References

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MOST MONOTONE FUNCTIONS ARE SINGULAR

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The history of singular functions, i.e., monotone continuous functions of one real variable with a.e. vanishing derivative, begins in 1904, when H. Lebesgue [1] and H. Minkowski [2] produced their well-known examples. Since then many interesting examples have been published. One of the most recent is due to L. Takács [3], who also presents a good bibliography.

The purpose of this note is to present the following result: *Most monotone functions are singular*. In fact we shall prove the slightly less easy assertion that most functions of uniformly bounded variation have an a.e. vanishing derivative. The monotone case can be treated analogously; therefore a separate proof will not be given. The word *most* has to be understood in the sense of *all, except those in a set of first Baire category*, and will be used only in a space of second Baire category.

Let $\mathcal{C}([0,1])$ be the space of continuous real functions on $[0,1]$ and $\mathcal{V}(V)$ be the family of all functions in $\mathcal{C}([0,1])$ with variation at most $V\in\mathbb{R}$. Clearly, $\mathcal{V}(V)$ is closed in $\mathcal{C}([0,1])$ and therefore is a complete metric subspace with respect to the usual supremum-distance. Thus $\mathcal{V}(V)$ is in itself a space of second Baire category.

Let f_i^- and f_s^- (f_i^+ and f_s^+) be the left (right) lower and upper Dini derivatives of $f:[0,1]\to\mathbb{R}$.

THEOREM 1. *For most functions $f\in\mathcal{V}(V)$, we have, at each point $x\in(0,1]$,*

$$f_i^-(x)=-\infty \quad \text{or} \quad f_i^-(x)\leqslant 0\leqslant f_s^-(x) \quad \text{or} \quad f_s^-(x)=\infty;$$

and, at each point $x\in[0,1)$,

$$f_i^+(x)=-\infty \quad \text{or} \quad f_i^+(x)\leqslant 0\leqslant f_s^+(x) \quad \text{or} \quad f_s^+(x)=\infty.$$

Proof. Let $n\in\mathbb{N}$ and let $\mathcal{V}_n$ be the family of all functions $f\in\mathcal{V}(V)$ such that there exists a point $x\in[0,1]$ for which (i) $x-n^{-1}\geqslant 0$ and

$$n^{-1}\leqslant\left|\frac{f(y)-f(x)}{y-x}\right|\leqslant n \tag{$*$}$$

for each point $y\in(x-n^{-1},x)$, or (ii) $x+n^{-1}\leqslant 1$ and ($*$) holds for each point $y\in(x,x+n^{-1})$.

We show that $\mathcal{V}_n$ is closed in $\mathcal{V}(V)$. Let $f_m\to f$ with $f_m\in\mathcal{V}_n$ and $f\in\mathcal{V}(V)$. There exists $x_m\in[0,1]$ with

$$n^{-1}\leqslant\left|\frac{f_m(y)-f_m(x_m)}{y-x_m}\right|\leqslant n \tag{$**$}$$

for each y in $(x_m-n^{-1},x_m)\subset[0,1]$ or each y in $(x_m,x_m+n^{-1})\subset[0,1]$. We may suppose that x_m converges to some point $x_0\in[0,1]$ and that one of the intervals $(x_m-n^{-1},x_m),(x_m,x_m+n^{-1})$, say the first of these, can be taken for all indices m. (Otherwise consider a subsequence.) Now let $y_0\in(x_0-n^{-1},x_0)$. For m large enough, $y_0\in(x_m-n^{-1},x_m)$ and ($**$) holds with $y=y_0$. Since $f_m\to f$ and $x_m\to x_0$, we get

$$n^{-1} \leqslant \left| \frac{f(y_0)-f(x_0)}{y_0-x_0} \right| \leqslant n,$$

which proves $f \in \mathcal{V}_n$.

Now, we show that $\mathcal{V}(V)-\mathcal{V}_n$ is dense in $\mathcal{V}(V)$. Let $\mathcal{O}$ be an open set in $\mathcal{V}(V)$, choose $f \in \mathcal{O}$, and let $\varepsilon>0$ be such that every function at distance at most ε from f lies in $\mathcal{O}$. Since f is uniformly continuous, there is a $\delta>0$ such that $|x-y|<\delta$ implies $|f(x)-f(y)|<\varepsilon$. We introduce a partition $0=a_0, a_1, \ldots, a_k, a_{k+1}=1$ of $[0,1]$ such that $a_i<a_{i+1}$ and $a_{i+1}-a_i<\delta (i=0,\ldots,k)$. If $f(a_i) \neq f(a_{i+1})$, let $b_i \in (a_i, a_{i+1})$ be such that

$$\left| \frac{f(a_{i+1})-f(a_i)}{b_i-a_i} \right| > n;$$

otherwise let $b_i=a_i$. We construct a continuous function $g:[0,1] \to \mathbb{R}$ in the following way: the restriction of g to $[a_i, a_{i+1}]$ is linear from a_i to b_i and constant on the rest, $g(a_i)=f(a_i)$ and $g(b_i)=g(a_{i+1})=f(a_{i+1})$ $(i=0,\ldots,k)$. Clearly, the distance from f to g does not exceed ε and the variation of g is not greater than that of f. Also, $g \notin \mathcal{V}_n$ by construction. Thus $g \in (\mathcal{V}(V)-\mathcal{V}_n) \cap \mathcal{O}$.

