as follows: if $c \in C$, the line through O and orthogonal to the line $\langle O, c\rangle$ cuts B in O and in another point, denoted by $f(c)$. Then f is continuous, monotone (with respect to the natural orders along C and B) and, if c tends to infinity on C, then $f(c)$ tends to O on B. So, there is a point c_0 of C such that, if $y(c) > y(c_0)$, then $0 < x(f(c)) < -x(c)$, which implies that the midpoint $m(c)$ of $cf(c)$ is in the half-plane $\{x < 0\}$ (see Figure 1). For every point c with this property, we make the following construction: first we remark that the circle with centre $m(c)$ passing through c also passes through O and $f(c)$, but does not contain the arc $\widehat{Of(c)}$ of B in its convex hull, because $B \cap \{x \geqslant 0\}$ and L_1 are tangent. Hence, the smallest circle with centre $m(c)$ surrounding this arc, say S, has O in its interior. Let s be any point of $S \cap \widehat{Of(c)}$ and t be the point of $C \cap \langle m(c), s\rangle$. It is clear that the segment st cannot be the side of a rectangle contained in M. As M is r-convex, st is the diagonal of a rectangle $R \subset M$. But M does not meet the sets $\{y < 0\}$ and $\{x < x(t), y < y(t)\}$. Hence R does not intersect these sets, so

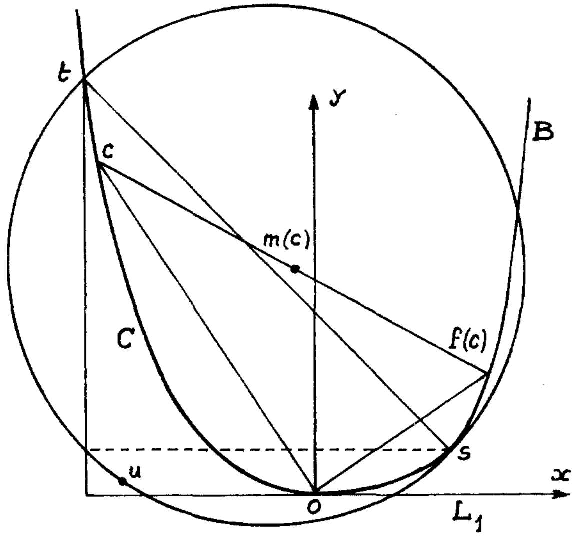

Fig. 1

that it has a vertex, say u, in $\{x(t) \leqslant x < x(s), 0 \leqslant y \leqslant y(s)\}$. Now, u belongs to the circle S' with diameter st. As the radius of S' is larger than that of S, u cannot be in $\{0 \leqslant x \leqslant x(s)\}$. As the centre $\frac{1}{2}(s + t)$ of S' is in $\{x < 0\}$, u cannot be in $\{x(s + t) < x < 0\}$. Hence u is a point of $\{x(t) \leqslant x \leqslant x(s + t)$, $0 \leqslant y \leqslant y(s)\}$, which means that M has points in this set. Finally, let c tend to infinity on C; then s tends to O and u tends to a point $(k, 0)$ of the negative x-axis (it is clear that u cannot tend to the point at infinity of the negative x-axis). For this reason, $x(t)$ has a lower bound, which must be k. As M is closed, this implies that

$$M \cap \{x \leqslant 0\} = \{k \leqslant x \leqslant 0, y \geqslant 0\}.$$

In both cases, we find the same conclusion. Transposing d_1 and d_2, we see that M must be a half-strip.

