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ON CONTINUOUS FAMILIES OF CURVES. VI

1. Introduction

This paper is about continuous families of curves in Grünbaum's sense [1], later called spreads [2].

Let C be a Jordan closed curve in the Euclidean plane and D be the bounded domain with boundary C. A family $\mathfrak{L}$ of simple arcs in $\bar{D}$ (further called *curves*) is a *spread* (*continuous family of curves*) provided [1]:

(i) each curve in $\mathfrak{L}$ (except its endpoints) lies in D and its endpoints belong to C,

(ii) each point $p \in C$ is the endpoint of exactly one curve $L(p) \in \mathfrak{L}$,

(iii) if L_1, L_2 are different curves of $\mathfrak{L}$, then $L_1 \cap L_2$ is a single point,

(iv) the curve $L(p)$ depends continuously on $p \in C$.

The reader is invited to verify that, on C, the endpoints of any curve separate those of any other curve and that $L(p) \cap L(q)$ depends continuously on p and q for $p \neq q$.

We recall [4], [6]:

$$M_x(\mathfrak{L}) = \{a \in D : \text{card}\,\{L \in \mathfrak{L} : a \in L\} \geqslant x\},$$

$$T_x(\mathfrak{L}) = \{a \in D : \text{card}\,\{L \in \mathfrak{L} : a \in L\} = x\}.$$

The bounded component of the complement of the union of three non-concurrent curves in $\mathfrak{L}$ is called *triangle*. Let $T(\mathfrak{L})$ be the union of all triangles in D (compare [5]).

A spread $\mathfrak{L}$ is called *degenerate* if $\cap\, \mathfrak{L} \neq \varnothing$.

We are mainly interested here in the following conjecture and in properties of $T_2(\mathfrak{L})$ and $T(\mathfrak{L})$.

CONJECTURE (*Grünbaum* [2]). *For each nondegenerate spread* $\mathfrak{L}$,

$$T_2(L) \neq \varnothing.$$

We say that the spread $\mathfrak{L}$ has *property* C_3 in $p \in C$ if

$$\left.\begin{array}{ll} p_n, q_n \in C & (n \in \mathbb{N}) \\ L(p_n) \neq L(q_n) & (n \in \mathbb{N}) \\ p_n, q_n \to p & \end{array}\right\} \Rightarrow \{L(p_n) \cap L(q_n)\}_{n=1}^{\infty} \text{ converges.}$$

$\mathfrak{L}$ is a C_3-*spread* if it has property C_3 in all points of C (see [7]).

For example, the family of all area-bisectors of a plane convex body, the family of all diameters of a plane convex curve with finite positive curvature

Geometriae Dedicata **10** (1981) 205–217. 0046–5755/81/0102–0205$01.95

everywhere and the family of all perimeter-bisectors of a plane convex curve of class C^1 are C_3-spreads. In view of an application of Theorem 1 we also remark that for an arbitrary (not necessarily of class C^1) plane convex curve, the last spread mentioned above has property C_3 almost everywhere.

Let

$$C_{(n)} = \{p \in C : L(p) \cap T_n(\mathfrak{L}) \neq \varnothing\}.$$

Also, let $\langle p, q\rangle$ be the single point of the set $L(p) \cap L(q)$ for $L(p) \neq L(q)$.

For $p \in C$, we denote by $-p$ the other endpoint of $L(p)$.

If $p, q \in C$ and $q \neq -p$, then pq denotes that arc of C with endpoints p and q, which does not contain $-p$. If $x, y \in L$ and $L \in \mathfrak{L}$, then xy denotes the subarc of L with endpoints x, y.

By int A, $\bar{A}$, ∂A we mean the interior, closure, frontier of A in the topology of the plane and by $\mathrm{int}^+ A$ the interior of A with respect to the topology of some $L \in \mathfrak{L}$ or C, in case $A \subset L$ or $A \subset C$.

Throughout the paper, only nondegenerate spreads will be considered, and this will not be mentioned further.

For a general discussion of spreads with historical and bibliographical notes see Grünbaum [2]. For further developments on arbitrary spreads and on spreads under continuity restrictions (like property C_3) as well see Zamfirescu [7].

2. On the existence of double points

In this section (only) C will be a circle, which is no topological restriction. The following result was promised (without being made precise) in [6].

THEOREM 1. *If $\mathfrak{L}$ is a spread with property C_3 in some point of C, then $C_{(2)}$ has positive Lebesgue inner measure and is not rare on C, whence $T_2(\mathfrak{L})$ is uncountable.*

Proof. Let

$$\varphi(q) = \{\lim_{n\to\infty} \langle p_n, q\rangle : p_n \to q, p_n \neq q \text{ and } \{\langle p_n, q\rangle\}_{n=1}^{\infty} \text{ converges}\},$$

where $q \in C$. We remark that $\varphi(q)$ is a closed set with at most two connected components, possibly reduced to a point, like for instance in case q equals the point p, in which $\mathfrak{L}$ has property C_3 (see Figure 1).

