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Shortness Exponents for Polytopes Which Are k-Gonal Modulo n

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INTRODUCTION

Barnette's famous conjecture states that the family, denoted here by $\mathscr{P}_2^0$, of all those simple 3-dimensional polytopes whose faces are even sided contains only Hamiltonian members. Malkevitch [8] raised the question whether the family $\mathscr{P}_5^0$ of all simple polytopes having only pentagons, decagons, 15-gons, etc., as faces contains only Hamiltonian members or not.

In order to generalize the preceding problems we consider the class $\mathscr{Q}_n^k$ of all (always 3-dimensional) polytopes which are k-gonal modulo n. We say that a polytope is *k-gonal modulo n* ($k < n$) if all vertices have the same valency and each of its faces is an m-gon with $m = k \pmod n$. The few regular polytopes are all Hamiltonian. We shall see that, for many values of k and n, the polytopes which are k-gonal modulo n may well be non-Hamiltonian, even if they are assumed to be simple (every vertex is 3-valent). The class of all simple k-gonal (modulo n) polytopes ($k < n$) will be denoted by $\mathscr{P}_n^k$.

Grünbaum and Walther [7] introduced the following "measure" of how short a longest circuit can be, called the *shortness exponent* and defined for any family $\mathscr{F}$ of graphs

$$\sigma(\mathscr{F}) = \liminf_{G \in \mathscr{F}} (\log h(G))/\log v(G),$$

where $v(G)$ is the number of all vertices of G and $h(G)$ the maximal circuit length in G.

We identify polytopes with the graphs of their vertices and edges.

The case of those simple polytopes which are k-gonal modulo 3 was treated by Zaks [14] and Walther [12]. They proved that

$$\sigma(\mathscr{P}_3^i) \leqslant \sigma(\mathscr{P}_3^0) \leqslant (\log 22)/\log 23 \qquad (i = 1, 2)$$

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which seems to be the best known bound for simple polytopes at all (cf. [7]). and

$$\sigma(\mathscr{P}_3^1) = \sigma(\mathscr{P}_3^2).$$

Two-gonal polytopes exist only modulo 3. It is indeed clear that each face of a 2-gonal (modulo n) polytope would be at least a hexagon if $n \geqslant 4$, which is impossible. The case $n = 3$ was mentioned above. Thus, special attention will be paid to cases $k = 0, 1, 3, 4, 5$. Clearly, there are no k-gonal (modulo n) polytopes for $k \geqslant 6$.

Preliminary Results

We present here three lemmas which will be repeatedly, sometimes tacitly, used.

Lemma 1 (Bielig [1]). *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two families of graphs and $\varphi: \mathscr{G}_2 \to \mathscr{G}_1$ an injective function. If there exist constants a, $b > 0$ such that, for each $G \in \mathscr{G}_2$,*

(1) *there is a circuit of length at least $ah(G)$ in $\varphi(G)$ and*

(2) $v(\varphi(G)) \leqslant bv(G)$,

then $\sigma(\mathscr{G}_1) \leqslant \sigma(\mathscr{G}_2)$.

Before we state Lemma 2, let us consider two polytopes A and B, each having two kinds of vertices: white and black. The black vertices are all 3-valent. Let v be a white 3-valent vertex of B; it will be called the *special vertex* of B. Let $B - v$ be the graph fragment obtained from B by deleting v. Let $B \to A$ denote the graph obtained by replacing each black vertex of A by $B - v$, the edges adjacent to any black vertex being identified with the edges which were adjacent to v. Now let G_0 and H_0 play the role of B, v and v' be the special vertices, and z and z' the number of all black vertices in G_0 and H_0, respectively. Suppose:

(a) each circuit of G_0 through v misses at least w black vertices ($z - w > 1$), or

(b) each circuit of H_0 through v' misses at least w' black vertices ($z' - w' > 1$).

Let $G_n = H_0 \to G_{n-1}$ and $H_n = H_{n-1} \to G_0$. The special vertex of G_0 becomes the special vertex of H_n ($n = 1, 2, 3, \ldots$).