Proof of Theorem 3. Let P be an r-convex polygon. Let p_1 and p_1' be the endpoints of a diameter of P and let m be the midpoint of p_1 and p_1'. Let K be a circle with centre m and passing through p_1 and p_1'. The segment p_1p_1' cannot be the side of a rectangle contained in P, and the other two vertices p_2 and p_2' of this rectangle are diametral points of K. If two points of $P \cap K$ are diametral points, then they are vertices of P. It follows that the number of pairs of diametral points of $P \cap K$ is at least two and is finite, say i_0. Let $\{p_1, p_1'\}$, $\{p_2, p_2'\}, \ldots, \{p_{i_0}, p_{i_0}'\}$ be these pairs. The edges of P passing through p_i or p_i' are lying in secants of K, so for each point p_i (resp. p_i'), there is a neighbourhood containing no point of $P \setminus \text{int conv } K$ (int conv K being the interior of the convex hull of K) different from p_i (resp. p_i'). Clearly

$P \supset \mathrm{conv}\{p_1, p'_1, \ldots, p_{i_0}, p'_{i_0}\}$, which is centrally symmetric and extremely circular, and it shall be proved that $P = \mathrm{conv}\{p_1, p'_1, \ldots, p_{i_0}, p'_{i_0}\}$.

Otherwise, it may be assumed that p_1p_2 is an edge of $\mathrm{conv}\{p_1, p'_1, \ldots, p_{i_0}, p'_{i_0}\}$, but not an edge of P. Let H (resp. H') be the half-plane determined by the line $\langle p_1, p'_1\rangle$ and containing p_2 (resp. p'_2). Let L be the intersection of H and a supporting line of P in p_1 such that L contains an edge of P (see Figure 2).

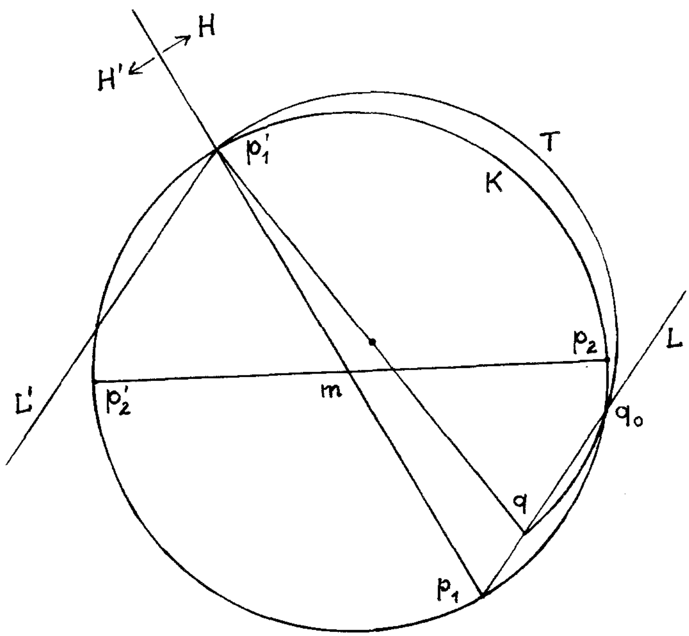

Fig. 2

Similarly, let L' be the intersection of H' and a supporting line of P in p'_1 such that L' contains an edge of P. Then L meets K in p_1 and in a point q_0 with $p_i \neq q_0 \neq p'_i$ $(1 \leqslant i \leqslant i_0)$. Let us choose $q \in L \cap P$ with $q \neq p_1$ and sufficiently close to p_1 that the angle defined by qp'_1 and L' is smaller than $\pi/2$. Then qp'_1 cannot be the side of a rectangle contained in P. As P is r-convex, qp'_1 is the diagonal of a rectangle contained in P, and the other two vertices u and u' of this rectangle are diametral points of the circle T with diameter qp'_1. Since qq_0 and p'_1q_0 are perpendicular, T contains q_0. Because of the supporting property of L, the open small arc of T between q and q_0 does not contain any point of P; it follows that, for example, u is contained in the small arc $\widehat{q_0p'_1}$ of T. Hence $u = q_0$ or u is a point in the exterior of K. Now $u, u' \in P$ and P is compact; thus, if we choose a suitable sequence of points q tending to p_1, the associated points u tend to a point $\bar{u} \in P$, and the associated points u' tend to a point $\bar{u}' \in P$. Because u and u' are diametral points of the circles T tending to K, $\bar{u}$ and $\bar{u}'$ are diametral points of K. As $p_i \neq q_0 \neq p'_i$ ($1 \leqslant i \leqslant i_0$) and because each point p_i (resp. p'_i) has a neighbourhood containing no point of $P \setminus \text{int conv } K$ different from p_i (resp. p'_i), it follows that $p_i \neq \bar{u} \neq p'_i$. This contradicts the fact that $\{p_1, p'_1\}, \ldots, \{p_{i_0}, p'_{i_0}\}$ are all pairs of diametral points of $P \cap K$.