Since $\mathfrak{L}$ is nondegenerate, $L(p) \cap M_2(\mathfrak{L})$ is a nondegenerate (not necessarily closed) arc γ. At least one of the endpoints of γ, say a, does not coincide with $\varphi(p)$ and thus belongs to γ. Obviously $\varphi(q) \to \varphi(p)$ for $q \to p$, because $\mathfrak{L}$ has property C_3 in p (the convergence is considered with respect to the Hausdorff metric in the space of compact sets). Let $L(r) \ni a$ with $r \neq p$. Since $\varphi(p) \notin L(r)$, there exists a neighbourhood A of p such that $\varphi(q)$ and $\varphi(p)$ are on the same

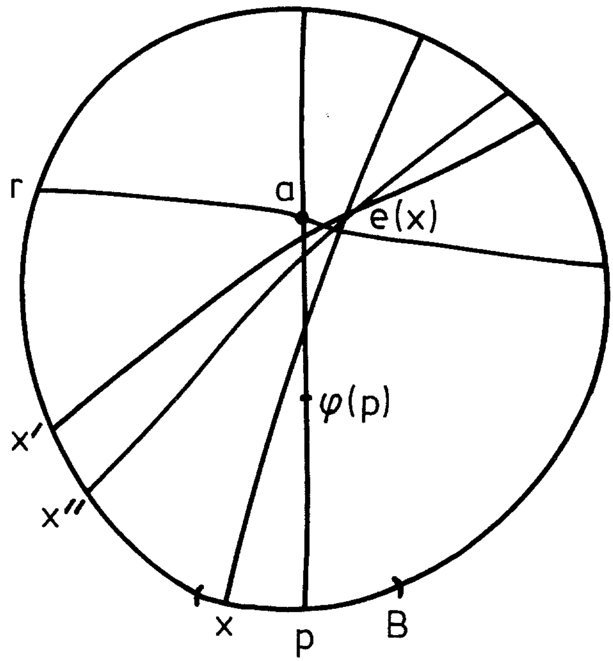

Fig. 1

side of $L(r)$ for all $q \in A$. One of the endpoints of $L(q) \cap M_2(\mathfrak{L})$ lies on $L(r)$ or is separated from $\varphi(q)$ by $L(r)$. Let $e(q)$ be this endpoint. We have $e(q) \in$ $\in M_2(\mathfrak{L})$.

Suppose for each neighbourhood B' of p, there are two distinct points x, $y \in B'$ such that $L(x)$ and $L(y)$ meet in a point on $L(r)$ or separated from $\varphi(p)$ by $L(r)$. But because $\mathfrak{L}$ has property C_3 in p,

$$\lim_{\substack{x,y \to p \\ x \neq y}} \langle x, y \rangle = \varphi(p)$$

and a contradiction is obtained. Thus there exists a neighbourhood $B \subset A$ of p, such that for each point $x \in B$ no other curve with an endpoint in B passes through $e(x)$. Suppose $e(x) \in M_3(\mathfrak{L})$ for some $x \in B \cap pr$. Then there are two curves $L(x')$, $L(x'')$ with x', $x'' \in pr - B$, which meet in $e(x)$. If now $y \in B$, $e(y) \in M_3(\mathfrak{L})$, and $L(y')$, $L(y'')$ are curves analogous to $L(x')$, $L(x'')$, then $x \neq y$ implies

$$\operatorname{int}^+ x'x'' \cap \operatorname{int}^+ y'y'' = \varnothing .$$

It follows that

$$\{x \in B : e(x) \in M_3(\mathfrak{L})\}$$

is countable. Thus, λ being the Lebesgue inner measure on C,

$$\lambda C_{(2)} \geqslant \lambda B > 0$$

and $C_{(2)} \cap B$ is dense on B. From $\lambda C_{(2)} > 0$ it follows that $C_{(2)}$ is uncountable, whence $T_2(\mathfrak{L})$ is uncountable too.

For arbitrary spreads Grünbaum's conjecture remains open; for spreads with line-segments as curves it is proved by Theorem 7.

3. About the set $T(\mathfrak{L})$

A Jordan arc is called *2-curve* if it is the union of two arcs lying on curves in $\mathfrak{L}$. A set $S \subset D$ is called an *$L_2(\mathfrak{L})$-set* if each pair of points in S are joined by a 2-curve lying in S (compare [1], [4]).

THEOREM 2. *In an arbitrary spread $\mathfrak{L}$, $T(\mathfrak{L})$ is a simply connected $L_2(\mathfrak{L})$-set.*

Proof. Theorem 1 from [4] states that $M_3(\mathfrak{L})$ is an $L_2(\mathfrak{L})$-set. In its proof it is shown that for each pair of points $x, y \in M_3(\mathfrak{L})$ there exists a 2-curve $xz \cup zy \subset T(\mathfrak{L}) \cup \{x, y\}$. Now, if we choose x, $y \in T(\mathfrak{L})$, we also have $x, y \in M_3(\mathfrak{L})$ (see [1]) and hence we get a 2-curve completely contained in $T(\mathfrak{L})$.

By Theorem 3 of Grünbaum [1], $M_2(\mathfrak{L})$ is $\mathfrak{L}$-convex (i.e. the intersection with every curve in $\mathfrak{L}$ is empty or connected) and connected. It follows that $M_2(\mathfrak{L})$ has a connected complement, for, if z were a point in some bounded component of the complement of $M_2(\mathfrak{L})$, then any curve of $\mathfrak{L}$ through z would intersect $M_2(\mathfrak{L})$ in more than one component. Then also int $M_2(\mathfrak{L})$ has a connected complement since a set remains connected when adding boundary points. Since $T(\mathfrak{L})$ is connected and equals int $M_2(\mathfrak{L})$ by Theorem 2 in [5], it follows from Theorem VI.4.1 in [3] that $T(\mathfrak{L})$ is simply connected.

