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*Inventiones mathematicae*


# Many Endpoints and Few Interior Points of Geodesics*

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This paper is about abnormal convex surfaces and their equally abnormal geodesics. We do not feel ashamed of studying them because, in the sense of Baire categories, most convex surfaces *are* abnormal! Thus, in fact, *they* should be considered normal and vice-versa.

By a *convex surface* we always mean a closed one (see Busemann [2], p. 3), by a *segment* a shortest path on the surface ([2], p. 75), by a *geodesic* a curve which is locally a segment (see for a precise definition [2], p. 77).

In spaces of second Baire category, we use the words *most* and *typical* in the sense of "all, except those in a set of first Baire category". The space of all convex surfaces in $\mathbb{R}^n$, endowed with Hausdorff's metric, is a Baire space. We shall see how abnormal convex surfaces may be, by proving that most of them are so.

## Results

Since any two points of a convex surface are joined by at least one segment, the union of all segments equals the surface. A point of a segment different from its two endpoints will briefly be called *interior*. Is each point of a surface an interior point? The answer is easy for non-smooth surfaces: no conical point is (for any segment) an interior point ([1], p. 155). Points which are not for any segment interior will be called *endpoints*. They are, of course, endpoints of lots of geodesics. Smooth surfaces with an endpoint are also known ([1], p. 58–59). But, for each convex surface of class $C^2$, every point is an interior point of a segment in each tangent direction. More generally, this happens at a point $x$ if the lower indicatrix at every point $y$ in some neighbourhood of $x$ does not contain $y$ as a boundary point (Busemann [2], p. 92). Clearly, the set of all interior points is uncountable and dense, for an arbitrary convex surface.

Thus, it seems that, usually, convex surfaces must have many interior points. But let us look more closely at a typical convex surface: it is of class $C^1$, but

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not of class $C^2$ (Klee [4], Gruber [3]) and at most points the lower indicatrix reduces to a point ([5], Theorem 2); thus, it becomes doubtful whether those points are all interior. In fact, we establish the following result.

**Theorem 1.** *On most convex surfaces, most points are endpoints.*

It is known that in a certain tangent direction at a point of a convex surface may not start any segment. Such a tangent direction is called by Aleksandrov *singular.* He shows that there are smooth convex surfaces with a dense set of singular tangent directions at a certain point ([1], p. 59). Also, non-smooth convex surfaces all points of which are of this kind do exist: take any convex surface with a dense set of conical points (see [1], p. 60). But, again, at any point of a $C^2$-surface or with the above indicatrix condition, a segment starts in each tangent direction. And for an arbitrary convex surface, at each point, the set of singular tangent directions has measure zero, as Aleksandrov proved ([1], p. 213). However we get the following theorem; it is given only in $\mathbb{R}^3$, because of the essential use of Aleksandrov's concepts and results.

**Theorem 2.** *On most convex surfaces, at each point, most tangent directions are singular.*

A *circle* (*disk*) on a convex surface is the set of all points at intrinsic distance equal to (less than or equal to) a certain positive number from a fixed point of the surface. A circle which is a closed Jordan curve will be called a *Jordan circle.*

It is known that a Jordan circle may possess *vertices*, i.e. points where the circle is not smooth (see [1], p. 61, though our definition slightly differs from Aleksandrov's).

**Corollary.** *On most convex surfaces no circular arc is smooth. Thus, every Jordan circle has infinitely (densely, countably) many vertices.*

## Proofs

*Proof of Theorem 1.* Let $S(C)$ be the set of all interior points of $C \in \mathscr{C}$, $\mathscr{C}$ being the space of all convex surfaces. We have

$$S(C) = \bigcup_{n=1}^{\infty} S_n(C),$$

where

$$S_n(C) = \bigcup_{\lambda(s) \geqq n^{-1}} t(s),$$

$t(s)$ denotes the closed middle third and $\lambda(s)$ the length of the segment $s$.