Proof of Theorem 4. Let K be the circle containing the extreme points of M. If a is an extreme point of M, let us choose a point b in $\{x \in M; d(a, x) \geqslant d(a, y) \text{ for all } y \in M\}$. Then b is also an extreme point of M and ab cannot be the side of a rectangle included in M. Therefore, ab is a diagonal of a rectangle included in M. Since the other diagonal must be contained in M, the circle with diameter ab must be equal to K. This implies that the set of extreme points of M is centrally symmetric, and the statement is proved.

Proof of Theorem 5. Let S be a compact r-convex set which is centrally symmetric. Let m be the centre of S and let K be the smallest circle such that $S \subset \text{conv } K$; then m is the centre of K and $S \cap K$ contains two diametral points, say p_1 and p'_1. The segment $p_1 p'_1$ cannot be the side of a rectangle contained in S. As S is r-convex, $p_1 p'_1$ is the diagonal of a rectangle contained in S, and the other two vertices p_2 and p'_2 of this rectangle are diametral points of K, hence $S \cap K$ contains at least two pairs of diametral points. Clearly $S \supset \text{conv}(S \cap K)$, which is centrally symmetric and extremely circular, and it shall be shown that $S = \text{conv}(S \cap K)$.

Otherwise, there exists a ray starting in m and meeting $\partial \text{ conv}(S \cap K)$ in a point c and ∂S in a point different from c (where ∂ means the boundary). Thus $c \notin S \cap K$ and it may be assumed that $c \in p_1 p_2$, hence $p_1 p_2 \subset \partial \text{ conv}(S \cap K)$. It follows that the open small arc of K between p_1 and p_2 does not contain any point of S, and that $p_1 p_2 \cap \partial S = \{p_1, p_2\}$. Let now H (resp. H') be the half-plane determined by the line $\langle p_1, p'_1 \rangle$ and containing p_2 (resp. p'_2). Let L be the intersection of H and the supporting line of S in p_1 for which the angle α between L and $p_1 p'_1$ is minimal. Let L' be the image of L under the central symmetry defined by m, and let α' be the angle between L' and $p_1 p'_1$. Clearly $\alpha = \alpha' \leqslant \pi/2$. The two cases $\alpha' < \pi/2$ and $\alpha' = \pi/2$ are treated separately.

(1) $\alpha' < \pi/2$: Let $q \in H \cap \partial S$, $q \neq p_1$ be sufficiently close to p_1 that the angle between $p'_1 q$ and L' is smaller than $\pi/2$. As L' is contained in a supporting line of S which does not meet the exterior of K, $p'_1 q$ cannot be the side of a rectangle contained in S. Since S is r-convex, $p'_1 q$ is the diagonal of a rectangle contained in S, and the other two vertices u and u' of this rectangle are diametral points of the circle T with diameter $p'_1 q$. T meets K, in addition to p'_1, in a point p. Clearly $q \in pp_1$. As q and p_1 are on the boundary of the convex set S, the open small arc of T between q and p does not contain any point of S; the open small arc of T between p'_1 and p is in the exterior of K, hence it too does not contain any point of S. It follows that u or u' is equal to p, hence

$p \in S$. Now p_1 is in the exterior of T, and because $q \notin p_1p_2$, p_2 is in the interior of T, thus p lies in the open small arc of K between p_1 and p_2, in contradiction to the fact that this arc does not contain any point of S.