We prove now a conjecture from [7].

THEOREM 3. *In each C_3-spread $\mathfrak{L}$, $T(\mathfrak{L})$ is a Jordan domain.*

Proof. We show that $T(\mathfrak{L})$ is uniformly locally connected. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences of points of $T(\mathfrak{L})$ converging to the same point z. Let $L(\xi_{1n})$, $L(\xi_{2n})$, $L(\xi_{3n})$ (respectively $L(\zeta_{1n})$, $L(\zeta_{2n})$, $L(\zeta_{3n})$) be the curves of $\mathfrak{L}$ forming a triangle containing x_n (respectively y_n). We can suppose without loss of generality that all six sequences $\{\xi_{1n}\}_{n=1}^{\infty}$, $\{\xi_{2n}\}_{n=1}^{\infty}$, $\{\xi_{3n}\}_{n=1}^{\infty}$, $\{\zeta_{1n}\}_{n=1}^{\infty}$, $\{\zeta_{2n}\}_{n=1}^{\infty}$, $\{\zeta_{3n}\}_{n=1}^{\infty}$ converge. Let ξ_1, ξ_2, ξ_3, ζ_1, ζ_2, ζ_3 be their respective limit points. There are several possibilities for the mutual position of $L(\xi_1)$, $L(\xi_2)$, $L(\xi_3)$, $L(\zeta_1)$, $L(\zeta_2)$, $L(\zeta_3)$, and z; however we consider here in detail just two of these.

What we do now is to show that in any case $T(\mathfrak{L})$ is locally connected in z.

The case where z lies on none of the six curves is trivial. All cases with z on the boundary of some triangle with sides on the six curves are analogous to the first one treated here. The differences between the proofs of these various cases are only of a technical nature and do not necessitate separate consideration. They do not use property C_3. In the (five) remaining cases at least two of the curves $L(\xi_1)$, $L(\xi_2)$, $L(\xi_3)$ and at least two of $L(\zeta_1)$, $L(\zeta_2)$, $L(\zeta_3)$ coincide. The proofs of these are exactly like our second proved case. They all use the property C_3.

We first consider the case shown in Figure 2.

Let $D(\eta)$ be that Jordan domain with $\partial D(\eta) \subset L(\eta) \cup C$, which contains $\langle \xi_2, \xi_3 \rangle$, for each $\eta \in C$ with $L(\eta) \cap L(\xi_2) \cap L(\xi_3) = \varnothing$.

Let δ be a Jordan domain containing z, such that $\partial\delta$ meets exactly twice each of the curves $L(\xi_1)$, $L(\zeta_1)$, and $\bar{\delta} \cap L(\xi_2) = \bar{\delta} \cap L(\xi_3) = \varnothing$. There exists a natural number m such that for all $n \geqslant m$, $x_n, y_n \in \delta$, $\langle \xi_{1n}, \zeta_{1n} \rangle \in \delta$, and $L(\xi_{2n}) \cap \delta = L(\zeta_{2n}) \cap \delta = L(\xi_{3n}) \cap \delta = L(\zeta_{3n}) \cap \delta = \varnothing$. For such an n, we have $x_n \in D(\xi_{1n})$.

Consider the point $v \in \delta \cap L(\xi_1)$ between ξ_1 and z. Since $v \in M_2(\mathfrak{L})$, there exists another curve than $L(\xi_1)$, say $L(\nu)$, passing through v. Let α be the connected component of $L(\nu) \cap \delta - D(\xi_1)$ containing v. There exists an $m' \geqslant m$ such that for all $n \geqslant m'$, x_n and z lie on the same side of $L(\nu)$ and either $x_n \in D(\xi_1)$ or $L(\xi_{1n})$ meets α. If $x_n \in D(\xi_1)$, then x_n and $\delta \cap D(\xi_1) \cap D(\zeta_1)$