Let

$$\mathscr{A} = \{C \in \mathscr{C} \colon S(C) \text{ is of 2nd category}\},$$

$$\mathscr{A}_n = \{C \in \mathscr{C} \colon S_n(C) \text{ is not nowhere dense}\},$$

$$\mathscr{A}_{n,m} = \{C \in \mathscr{C} \colon \exists \text{disk } D \subset C \text{ of radius } m^{-1} \text{ with } \overline{S_n(C)} \supset D\}.$$

Since

$$\mathscr{A} \subset \bigcup_{n=1}^{\infty} \mathscr{A}_n, \quad \mathscr{A}_n = \bigcup_{m=1}^{\infty} \mathscr{A}_{n,m},$$

we are ready if it is proved that $\mathscr{A}_{n,m}$ is nowhere dense; this is provided by the following lemma.

**Lemma.** *$\mathscr{A}_{n,m}$ is nowhere dense.*

*Proof.* Let $\mathcal{O}$ be an open set in $\mathscr{C}$. Consider a polytopal surface $P \in \mathcal{O}$. By adding if necessary new vertices near the faces of $P$, we get a new polytopal surface $P' \in \mathcal{O}$ such that each disk of radius $m^{-1}$ on $P'$ contains a vertex $v$ of $P'$. As a conical point, $v \notin S_n(P')$. The set $S_n(P')$ is compact. Indeed, let $y_i \in S_n(P')$ converge to some point $y$. Each $y_i$ lies on $t(s_i)$ for some segment $s_i \subset P'$ with $\lambda(s_i) \geqq n^{-1}$. By taking a subsequence if necessary, we may assume that $\{s_i\}_{i=1}^{\infty}$ converges to a curve $s$, i.e. there are parametrizations $x_i(t)$ of $s_i$ and $x(t)$ of $s$, where $\alpha \leqq t \leqq \beta$, such that $x_i(t) \to x(t)$ for every $t$ (see Busemann [2], (10. 5')). Then $s$ is a segment with $\lambda(s) \geqq n^{-1}$ (use Busemann [2], (10. 5) and (11. 3)). The same theorems of [2] imply $y \in t(s)$; hence $S_n(P')$ is closed. Thus, there exists a ball $O_v \subset \mathbb{R}^n$ with centre $v$ such that $O_v \cap S_n(P') = \emptyset$. We claim that there is an open neighbourhood $\mathcal{O}'_v$ of $P'$ included in $\mathcal{O}$, such that, for all $Q \in \mathcal{O}'_v$,

$$O_v \cap S_n(Q) = \emptyset.$$

To prove that this can be assured, suppose there is a sequence $\{Q_i\}_{i=1}^{\infty}$ converging to $P'$ such that $O_v \cap S_n(Q_i) \neq \emptyset$ for all $i$'s. Then there is a sequence of segments $s_i \subset Q_i$ such that $O_v \cap t(s_i) \neq \emptyset$ for all $i$'s. By taking again a subsequence if necessary, we may assume that $\{s_i\}_{i=1}^{\infty}$ converges to a curve $s$ (see [2], (10. 5')), which is a segment, and $t(s_i) \to t(s)$ (use [2], (10. 5) and (11. 3)), whence $O_v \cap t(s) \neq \emptyset$, which is impossible.

Now let $V(P')$ be the set of vertices of $P'$ and

$$\mathcal{O}'' = \bigcap_{v \in V(P')} \mathcal{O}'_v.$$

There exists a new neighbourbood $\mathcal{O}''' \subset \mathcal{O}''$ of $P'$ such that, for each $C \in \mathcal{O}'''$ and every disk $D$ of radius $m^{-1}$ on $C$, int $D$ meets $\cup \{O_v \colon v \in V(P')\}$. For, otherwise, we find a sequence $\{T_i\}_{i=1}^{\infty}$ converging to $P'$ and disks $D_i \subset T_i$ of radius $m^{-1}$, such that int $D_i \cap O_v = \emptyset$ for every $v$; this implies that for any limit disk $D \subset P'$,

$$D \cap \operatorname{int} O_v = \emptyset$$

for every $v$, in contradiction with the construction of $P'$.