Now, since $\mathcal{V}_n$ is closed and has a dense complement, it is nowhere dense in $\mathcal{V}(V)$.

Let $\mathcal{V}^*$ be the family of those functions $f \in \mathcal{V}(V)$ for which, at some point $x \in (0,1]$,

$$f_i^-(x) > -\infty \quad \text{and} \quad f_s^-(x) < 0$$

or

$$f_i^-(x) > 0 \quad \text{and} \quad f_s^-(x) < \infty,$$

or, at some point $x \in [0,1)$,

$$f_i^+(x) > -\infty \quad \text{and} \quad f_s^+(x) < 0$$

or

$$f_i^+(x) > 0 \quad \text{and} \quad f_s^+(x) < \infty.$$

It is easily seen that each $f \in \mathcal{V}^*$ belongs to $\mathcal{V}_n$ for a certain $n \in \mathbb{N}$. We have

$$\mathcal{V}^* = \bigcup_{n=1}^{\infty} \mathcal{V}_n;$$

hence $\mathcal{V}^*$ is of first Baire category and the theorem follows.

Now, if $f \notin \mathcal{V}^*$, then, for every point $x \in [0,1]$, either f is not differentiable at x, or $f'(x)=0$. Since f is differentiable a.e., we have the following:

THEOREM 2. *For most functions $f \in \mathcal{V}(V)$, $f'=0$ a.e..*

With the topology derived from the supremum-metric, we can say nothing similar about the whole space $\mathcal{V}$ of functions of bounded variation from $\mathcal{C}([0,1])$, because

$$\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}(n)$$

and each $\mathcal{V}(n)$ is nowhere dense in $\mathcal{V}$; hence $\mathcal{V}$ is of the first category itself.

However, we can consider the space $\mathcal{M}$ of all functions in $\mathcal{C}([0,1])$ that are increasing. Here, without any restriction on the variation, $\mathcal{M}$ is closed in $\mathcal{C}([0,1])$, hence is of second Baire category in itself. In a similar way as for Theorem 1 we can prove:

THEOREM 3. *For most functions $f \in \mathcal{M}$, we have, at each point $x \in (0,1]$,*

$$f_i^-(x)=0 \quad or \quad f_s^-(x)=\infty,$$

and, at each point $x \in [0,1)$,

$$f_i^+(x)=0 \quad or \quad f_s^+(x)=\infty.$$

Since every function $f \in \mathcal{M}$ is differentiable a.e., the following holds.

THEOREM 4. *For most functions* $f \in \mathcal{M}$, $f' = 0$ *a.e.*

Analogous phenomena happen for first-order Lipschitz maps. We consider those functions $f \in \mathcal{C}([0,1])$ for which

$$\alpha \leqslant \frac{f(y)-f(x)}{y-x} \leqslant \beta$$

for all pairs of distinct points x,y in $[0,1]$, α and β being fixed real numbers ($\alpha < \beta$). Let $\mathcal{V}_{\alpha,\beta}$ be the family of all such functions; obviously $\mathcal{V}_{\alpha,\beta} \subset \mathcal{V}$ $(\max\{|\alpha|,|\beta|\})$. $\mathcal{V}_{\alpha,\beta}$ is of the second category. We get analogously:

THEOREM 5. *For most functions* $f \in \mathcal{V}_{\alpha,\beta}$, *we have, at each point* $x \in (0,1]$,

$$f_i^-(x) = \alpha \quad \text{or} \quad f_s^-(x) = \beta,$$

and, at each point $x \in [0,1)$,

$$f_i^+(x) = \alpha \quad \text{or} \quad f_s^+(x) = \beta.$$

THEOREM 6. *For most functions* $f \in \mathcal{V}_{\alpha,\beta}$, *the set*

$$f'^{-1}(\alpha) \cup f'^{-1}(\beta)$$

has measure 1.

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AN ALGORITHM-INSPIRED PROOF OF THE SPECTRAL THEOREM IN E^n

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THEOREM. *If A is a real symmetric matrix, there is a real orthogonal matrix Q such that Q^TAQ is diagonal.*

Of course, this is the spectral theorem. It implies that the eigenvalues are real, that there is a pairwise orthogonal complete set of eigenvectors—namely, the columns of Q—and that the dimension of an eigenspace is equal to the algebraic multiplicity of the eigenvalue.

Many proofs grapple with the question of finding enough independent eigenvectors for a multiple eigenvalue, usually one at a time. We shall find the whole matrix Q at once by using the main idea of Jacobi's numerical method for calculating the eigenvalues and vectors, together with a little compactness.

For an $n \times n$ real matrix A we shall use $\mathrm{Od}(A)$ for the sum of the squares of the off-diagonal elements of A, and $O(n)$ will denote the set (group) of $n \times n$ orthogonal matrices.

Suppose we can prove the following.

LEMMA. *If A is a nondiagonal real symmetric matrix, then there is a real orthogonal matrix J such that* $\mathrm{Od}(J^TAJ) < \mathrm{Od}(A)$.

Then the theorem would follow quickly, for let A be real and symmetric. Consider the mapping f that sends an orthogonal matrix P into $f(P) = P^TAP$. For fixed A this is a continuous mapping of $O(n)$, a compact set, and so $f(O(n))$ is compact. Let $f(Q) = D$ be a point at which the continuous function Od attains its minimum value on the image set of f. This value must be