(2) $\alpha' = \alpha = \pi/2$: Let a_1 be a point of the boundary curve of S between p_1 and p_2 such that the angle β between a_1m and p_1m is smaller than $\pi/2$. Let a_2 be the unique point of $K \cap H' \cap \langle a_1, m\rangle$. Let A be the intersection of H' and the line bisecting the angle between p_1m and a_2m. Let W_1, W_2, W_3, W_4 be the cones with vertex m as in Figure 3. Also, $v \in A$, $v \neq m$, $v \in \operatorname{int} S$. We choose now a point $z_0(v) \in \partial S \cap W_1$ with

$$d(z_0(v), v) = \sup\{d(z, v); z \in S \cap W_1\}.$$

From $\alpha = \pi/2$ it follows that $z_0(v) \neq p_1$. Let $z_0'(v) \in \partial S$ be the image of $z_0(v)$ under the central symmetry defined by m. The line $\langle z_0(v), v\rangle$ meets ∂S, in addition to $z_0(v)$, in a point $z_1(v)$. If $m_1(v)$ is the midpoint of $z_0(v)z_1(v)$, then $\langle m, m_1(v)\rangle$ is parallel to $\langle z_0'(v), z_1(v)\rangle$. Let $\gamma(v)$ be the angle between $z_1(v)z_0'(v)$ and $z_0(v)z_0'(v)$, which is also the angle between $m_1(v)m$ and $z_0(v)m$. Let now v tend to m.

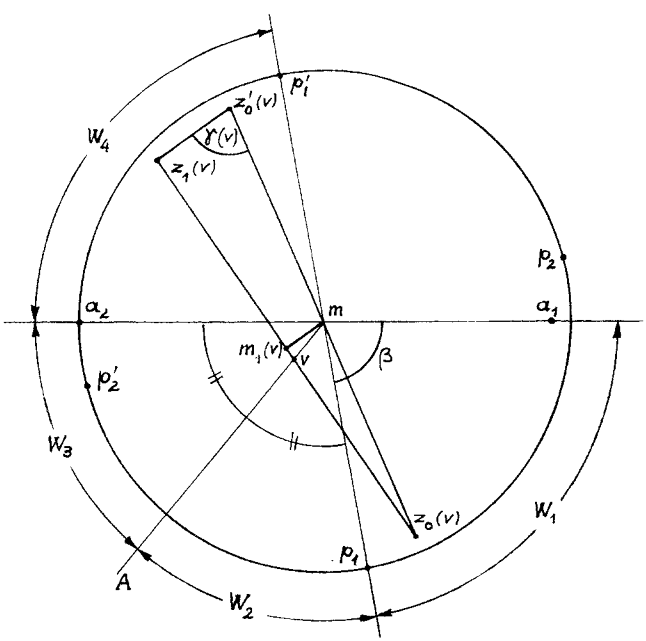

Fig. 3

As K is the smallest circle such that $S \subset \operatorname{conv} K$, we have $d(z_0(v), m) \leqslant d(p_1, m)$; on the other hand, $d(v, p_1) \leqslant d(v, z_0(v))$ for all v, hence $d(\lim_{v \to m} z_0(v), m) = d(p_1, m)$. As the open small arc of K between p_1 and p_2 does not contain any point of S, it follows that $\lim_{v \to m} z_0(v) = p_1$. Then

$\lim_{v \to m} z_0'(v) = p_1'$ and $\lim_{v \to m} z_1(v) = p_1'$. Thus, if v tends to m, the line $\langle z_0'(v), z_1(v)\rangle$ tends to the line containing L', hence $\gamma(v)$ tends to $\alpha' = \pi/2$. Taking into account those limits, we conclude that there is a $\bar{v} \in A$ with $z_0(\bar{v}) \neq a_1$, $z_1(\bar{v}) \in W_4$, and $m_1(\bar{v}) \in \operatorname{int} W_3$.