Lemma 2 (Bielig [1], see also [9]). *If* (b) *holds, then*

$$\sigma(\{G_n : n = 1, 2, 3, \ldots\}) \leqslant (\log(z' - w'))/\log z'.$$

We shall also use the following related lemma:

LEMMA 3. *If* (a) *holds, then*

$$\sigma(\{H_n : n = 1, 2, 3, \ldots\}) \leqslant (\log(z - w))/\log z.$$

Proof. Consider the family $\{H'_n : n = 1, 2, 3, \ldots\}$ defined as follows: $H'_1 = G_0$; $H'_{n+1} = H'_n \to G_0 = G_0 \to H'_n$. By Lemma 2,

$$\sigma(\{H'_n : n = 1, 2, 3, \ldots\}) \leqslant (\log(z - w))/\log z.$$

Now let φ be defined as $\varphi(H'_n) = H_n$. The relationship between H'_n and H_n is

$$H_n = H_0 \to H'_n.$$

Obviously, for each n there exists a circuit of length at least $h(H'_n)$ in H_n; also $v(H_n) \leqslant (v(H_0) - 1)\, v(H'_n)$. Thus, by Lemma 1,

$$\sigma(\{H_n : n = 1, 2, 3, \ldots\}) \leqslant (\log(z - w))/\log z$$

Besides these lemmas, Theorem 1 will be useful. In analogy with the equality $\sigma(\mathscr{P}^1_3) = \sigma(\mathscr{P}^2_3)$, we have some further equalities among shortness exponents for classes of k-gonal (modulo n) polytopes.

THEOREM 1. $\sigma(\mathscr{P}^3_5) = \sigma(\mathscr{P}^4_5)$, $\sigma(\mathscr{P}^3_7) = \sigma(\mathscr{P}^5_7)$, $\sigma(\mathscr{P}^4_{11}) = \sigma(\mathscr{P}^5_{11})$.

Proof. Throughout the paper T, C, and D will be the tetrahedron, the cube, and the dodecahedron, respectively. In each of them let some vertex be special. Let P be a polytope with all vertices black. We have the implications:

$$P \in \mathscr{P}^3_5 \Rightarrow C \to P \in \mathscr{P}^4_5,$$
$$P \in \mathscr{P}^4_5 \Rightarrow T \to P \in \mathscr{P}^3_5,$$
$$P \in \mathscr{P}^5_7 \Rightarrow T \to P \in \mathscr{P}^3_7,$$
$$P \in \mathscr{P}^3_7 \Rightarrow D \to P \in \mathscr{P}^5_7,$$
$$P \in \mathscr{P}^4_{11} \Rightarrow D \to P \in \mathscr{P}^5_{11},$$
$$P \in \mathscr{P}^5_{11} \Rightarrow C \to P \in \mathscr{P}^4_{11}.$$

Now, by Lemma 1, the three equalities hold.

0-Gonal Polytopes

Barnette's conjecture, for which only partial answers are known (see [2, 4]), suggests the following weaker conjecture (Malkevitch [8, Problem 1]):

Conjecture. The family $\mathscr{P}_4^0$ consists only of Hamiltonian polytopes.

The case $n = 3$ (i.e., of $\mathscr{P}_3^0$), mentioned in the Introduction, was settled by Walther [12] and also considered before by Grünbaum (private communication to Zaks).

Malkevitch's question about $\mathscr{P}_5^0$ was answered in the negative by Zaks [15], but the problem of estimating the shortness exponent of $\mathscr{P}_5^0$ remained open.

Theorem 2.

$$\sigma(\mathscr{P}_5^0) \leqslant (\log 78)/\log 79.$$

Proof. To prove this we use the graph Q_1 of Fig. 1. This non-Hamiltonian graph, which has only pentagons and 15-gons as faces, was constructed in [16]. It was shown there that every circuit in Q_1 misses at least one vertex. Let $H_0 = D$, with an arbitrarily chosen special vertex, and G_0 be the graph obtained from Q_1 by replacing w by $D - z$, where z is again some vertex of D. Now let v, some vertex of D which is not z or one of its neighbours, be the special vertex of G_0. Colour in black all the 79 vertices of $Q_1 - w$ in G_0. Then, by Lemma 3, we get a sequence $\{H_n : n = 1, 2, 3,...\}$ with shortness exponent at most (log 78)/log 79 and the graphs of which have just penta-, 20-, and 60-gons as faces. Thus the theorem is proved.

