Three Small Cubic Graphs with Interesting Hamiltonian Properties

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In memory of my father

ABSTRACT

We present here three graphs, which are the smallest known ones of their kind: a cubic three-connected planar nontraceable graph, a cubic three-connected planar graph which is not homogeneously traceable, and a cubic one-Hamiltonian graph which is not Hamiltonian connected.

1. SMALL CUBIC 3-CONNECTED PLANAR GRAPH THAT IS NOT TRACEABLE

The determination of the minimum order (i.e., number of vertices) of a cubic 3-connected planar graph without Hamiltonian paths is of special interest in connection with a classification scheme for organic compounds (see Klee, in [5], Chap. 17). The first examples of such graphs have been found independently by Brown [3] and Grünbaum and Motzkin [7]. These graphs had rather large order. Later, Brown constructed a much smaller example, with 90 vertices only (see [5], Fig. 17.1.6).

A graph admitting a Hamiltonian path is called *traceable.*

The graph of Figure 1 was discovered some ten years ago and communicated to Grünbaum, who kindly published it in [6]. It is a cubic 3-connected planar nontraceable graph of order 88. (I thought I should delay publishing a proof since it did not (and still does not) seem to me to be the smallest such graph, and because I thought a smaller one would soon be discovered. But this appears not to have happened for a decade!) A proof now follows:

The Tutte "triangle" *abc* (see Fig. 1) has the property that no path joining *b* with *c* visits all the vertices of *abc* (see ref. [11]). If we contract

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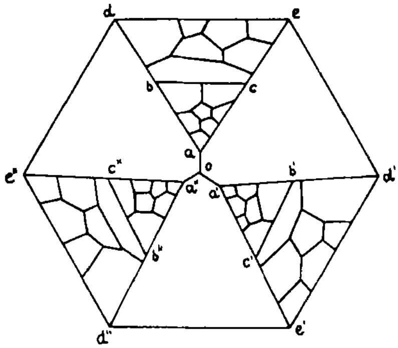

FIGURE 1

abc, the triangle ade is also a Tutte triangle; thus, with or without the contraction, no path joining a with d within ade visits all its vertices. Also, if a path joining d with e would contain all vertices of ade, this path would visit all vertices of abc between b and c, which is impossible. Therefore, each Hamiltonian path of ade joining two of the vertices a, d, e necessarily joins a with e.

Suppose now the graph possesses a Hamiltonian path H. Because H has at most two endpoints, at least one of the triangles ade, $a'd'e'$, $a''d''e''$, say $a'd'e'$, contains no endpoint of H. Then $H \cap a'd'e'$ is a Hamiltonian path of $a'd'e'$ with endpoints a' and e'. The graph $H \cap a''d''e''$ has at most two components; otherwise, H would have more than two endpoints.

If $H \cap a''d''e''$, includes a path joining d'' with e'', then H has an endpoint in $a''d''e''$, de'' must lie on H, and $H \cap ade$ has to be a Hamiltonian path joining d with a, which is impossible.

If $H \cap a''d''e''$ includes a path joining a'' with e'', then either oa'' does not lie on H and $H \cap ade$ must be a Hamiltonian path of ade joining d with a (which is not possible) or oa'' lies on H and $H \cap ade$ must be a Hamiltonian path of ade with an endpoint in d. But, in the second case, either $H \cap abc$ is a Hamiltonian path of abc joining b with c, or $H \cap ade$ has its second endpoint in abc, both impossible.

Finally if $H \cap a''d''e''$ includes a path joining d'' with e'', then H has an endpoint in $a''d''e''$, edges oa'' and $e''d$ lie on H, and therefore $H \cap ade$ must be a Hamiltonian path of ade with an endpoint in d, which is impossible as seen above.