From the definition of $z_0(v)$, it follows that $z_0(\bar{v})z_1(\bar{v})$ cannot be the side of a rectangle contained in S. As S is r-convex, $z_0(\bar{v})z_1(\bar{v})$ is the diagonal of a rectangle contained in S, and the other two vertices of this rectangle are diametral points of the circle T with centre $m_1(\bar{v})$ and passing through $z_0(\bar{v})$ and $z_1(\bar{v})$. Because $T \cap (W_1 \cup W_2 \cup W_3)$ contains a half-circle, we get the intended contradiction in showing that this arc of T contains no point of S except $z_0(\bar{v})$.

Because of $m_1(\bar{v}) \in \operatorname{int} W_3$, we have $\bar{v} \in z_0(\bar{v})m_1(\bar{v})$, $\bar{v} \neq m_1(\bar{v})$. Hence it follows from the construction of $z_0(v)$ that $T \cap W_1$ does not contain a point of S except $z_0(\bar{v})$. Furthermore, p_1 is in the interior of T. As A bisects the angle between p_1m and a_2m, and because of $d(p_1, m) = d(m, a_2)$ and $m_1(\bar{v}) \in W_3$, we have $d(m_1(\bar{v}), p_1) \geqslant d(m_1(\bar{v}), a_2)$, thus a_2 is also in the interior of T. Hence $T \cap (W_2 \cup W_3)$ is lying in the exterior of K and does not contain a point of S.

3. Rectangular convexity in 3-space

Proof of Theorem 6. 'If': *Suppose A is q-large and prove that B is r-convex.*

It suffices to prove that for each pair of points $x, y \in \partial B$, there is a rectangle included in B and having x, y as vertices. Let $\xi = (x - y)/d(x, y)$. Since B is strictly convex, $\xi \neq A$. Let Γ_1, Γ_2 be the great circles through ξ tangent to A and r_1, r_2 the contact points of Γ_1 and Γ_2, respectively. For each point $r \in \partial A \setminus \{r_1, r_2\}$, let $j(r)$ be the other intersection point of ∂A with the great circle through ξ and r. The function j, extended to ∂A by setting $j(r_i) = r_i$ $(i = 1, 2)$, is then a continuous involution on ∂A with fixed points r_1, r_2. Now, let $\beta \in \partial A$. The set of all farthest points from β on A is a connected subset of ∂A, since A is q-large. Moreover, this set has only a single point $k(\beta)$, because A is strictly convex. The function k, from ∂A onto itself, is fixed-point-free and continuous. The functions j and k must then coincide at some point $\alpha \in \partial A$. Let Γ be the great circle through ξ and α. Also, let Π be the plane through x parallel to the plane of Γ. The asymptotic cone of $\Pi \cap B$ is $\Gamma \cap A$, whose angular measure is at least $\pi/2$. Hence, by Theorem 1 there is a rectangle containing x, y and entirely lying in $\Pi \cap B$.

'Only if': *Suppose B is r-convex and prove that A is q-large.*

Suppose on the contrary A is not q-large, i.e. there is a point $p \in \partial A$ such that the distance δ on S_2 between p and the farthest point of ∂A is less than $\pi/2$. Consider the point $v \in \partial B$ having $-p$ as spherical image.[1] Let Γ_p be a

[1] The exterior normal at v to ∂B is parallel to and has the same orientation as the vector $-p$.

great circle of S_2 supporting A at p and only at p. Let Π be the plane through v orthogonal to the tangent in p to Γ_p. Π contains the normal N in v to ∂B. Let Π_+ be the closed half-plane with boundary N that contains all half-lines through v included in $\Pi \cap B$ (if there is only one such half-line, choose Π_+ to be one of the two half-planes with boundary N). Let Π_- be the closure of $\Pi \setminus \Pi_+$. The curve $\Pi_- \cap \partial B$ either has an asymptote L' parallel (but not identical) with N, or has no asymptote. Let L be a line in Π_- different from and parallel to N such that, if L' exists, the distance between L and L' is greater than that between L and N (see Figure 4). Let $w = L \cap \partial B$.(2)