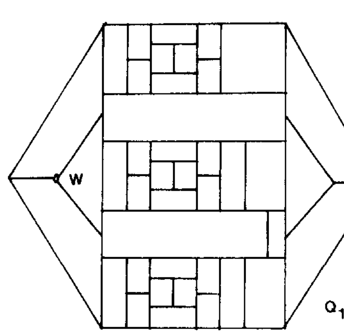

Figure 1

MONOGONAL POLYTOPES

Bielig and Schulz [2] observed that $\sigma(\mathscr{P}_2^1) < 1$. Their construction leads more precisely to $\sigma(\mathscr{P}_2^1) \leqslant (\log 42)/\log 43$.

THEOREM 3.

$$\sigma(\mathscr{P}_2^1) \leqslant (\log 26)/\log 27.$$

Proof. Let $G_0 = T$ with one vertex black, and H_0 be the graph of Fig. 2. This H_0 is a slight modification of the famous non-Hamiltonian graph of Tutte [10]. The fact that no path from b to c contains all black vertices of abc, observed and used by Walther [12] and Grünbaum–Walther [7], implies that the hypotheses of Lemma 2 are fulfilled. The theorem now follows.

Case $n = 3$ was mentioned in the Introduction.

THEOREM 4.

$$\sigma(\mathscr{P}_4^1) \leqslant (\log 100)/\log 102.$$

Proof. Let G_0 be the graph of Fig. 3, where each square vertex is replaced with the graph F of Fig. 4. Now, H_0 is obtained from G_0 by replacing the vertex v with the graph fragment of Fig. 5. The black vertices