Thus, since $O_v \cap S_n(Q) = \emptyset$ for any $Q \in \mathcal{O}'''$ and $v \in V(P')$,

$$\mathcal{O}''' \cap \mathscr{A}_{n,m} = \emptyset$$

and $\mathscr{A}_{n,m}$ is nowhere dense.

*Proof of Theorem 2.* Let $G_x$ be the set of directions of segments starting in $x$. Let

$$\mathscr{B} = \{C \in \mathscr{C} \colon \exists x \in C \text{ with } G_x \text{ of 2nd category}\},$$

$$\mathscr{B}_n = \{C \in \mathscr{C} \colon \exists x \in C \text{ and angle } A \text{ at } x \text{ with } \overline{G_{x,n^{-1}}} \supset A\},$$

where $G_{x,\rho}$ is the set of directions of the segments $xy$ with $\lambda(xy) \geqq \rho$, at $x$,

$$\mathscr{B}_{n,m} = \{C \in \mathscr{C}\colon \exists x \in C \text{ and angle } A \text{ at } x \text{ with } \mu(A) = m^{-1} \text{ and } \overline{G_{x,n^{-1}}} \supset A\},$$

where $\mu$ means measure of angles. Then

$$\mathscr{B} \subset \bigcup_{n=1}^{\infty} \mathscr{B}_n; \qquad \mathscr{B}_n = \bigcup_{m=1}^{\infty} \mathscr{B}_{n,m}$$

and we only have to prove that $\mathscr{B}_{n,m}$ is nowhere dense. In view of the Lemma, the set $\mathscr{A}_{k,r}$ is nowhere dense for arbitrary $k$ and $r$; thus it suffices to show that

$$\mathscr{B}_{n,m} \subset \mathscr{A}_{k,r}$$

for suitable $k$ and $r$.

Let $C \in \mathscr{B}_{n,m}$. Consider the point $x$ and the angle $A$ from the definition of $\mathscr{B}_{n,m}$. For $k \geqq n$ large enough, the circle with centre $x$ and radius $k^{-1}$ on $C$ is a Jordan circle (see [1], p. 383). Suppose in the circular sector $\Delta = xzz'$ composed by the segments $xz$, $xz'$ and the arc $zz'$ of the above Jordan circle, satisfying

$$\lambda(xz) = \lambda(xz') = k^{-1},$$
$$\angle zxz' = A,$$

there is a point $p$ not lying on any segment $xy$ with $y$ on the circular arc $zz'$ of the boundary of $\Delta$. Then there exists a point $y_0$ on $zz'$ admitting (at least) two segments from $x$ to $y_0$ (see [1], p. 58). The angle between them at $x$ is not zero (see [1], p. 131), whence the direction of no segment of length $n^{-1}$ starting at $x$ lies in that angle, which contradicts $C \in \mathscr{B}_{n,m}$. Thus, all points of $\Delta$ are on segments $xy$. Take a point $p$ on a segment $xz''$ with $z''$ between $z$ and $z'$ on the arc $zz'$, such that $\lambda(xp) = (2k)^{-1}$, and a disk $D$ around $p$ of radius $r^{-1} < (6k)^{-1}$, contained in $\Delta$. Then $S_k(C)$ includes $D$ and therefore $C \in \mathscr{A}_{k,r}$. The theorem is proved.

*Proof of the Corollary.* In every circular sector $xzz'$ there is a singular direction at $x$. Thus, there is a point $y_0$ between $z$ and $z'$ on the circular arc $zz'$ admitting at least two segments from $x$ to $y_0$. Consider those two which produce the maximal angle. Since these are orthogonal to the two arcs $y_0 z$ and $y_0 z'$ in $y_0$ ([1], p. 381), the arc $zz'$ is not smooth at $y_0$.

## Open Problems

Several problems arise naturally in connection with this topic. We select two of them, which are very easy to state. The first one is suggested by some examples of "smooth" endpoints.

1. Is each point with infinite sectional curvature in every tangent direction an endpoint?

Even stronger:

2. Is each point with the lower indicatrix reduced to a point an endpoint?

An affirmative answer to the second question would permit us to deduce Theorem 1 directly from Theorem 2 in [5].

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