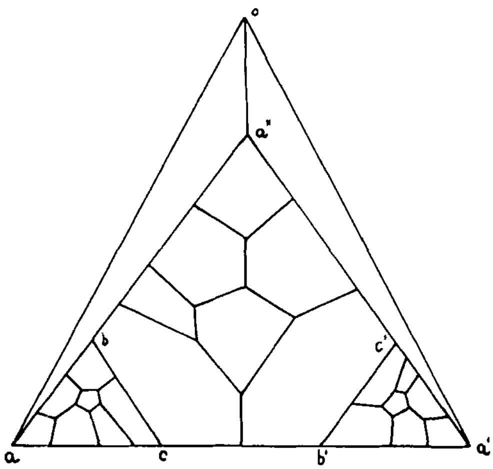

FIGURE 2

2. SMALL CUBIC 3-CONNECTED PLANAR GRAPH THAT IS NOT HOMOGENEOUSLY TRACEABLE

This section deals with homogeneously traceable graphs, introduced by Skupień [10] (see also [4]), and lying in their degree of generality between Hamiltonian and traceable graphs. A graph is *homogeneously traceable* if in each vertex starts a Hamiltonian path of the graph. As is well-known, the 3-connected planar graphs are quite meaningful geometrically. It is therefore a natural question to ask again about the smallest cubic graph of this kind, which is not homogeneously traceable.

It can be easily shown that the famous graph of Tutte from [11] is not homogeneously traceable and that the smallest known non-Hamiltonian cubic 3-connected planar graph of Lederberg [9], Bosak [2], and Barnette [5] is homogeneously traceable. We present here a graph of the mentioned kind, which is not homogeneously traceable and has 44 vertices (two less than Tutte's). This is the graph of Figure 2, which also contains Tutte triangles, already used in Sec. 1.

We show that no Hamiltonian path starts at o. Suppose H would be a Hamiltonian path with o as endpoint.

If oa lies on H, then either H has the other endpoint e in $a'b'c'$ and $H \cap aa'a''$ is a Hamiltonian path of $aa'a''$ joining a with e (which is impossible), or $H \cap a'b'c'$ is a Hamiltonian path of $a'b'c'$ joining b' with c' (which is equally impossible).

The case that oa' lies in H is analogous. Let oa' lie on H. Since at least one of the Tutte triangles abc and $a'b'c'$, say abc, does not contain an

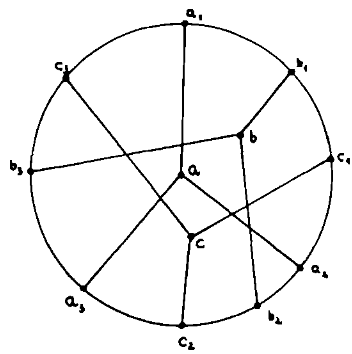

FIGURE 3

endpoint of H, $H \cap abc$ is a Hamiltonian path of abc joining b with c, which is impossible.

3. SMALL CUBIC 1-HAMILTONIAN GRAPH THAT IS NOT HAMILTONIAN CONNECTED

A Hamiltonian graph which remains Hamiltonian after dropping any vertex is called 1-*Hamiltonian*. A graph possessing a Hamiltonian path between any pair of vertices is called *Hamiltonian connected.*

In answering a question of Bermond [appeared also as a problem in *Graph Theory Newslett.* 3(4) (March 1974)] whether 1-Hamiltonian graphs which are not Hamiltonian connected exist, Danzer constructed a cubic graph of this kind with 18 vertices (private communication). Shortly thereafter, I found and communicated to Bermond a smaller example of a cubic graph satisfying the preceding conditions, which will be presented here. Recently, Bermond claimed to have discovered infinitely many (noncubic) graphs of this type (see Theorem 10.3 in [1]), but, unfortunately, the proof is in error. In particular, the smallest graph obtainable via Theorem 10.3, which has nine vertices, is Hamiltonian connected. I conjecture that the graph G of Figure 3, which has 12 vertices, is the smallest example (or at least the smallest cubic one).

For a proof that G has the stated properties, let x be an extra vertex and join x with a, b, c. The new graph G' coincides with the hypohamiltonian graph of order 13 of Herz et al. [8]. Thus G is Hamiltonian. Also,

$$bb_1a_1c_3cc_1a_2b_2c_2a_3b_3b$$

is a Hamiltonian cycle of $G-a$ and

$$aa_2c_1b_1bb_2c_2cc_3b_3a_3a$$

is a Hamiltonian cycle of $G-a_1$. Hence, by symmetry, G is 1-Hamiltonian. On the other hand a and b cannot be joined by a Hamiltonian path, because otherwise G' would be Hamiltonian.

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