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Nonexistence of Curvature in Most Points of Most Convex Surfaces

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Busemann and Feller [3] and Aleksandrov [1] have shown that each convex surface has a (sectional) curvature almost everywhere (in the usual sense of Hausdorff measure) in each tangent direction. Also, examples of convex surfaces S without curvature in any tangent direction at a dense set E of points are known. Schneider [6] proved that most (meaning always in this paper "all, except those in a set of first Baire category") closed convex surfaces have this property. It seems unknown how large – except of having measure zero – the set E might be. For instance, can E be residual? We prove in this paper that, for most (always closed) convex surfaces, at most points and every tangent direction there exists no curvature.

To formulate the precise (stronger) main theorem, we need some notations. First of all, everything happens in $\mathbb{R}^n$ ($n \geqq 2$). Let x be a point on the convex surface S at which S is smooth. Let τ be a tangent direction at x and $S(\tau)$ the arc along which S intersects the halfplane determined by the tangent halfline originating at x in direction τ and the normal in x to S. Let $\varrho_i(\tau)$ and $\varrho_s(\tau)$ be the lower and upper radii of curvature (see [2], p. 14 for a definition) of $S(\tau)$ at x. If $\varrho_i(\tau) = \varrho_s(\tau)$, we denote by $\varrho(\tau)$ the common value.

Klee [5] and Gruber [4] showed that most convex surfaces are smooth and strictly convex. A description of the curvature of most convex surfaces is given by the following theorem.

Theorem 0 [7]. *On most convex surfaces, for each point x and tangent direction τ at x, $\varrho_i(\tau)=0$ or $\varrho_s(\tau)=\infty$ (or both).*

This yields the following description in terms of Hausdorff measure.

Theorem 1 [7]. *On most convex surfaces $\varrho(\tau)=\infty$ a.e. in all tangent directions τ.*

We complete here these results with the following description in terms of Baire categories, which strengthens the Theorem in [6].

Theorem 2. *On most convex surfaces, at most points and for every tangent direction τ, $\varrho_i(\tau)=0$ and $\varrho_s(\tau)=\infty$.*

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Proof. Let $\mathscr{S}$ be the set of all smooth convex surfaces S such that the set E of all points in which $\varrho_i(\tau)=0$ and $\varrho_s(\tau)=\infty$ for every tangent direction τ is dense on S. By the results of Klee [5], Gruber [4] and Schneider [6], $\mathscr{S}$ is residual in the space of all convex surfaces in $\mathbb{R}^n$. We show that, for this set $\mathscr{S}$, at most points of any surface in $\mathscr{S}$ and for every tangent direction τ, $\varrho_i(\tau)=0$ and $\varrho_s(\tau)=\infty$. Choose $S\in\mathscr{S}$.

Let C be a circle and n a natural number. Let F_n be the set of all points x on S such that the circle $C(\tau)$ lying in the plane of $S(\tau)$ internally tangent at x to $S(\tau)$, congruent with C, meets $S(\tau)$ in a point $y(\tau)\neq x$ at distance at most n^{-1} from x, for every tangent direction τ at x. We show that $\complement F_n$ is nowhere dense on S.

Let O be an open set on S. Let $x_0\in O\cap E$. Since $x_0\in E$, there exist two points $z_i(\tau_0)$, $z_e(\tau_0)\in C(\tau_0)$, the first in the interior, the second in the exterior of S, at distances $d_i(\tau_0)$, $d_e(\tau_0)<n^{-1}$ from x_0, for each tangent direction τ_0 at x_0. Since S is of class C^1, there exists an open set $O'_{x_0,\tau_0}\subset O$ containing x_0 and an open set $O''_{x_0,\tau_0}\subset S^{n-1}$ containing τ_0, S^{n-1} being the unit sphere in $\mathbb{R}^n$, such that, for every point $x\in O_{x_0,\tau_0}$ and all those directions $\tau\in O''_{x_0,\tau_0}$ which are tangent at x, the two points $z_i(\tau)$, $z_e(\tau)\in C(\tau)$ at distances $d_i(\tau_0)$, $d_e(\tau_0)$ from x still lie in the interior, respectively exterior of S. The sets O''_{x_0,τ_0} obtained in this way for all tangent directions τ_0 at x_0 form an open covering of the $(n-2)$-dimensional subsphere S^{n-2} of S^{n-1} of all tangent directions at x_0. We select a finite subcovering $O''_{x_0,\tau_1},\ldots,O''_{x_0,\tau_m}$ and consider the corresponding open sets $O'_{x_0,\tau_1},\ldots,O'_{x_0,\tau_m}$ on S. Also, let O''' be an open set containing x_0 such that, for every point $x\in O'''$ and every tangent direction τ at x,

$$\tau\in\bigcup_{i=1}^{m}O''_{x_0,\tau_i}.$$

Let

$$O^*=O'''\cap O'_{x_0,\tau_1}\cap\ldots\cap O'_{x_0,\tau_m}.$$

For each $x\in O^*$, every tangent direction at x lies in some O''_{x_0,τ_i} and x belongs to the corresponding O'_{x_0,τ_i}. Thus, for each $x\in O^*$ and every tangent direction τ at x, the circle $C(\tau)$ has two points at distance less than n^{-1} from x, different from x, one in the interior and the other in the exterior of S. Then, obviously, $O^*\subset F_n$.

Thus $\complement F_n$ is nowhere dense and $\complement\cap F_n=\cup\complement F_n$ is of first Baire category on S. Hence, for most points x of S, in every tangent direction τ and for every natural number n, there exists $y(\tau)\neq x$ on $S(\tau)\cap C(\tau)$ at distance at most n^{-1} from x; then there exists on $S(\tau)\cap C(\tau)$ a sequence of points converging to x. This evidently implies that the radius r of C lies between $\varrho_i(\tau)$ and $\varrho_s(\tau)$. Now let r take consecutively the values 2, 1/2, 3, 1/3, 4, 1/4, ...; each time, for most points of S and every tangent direction τ,

$$\varrho_i(\tau)\leqq r\leqq\varrho_s(\tau).$$

Since any countable intersection of residual sets is residual, it follows that, for most points of S and every tangent direction τ, $\varrho_i(\tau)=0$ and $\varrho_s(\tau)=\infty$.

The proof is complete.

Theorem 2 makes more plausible Theorem 3 in [8], asserting that most pairs of internally tangent convex curves, one of which is a circle, meet at infinitely many points.

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