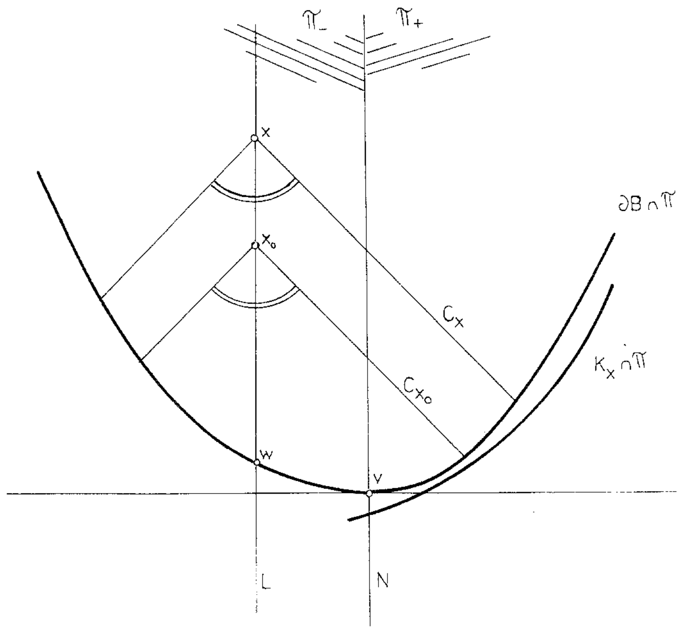

Fig. 4

Let $\varepsilon = (\pi/2) - \delta$ and suppose there exist two sequences of points $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ such that $x_n \in L \cap B$, $y_n \in B$, $d(w, x_n) = d(x_n, y_n)$, $d(w, x_n) \to \infty$, and the measure of the angle wx_ny_n equals ε. Then a certain subsequence of $(wy_n)_{n=1}^{\infty}$ converges to a half-line originating at w, included in B and forming with L an angle of measure $(\pi - \varepsilon)/2$. This half-line would correspond to a point in A at the distance $(\pi - \varepsilon)/2 > \delta$ from p, but such a point does not exist.

Hence, for some point $x_0 \in L \cap B$, each solid circular cone C_x with apex x such that $xw \supset x_0w$, with axis L and whose generators make an angle ε with $(L - B) \cup xw$ has, as intersection with B, a set completely contained in the solid ball K_x of centre x and radius $\max\{d(x, y) : y \in C_{x_0} \cap \partial B\}$.

(2) We identify a single point set with the point itself.

Let now x be such that $xw \supset x_0w$ and let $z_x \in \partial K_x \cap C_{x_0} \cap \partial B$. It is obvious that $d(w, x) \to \infty$ implies $z_x \to v$. Let z'_x be an intersection different from z_x (if any) of the line through x and z_x with ∂B. When z_x is sufficiently close to v, z'_x exists and the ball J_x with diameter $z_xz'_x$ contains K_x.

Let G_x be the great circle of J_x tangent in z_x to the line orthogonal to L and xz_x. For z_x sufficiently close to v, let H_x be the half-sphere bounded by G_x, containing w in its convex hull. Let M_x be the set of points on H_x, the angular distance of which to z'_x on J_x is smaller than ε (see Figure 5).