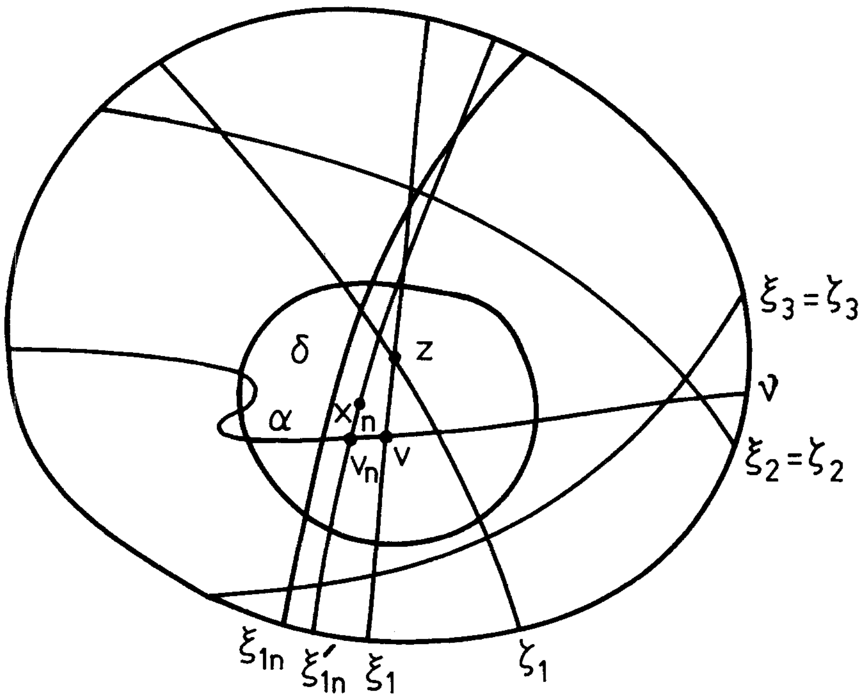

Fig. 2

lie both in the connected set $\delta \cap D(\xi_1) \subset T(\mathfrak{L})$. If $x_n \notin D(\xi_1)$, there is a point $\xi'_{1n} \in C$ between ξ_{1n} and ξ_1 such that $x_n \in L(\xi'_{1n})$. Obviously $L(\xi'_{1n})$ also meets α in a point v_n and $x_n v_n \to zv$. Thus for n large enough $x_n v_n \subset \delta$ and $x_n v_n \cup v_n v \cup (\delta \cap D(\xi_1))$ is a connected set included in $T(\mathfrak{L}) \cap \delta$ and containing both x_n and $\delta \cap D(\xi_1) \cap D(\zeta_1)$. Analogously, for n sufficiently large, y_n and $\delta \cap D(\xi_1) \cap D(\zeta_1)$ both lie in a connected subset of $T(\mathfrak{L}) \cap \delta$.

Hence x_n and y_n belong (for n large enough) to a connected subset of δ, which is included in $T(\mathfrak{L})$.

For the previous case we did not in fact use the property C_3 of $\mathfrak{L}$; thus we also briefly present a case the proof of which essentially needs it. Look at Figure 3. We choose again the Jordan domain δ around z avoiding ξ_3 and ζ_3; let $x_n, y_n \in \delta$. Since $x_n \to z$ and the vertices of the triangle determined by $L(\xi_1)$, $L(\xi_{1n})$, $L(\xi_{2n})$ must converge to a unique point by property C_3, this point must be z. Thus $x_n u_n \to z$, where u_n is the intersection of $L(\xi_1)$ with a curve $L(\xi'_n) \ni x_n$ with $\xi'_n \in \xi_{1n}\xi_{2n}$. We define v_n analogously and get $y_n v_n \to z$, hence $u_n v_n \to z$ too. Thus, for large n, $x_n u_n \cup u_n v_n \cup v_n y_n \subset T(\mathfrak{L}) \cap \delta$, whence again x_n and y_n belong to a connected subset of $T(\mathfrak{L}) \cap \delta$.

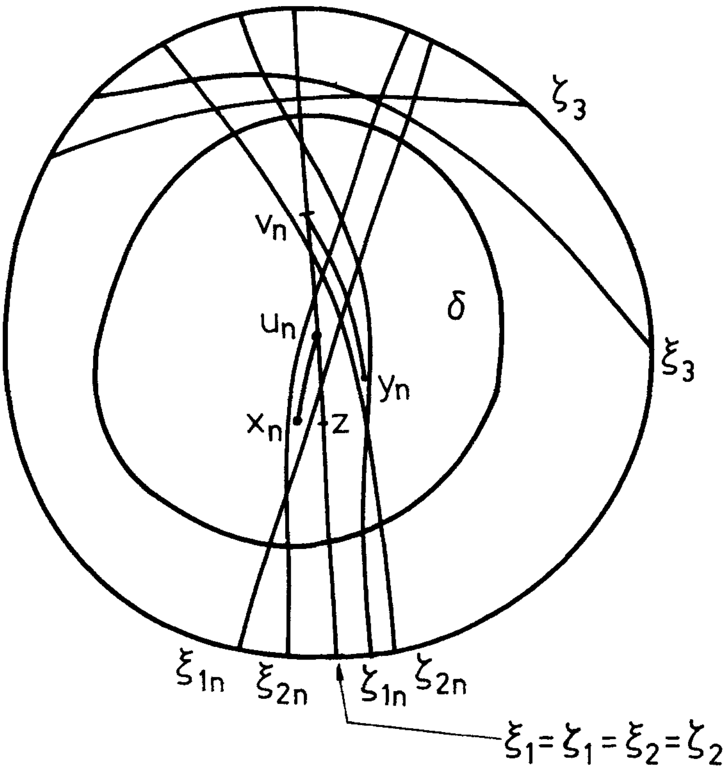

Fig. 3

Thus it is shown that $T(\mathfrak{L})$ is locally connected in the arbitrary point $z \in \overline{T(\mathfrak{L})}$. Since $T(\mathfrak{L})$ is bounded, it is also uniformly locally connected by Theorem 13.1 in [3].

The simple connectedness guaranteed by Theorem 2 and the uniform local connectedness of $T(\mathfrak{L})$ imply together, by Theorem 16.2 from [3], that $T(\mathfrak{L})$ is a Jordan domain, and the proof is achieved.

It is remarkable that asking $\mathfrak{L}$ to be a C_3-spread in Theorem 3 is not superfluous. This follows from the existence of spreads $\mathfrak{L}$ for which $T(\mathfrak{L})$ is not locally connected (see [7]).

We also notice that from Grünbaum's inclusions $M_2(\mathfrak{L}) \subset \overline{T(\mathfrak{L})}$ and $T(\mathfrak{L}) \subset M_3(\mathfrak{L})$ [1] it follows $T_2(\mathfrak{L}) \subset \partial T(\mathfrak{L})$.

4. Relationship between $T_2(\mathfrak{L})$ and $T(\mathfrak{L})$

PROPOSITION 1. *In an arbitrary spread $\mathfrak{L}$, for each point $a \in \partial T(\mathfrak{L})$ there exists a point $p \in C$ such that*

$$a \in L(p) - \operatorname{int}^+(L(p) \cap M_2(\mathfrak{L})).$$

Proof. Let $a \in \partial T(\mathfrak{L})$. Since $a \in \bar{D}$, $a \in L(x)$ for some $x \in C$. We choose $p = x$ if a is an endpoint of or does not belong to $\overline{L(x) \cap M_2(\mathfrak{L})}$. Otherwise there are two curves $L(y)$, $L(z)$ such that $a \in \operatorname{int}^+\langle x, y\rangle\langle x, z\rangle$ and $y \in xz$ (see Figure 4). We put $t < t'$ on yz if $t \in \operatorname{int}^+ yt'$. Let

$$H = \{t \in yz : a \in L(t)\}.$$

Obviously $H \neq \varnothing$. Let $h \in H$. If a is an endpoint of $\overline{L(h) \cap M_2(\mathfrak{L})}$, we choose $p = h$. If not, there is another point $h' \in yz$ such that $a \in \operatorname{int}^+\langle h', h\rangle\langle y, h\rangle$ (if h' were to lie on $-zy$, a would belong to the triangle determined by $L(h')$, $L(y)$, $L(z)$, which is false). Suppose, to make a choice, that $h' \in yh$. Let

$$H_1 = \{x \in H : \exists x' \in yx \text{ such that } a \in \operatorname{int}^+\langle x', x\rangle\langle y, x\rangle\},$$

$$H_2 = \{x \in H : \exists x' \in xz \text{ such that } a \in \operatorname{int}^+\langle x', x\rangle\langle y, x\rangle\}.$$

Put $h_1 = \inf H_1$. Since H is closed, $h_1 \in H$.

Case I. $h_1 \in H_1$. In this case, there is a point $h_1' \in yh_1$ such that $a \in \operatorname{int}^+ \langle h_1', h_1\rangle\langle y, h_1\rangle$. Thus there is another point $h_1'' \in yh_1'$ such that $a \in L(h_1'')$. Since $h_1'' < h_1$, $h_1'' \notin H_1$. If $h_1'' \notin H_2$, then we choose $p = h_1''$. If $h_1'' \in H_2$, put $h_2 = \sup(H_2 \cap yh_1)$. We have $h_1 \neq h_2$, otherwise $a \in T(\mathfrak{L})$, which is not true. Hence $h_2 < h_1$ and $h_2 \notin H_1$. If $h_2 \notin H_2$, then we can choose $p = h_2$. Suppose now $h_2 \in H_2$. Because for each $x \notin \operatorname{int}^+ h_1'h_1$, $a \in \langle x, h_1\rangle\langle y, h_1\rangle$, whence $\langle x, h_2\rangle \in ah_2$, there exists $h_2' \in h_2h_1'$ such that $a \in \operatorname{int}^+\langle h_2', h_2\rangle\langle y, h_2\rangle$. Thus there exists a point $h^+ \in C$ such that $h_2' < h^+ < h_1'$ and $a \in L(h^+)$. It follows $h^+ \notin H_1 \cup H_2$, and so we can choose $p = h^+$.

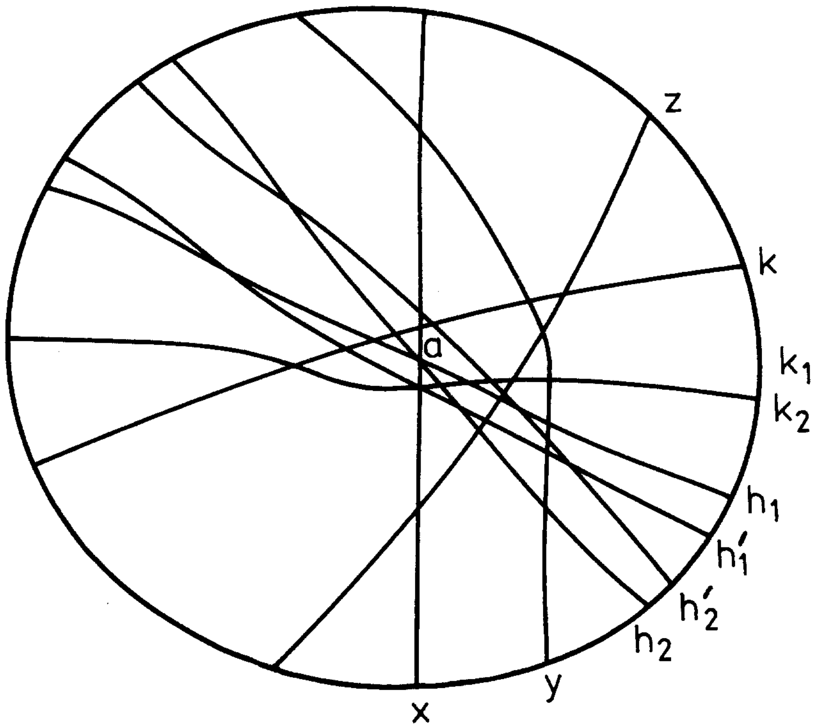

Fig. 4

Case II. $h_1 \notin H_1$. We have only the case $h_1 \in H_2$ to consider. Let $k \in h_1 z$ be such that $a \in \operatorname{int}^+\langle k, h_1\rangle\langle y, h_1\rangle$. Consider the points k_1 and k_2 such that $h_1 < k_2 < k_1 < k$, $k_1 \in H_1$ and $a \in \operatorname{int}^+\langle k_2, k_1\rangle\langle y, k_1\rangle$. It follows that a lies in the triangle determined by $L(k)$, $L(k_2)$, $L(y)$, which is false.