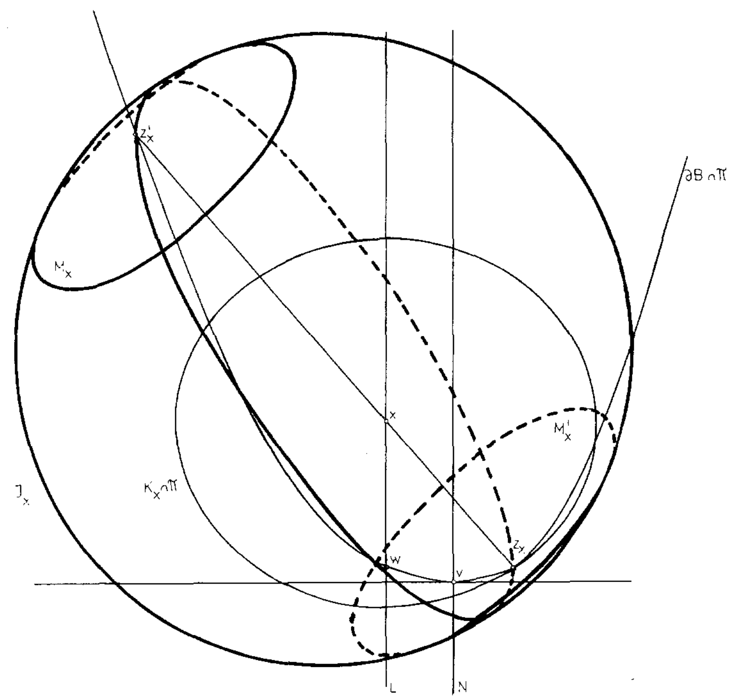

Fig. 5

Suppose there exist two sequences $(x_n)_{n=1}^{\infty}$ and $(u_n)_{n=1}^{\infty}$ such that $x_n \in L \cap B$, $d(w, x_n) \to \infty$ and $u_n \in B \cap H_{x_n} \setminus M_{x_n}$. Then a certain subsequence of $(z_{x_n}u_n)_{n=1}^{\infty}$ converges to a half-line originating in v, included in B, lying in the half-space containing w and bounded by the plane through N orthogonal to Π, and forming with N an angle of measure at least $\varepsilon/2$. This half-line would correspond to a point of S_2 different from p and lying on Γ_p or on the open half-sphere bounded by Γ_p and disjoint from A, but there is no such point. Hence, there exists a point $x'_0 \in L$ such that $x_0 \in x'_0w$, and for each $x \in L$ with $xw \supset x'_0w$, $z_x = B \cap H_x \setminus M_x$.

Let M'_x be the set symmetric with M_x with respect to the centre of J_x. Suppose again there exist two sequences $(x_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ such that $x_n \in L \cap B$, $d(w, x_n) \to \infty$ and $t_n \in M'_{x_n} \cap B \setminus C_{x_n}$. Let α_n be the angle between $\langle x_n, z_{x_n} \rangle$ and $\langle z_{x_n}, t_n \rangle$. Then, on the one hand, $d(v, t_n) \to \infty$ since $t_n \notin C_{x_n}$, and on the other some subsequence of $(\alpha_n)_{n=1}^{\infty}$ converges to a value $\nu \geqslant (\pi - \varepsilon)/2$. This means that some subsequence of $(z_{x_n} t_n)_{n=1}^{\infty}$ converges to a half-line originating in v, included in B and forming with N the angle ν, which is impossible. Thus, for some $x''_0 \in L$ and for all $x \in L$ with $xw \supset x''_0 w$, $M'_x \cap B \setminus C_x = \varnothing$. Since for these points x, $C_x \cap B \subset K_x$, we also have $M'_x \cap B \cap C_x = z_x$, hence $M'_x \cap B = z_x$.

It follows that if $x'_0, x''_0 \in xw$, then $M'_x \cap B = z_x$ and $B \cap H_x \setminus M_x = z_x$, i.e.

$$B \cap (H_x \cup M'_x) \setminus M_x = z_x.$$

But since B is r-convex, and since the line through x and z_x is normal in z_x to ∂B, the segment $z_x z'_x$ should be the diagonal of a rectangle included in B. The other two vertices of that rectangle must be diametral opposite points of J_x, whence one of them must lie on $(H_x \cup M'_x) \setminus M_x$ and a contradiction is obtained.

Authors' addresses:

R. Blind,
Mathematisches Institut B,
Universität Stuttgart,
Pfaffenwaldring 57,
7 *Stuttgart* 80,
W. Germany

G. Valette,
Dept. v. Wiskunde,
Vrije Universiteit Brussel,
Pleinlaan 2,
1050 Brussels,
Belgium

T. Zamfirescu,
Abteilung Mathematik,
Universität Dortmund,
46 Dortmund-Hombruch,
Postfach 500,
W. Germany

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