The proof is achieved.

PROPOSITION 2. *For each C_3-spread $\mathfrak{L}$ and point $p \in C$,*

$$L(p) \cap \overline{T(\mathfrak{L})} = \overline{L(p) \cap M_2(\mathfrak{L})}$$

Proof. Since $\overline{T(\mathfrak{L})} = \overline{M_2(\mathfrak{L})}$ by Theorem 3 in [5], the inclusion

$$L(p) \cap \overline{T(\mathfrak{L})} \supset \overline{L(p) \cap M_2(\mathfrak{L})}$$

is trivial. The proof of the converse inclusion parallels the proof of Theorem 8 in [7] and is therefore omitted.

THEOREM 4. *In a C_3-spread $\mathfrak{L}$, for each point $a \in \partial T(\mathfrak{L})$, there exists a point $p \in C$ such that a is an endpoint of the arc $L(p) \cap M_2(\mathfrak{L})$.*

Proof. Theorem 4 follows from Propositions 1 and 2.

THEOREM 5. *For each C_3-spread $\mathfrak{L}$, $\partial T(\mathfrak{L}) - T_2(\mathfrak{L})$ is the union of a rare with a finite or countable set; thus, in the sense of Baire categories, most of the points of $\partial T(\mathfrak{L})$ belong to $T_2(\mathfrak{L})$.*

Proof. First we make some remarks concerning the proof of Theorem 1. With the notations from there, $e(q) \in \partial T(\mathfrak{L})$ for all $q \in A$, since on one hand $e(q) \in \overline{L(q) \cap M_2(\mathfrak{L})}$ and $M_2(\mathfrak{L}) \subset \overline{T(\mathfrak{L})}$, on the other hand $e(q) \in \overline{L(q) - M_2(\mathfrak{L})}$ and $L(q) - M_2(\mathfrak{L}) \subset \complement\operatorname{int} M_2(\mathfrak{L}) = \complement T(\mathfrak{L})$ by Theorem 2 in [5]. Also, property C_3 implies $e(q) \to a$ for $q \to p$. More generally, e is continuous on A.

Let

$$Z = \{z \in \partial T(\mathfrak{L}) : \exists p \in C \text{ such that } z \text{ is an endpoint of } \overline{L(p) \cap M_2(\mathfrak{L})} \text{ and } z \neq \varphi(p)\}$$

and

$$F = \{z \in \partial T(\mathfrak{L}) : \exists \text{ nondegenerate arc } \beta \subset C \text{ such that } \forall x \in \beta,\ z \in L(x)\}.$$

Consider the points $a \in \partial T(\mathfrak{L})$ and $p \in C$ such that $a \in L(p)$. We can arrange that a is one of the endpoints of $\overline{L(p) \cap M_2(\mathfrak{L})}$, by Theorem 4.

Suppose $a \in Z$ and $p \in C$ is the point from the definition of Z. Then (see the proof of Theorem 1), there exists a neighbourhood B of p such that the function e is injective and continuous on B and $\{x \in B : e(x) \in M_3(\mathfrak{L})\}$ equals $\{x \in B : e(x) \notin T_2(\mathfrak{L})\}$ and is at most countable. Thus $e(B)$ is a neighbourhood of a on $\partial T(\mathfrak{L})$, on which all points except at most countably many belong to $T_2(\mathfrak{L})$.

Now consider the case $a \in \partial T(\mathfrak{L}) - (Z \cup F)$ (see Figure 5). By Theorem 4, by the definition of Z and with the notation of the proof of Theorem 1, there exists $p \in C$ such that $a = \varphi(p)$ is an endpoint of $\overline{L(p) \cap M_2(\mathfrak{L})}$. Let $t \in C$ be such that $\langle x, p\rangle \in \langle t, p\rangle\, a$ for all $x \in tp$. Such a t exists because $\mathfrak{L}$ has property C_3 in p and $a = \varphi(p)$. Let $x_0 \in tp$ be such that $\langle x_0, p\rangle \in \operatorname{int}^+\langle t, p\rangle\, a$ and $s \in tp$ such that $\langle s, x_0\rangle \in \operatorname{int}^+\langle p, x_0\rangle\langle t, x_0\rangle$. Then $\langle s, p\rangle \in \operatorname{int}^+ a\langle x_0, p\rangle$. Let G be the bounded domain with

$$\partial G = \omega \cup \langle x_0, p\rangle x_0 \cup x_0 t \cup t\langle t, p\rangle,$$

where $\omega = \langle t, p\rangle\langle x_0, p\rangle$. Suppose for each $z \in x_0 t$, $\varphi(x) \in \overline{G}$. Then, locally, $\langle x, s\rangle$ goes in direction $-s$ to s on $L(s)$ when x runs in direction x_0 to t. And since $x_0 t$ is compact, this implies that $-s, \langle x_0, s\rangle, \langle t, s\rangle, s$ lie in this order on $L(s)$, which is false. Thus, there is a point $y_0 \in x_0 t$ such that $\varphi(y_0) \notin \overline{G}$. Since φ is continuous and $\mathfrak{L}$ has property C_3, there exists an open arc $N_t \subset C$ containing y_0, such that $\varphi(x) \notin \overline{G}$ and $\underline{\langle x, x'\rangle \notin \overline{G}}$ for all distinct $x, x' \in N_t$. Denote by $-a$ the other endpoint of $\overline{L(p) \cap M_2(\mathfrak{L})}$. By Proposition 2, a and $-a$ divide $\partial T(\mathfrak{L})$ in two arcs, each of which lies in the closure of a component

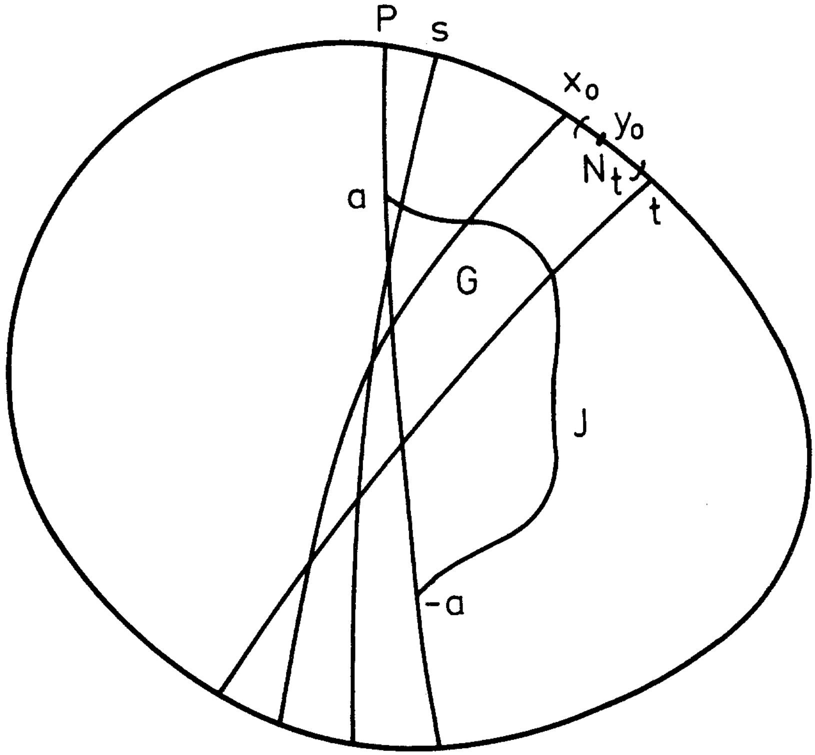

Fig. 5

of $D - L(p)$. Because int $\langle s, p\rangle\langle s, t\rangle\langle t, p\rangle \subset T(\mathfrak{L})$, only one of the above two arcs meets $G \cup \text{int}^+ \omega$; let J be this arc. Clearly the endpoint $e(x)$ of $\overline{L(x) \cap M_2(\mathfrak{L})}$ belonging to $G \cup \text{int}^+ \omega$ lies on J for all $x \in N_t$. Since $L(x)$ and $L(x')$ do not meet in $G \cup \omega$ for $x, x' \in N_t$, e is injective on N_t. Let j be a subarc of J with a as an endpoint. There exists a point t_0 on pt such that $L(t_0)$ meets j. For, if for all $x \in pq$ where $q \in pt$, $L(x) \cap j = \varnothing$, then j must lie along $L(p)$ on the arc ap, which contradicts Proposition 2. If we now rechoose $t' \in pt_0$ with the same property as t, we get that j contains the arc $e(N_{t'})$. Since $e(y) \neq \varphi(y)$, we have $e(y) \in Z$ for all $y \in N_{t'}$.

Let now E be an arc of $\partial T(\mathfrak{L})$. We have either $E \subset Z \cup F$ or $E - (Z \cup F) \neq \varnothing$. In the second case we saw that there exists a nondegenerate arc in $E \cap Z$. Thus $\partial T(\mathfrak{L}) - (Z \cup F)$ is rare on $\partial T(\mathfrak{L})$. F is at most countable. Z is obviously open on $\partial T(\mathfrak{L})$. Thus Z is a countable union of open arcs. Let z be such an arc. For each point $a \in z$, we found a neighbourhood of a in z on which all points except at most countably many belong to $T_2(\mathfrak{L})$. This obviously implies that $z - T_2(\mathfrak{L})$ and hence also $Z - T_2(\mathfrak{L})$ is at most countable.

Concluding, $\partial T(\mathfrak{L}) - T_2(\mathfrak{L})$ is the union of a rare with a finite or countable set, hence of first Baire category. The theorem is proved.

5. Double points in straight spreads

If $L^* \to L(x)$ (with $L^* \neq L(x)$) implies the convergence of $L^* \cap L(x)$, then $\mathfrak{L}$ is said to have *property* C_2 at x (compare [7]).

By Theorem 1, $T_2(\mathfrak{L})$ is uncountable if $\mathfrak{L}$ has property C_3 at some point of C. Assuming only property C_2, we get the following result.

THEOREM 6. *If the spread $\mathfrak{L}$ has property C_2 at uncountably many points, then $T_2(\mathfrak{L})$ is uncountable.*

Proof. Let A be an arc on C which has more than $\aleph_0$ points in which $\mathfrak{L}$ has property C_2, such that A does not contain x, $-x$ for any x. Let Φ be the set of all points in A where $\mathfrak{L}$ has property C_2. For every point $x \in \Phi$, let $\varphi(x)$ be again the point to which $L(x) \cap L(y)$ converges when $y \to x$, $y \neq x$, and let

$$\Sigma(x) = \overline{M_2(\mathfrak{L}) \cap L(x)}.$$

Clearly, $\Sigma(x)$ is a nondegenerate arc and $\varphi(x) \in \Sigma(x)$. Let $\sigma(x)$ be the endpoint of $\Sigma(x)$ separating int $\Sigma(x)$ from x (see Figure 6). Let

$$B_+ = \{x \in \Phi : \varphi(x) \neq \sigma(-x)\},$$
$$B_- = \{x \in \Phi : \varphi(x) \neq \sigma(x)\}.$$

Of course, one of these sets, say B_+, is uncountable.

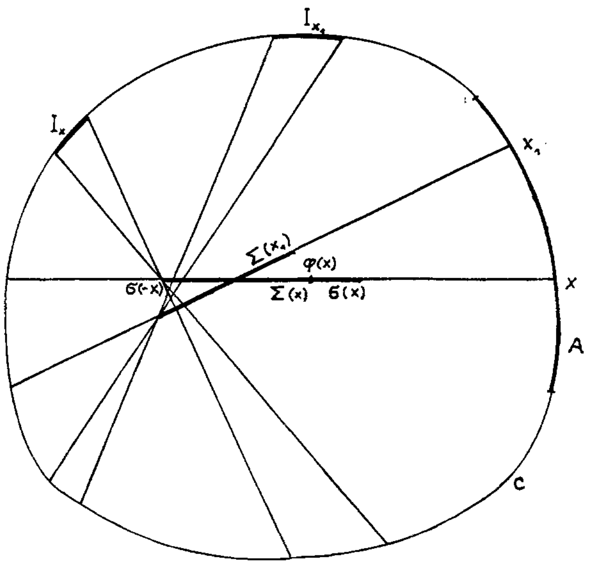

Fig. 6

It follows from the continuity of $\langle x, y\rangle$ as function of y that, for each $x \in B_+$, $\sigma(-x) \in M_2(\mathfrak{L})$.

Every (nondegenerate) subarc (with endpoints) of A containing just countably many points of B_+ may be extended to a maximal such subarc A_ι. The family $\{A_\iota\}_{\iota \in I}$ of all such subarcs is obviously countable. Then

$$B^* = B_+ - \bigcup_{\iota \in I} A_\iota$$

is uncountable.

We observe that $\sigma(-x_1) \neq \sigma(-x_2)$ if $x_1, x_2 \in B^*$, $x_1 \neq x_2$. Suppose indeed that $\sigma(-x_1) = \sigma(-x_2)$. Then $\sigma(-x_1) \in L(z)$ for all points z between x_1 and x_2 on A, which implies

$$\sigma(-x_1) = \sigma(-z) = \varphi(z),$$

whence $z \notin B^*$, in contradiction to the definition of B^*.

Let

$$E = \{x \in B^* : \sigma(-x) \in M_3(\mathfrak{L})\}.$$

We consider $x \in E$ and denote by C^* one of the arcs of C with endpoints x, $-x$. Let I_x be an arc in int C^* such that $\sigma(-x) \in L(u)$ for each endpoint u of I_x. Since $x \in B^*$, the set

$$B = \{y \in B^* \cap C^* : y \text{ separates } x \text{ from } I_x\}$$

is uncountable.

After taking a point $x_1 \in B \cap E$ and the corresponding arc I_{x_1}, it becomes clear that I_{x_1} separates x_1 from I_x; also, for each point $y \in B \cap E$ between x_1, $x_2 \in B \cap E$, the associated arc I_y lies between I_{x_1} and I_{x_2}, disjoint from them. Thus it is seen that the application associating I_x to x is injective into a countable space. Therefore $B \cap E$ is countable and $B - E$ uncountable. Since the involution $x \mapsto -x$ and σ are injective, $T_2(\mathfrak{L})$ is uncountable.

For *straight* spreads, i.e. spreads the curves of which are line-segments, we showed in [7] that $T_1(\mathfrak{L}) \neq \varnothing$. The next theorem completes this result.

THEOREM 7. *For every straight spread $\mathfrak{L}$, $T_2(\mathfrak{L})$ is uncountable and $\overline{T_2(\mathfrak{L})}$ meets every curve of $\mathfrak{L}$.*

Proof. Every straight spread has property C_2 almost everywhere [7]. Thus the hypotheses of Theorem 6 are verified. Also, we see from the above proof that there exists a point p_A in $T_2(\mathfrak{L}) \cap L(a)$, for some $a \in A$, if the arc A on C has uncountably many points at which $\mathfrak{L}$ has property C_2. In the present case, each arc of C has uncountably many points at which $\mathfrak{L}$ has property C_2. Let $x \in C$ and $\{A_n\}_{n=1}^{\infty}$ be a sequence of arcs on C converging to $\{x\}$; then $\{p_{A_n}\}_{n=1}^{\infty}$ has a limit point on $L(x)$. Hence $\overline{T_2(\mathfrak{L})}$ meets every member of $\mathfrak{L}$.

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(Received 16 June 1977)