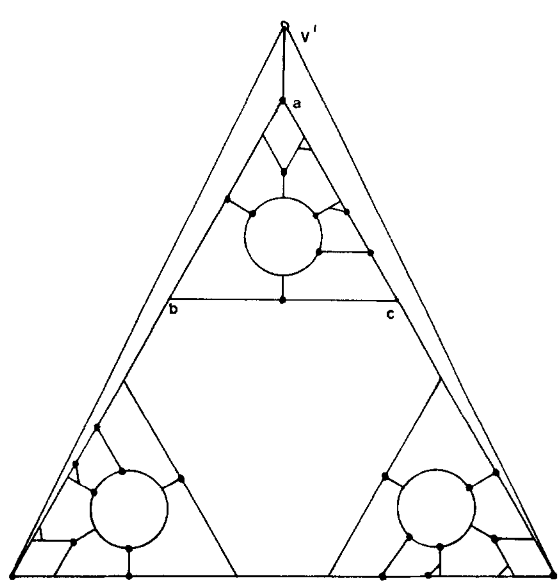

FIGURE 2

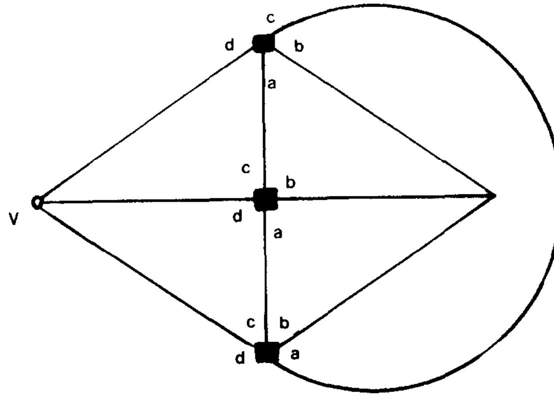

FIGURE 3

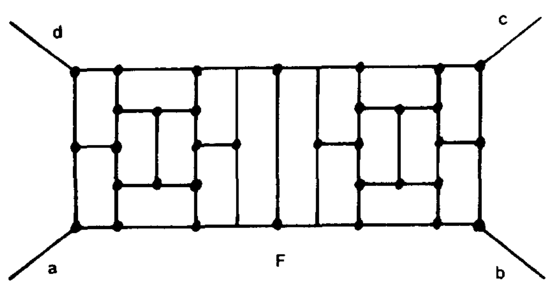

FIGURE 4

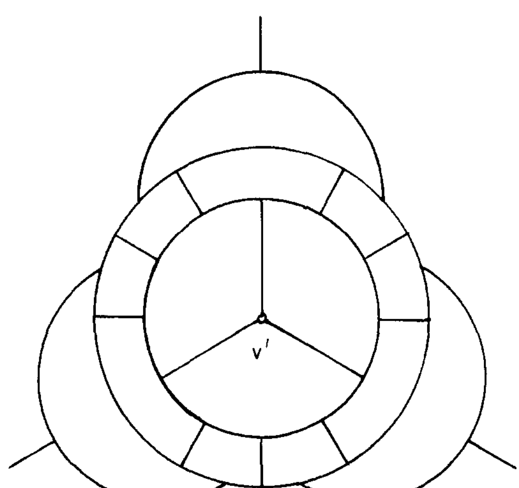

FIGURE 5

of G_0 and H_0 are in each copy of F those marked on Fig. 4. In a graph containing F as a subgraph, every circuit traversing F and passing through all black vertices of F must go through F from a to c or from b to d (see [3, 15, 16]). It follows that every circuit of H_0 through v' misses in at least two F-graphs, at least one black vertex, or misses completely one copy of F. There are 102 black vertices in H_0. It is easy to see that each G_n is a member of $\mathscr{P}_4^1$. Thus, by Lemma 2, $\sigma(\mathscr{P}_4^1) \leqslant (\log 100)/\log 102$.

Theorem 4 concludes this section, because no polytopes are monogonal modulo n for $n \geqslant 5$.

Triangular Polytopes

The first class of polytopes to be considered is $\mathscr{P}_4^3$.

Theorem 5.

$$\sigma(\mathscr{P}_4^3) \leqslant (\log 26)/\log 27.$$

Since the proof is completely analogous to that of Theorem 3, we only mention that the graph H_0 will be taken from Fig. 6 instead of Fig. 2.

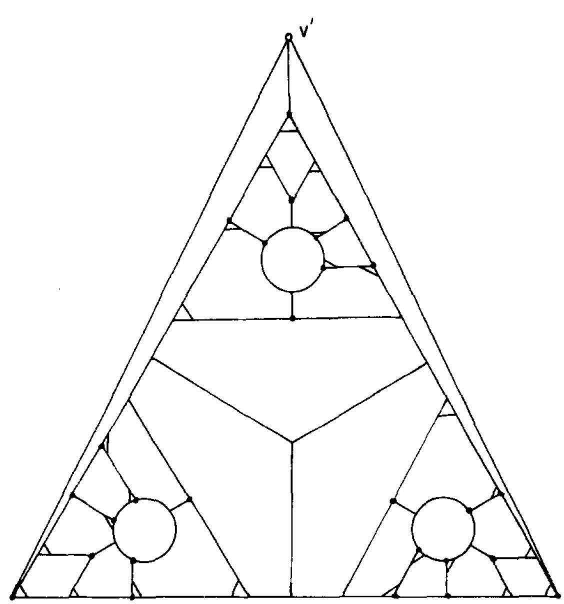

Figure 6

THEOREM 6.

$$\sigma(\mathscr{P}_5^3) \leqslant (\log 36)/\log 37.$$

The proof uses the equality $\sigma(\mathscr{P}_5^3) = \sigma(\mathscr{P}_5^4)$ of Theorem 1 and also Theorem 9 from the following section.

THEOREM 7.

$$\sigma(\mathscr{P}_6^3) \leqslant (\log 42)/\log 43.$$

Proof. To construct G_0 and H_0 we use the graph Q_2 of Fig. 7, which allows no Hamiltonian circuit (see Grinberg [5] and Tutte [11], or [6, p. 1145]). We replace every marked vertex of Q_2 with $T - w$, in which one vertex is black, and obtain in this way $G_0 \in \mathscr{P}_6^3$, in which the nonmarked vertices of Q_2 have also to be black. To get H_0 we contract in G_0 the triangle at v to the special vertex v'. Every circuit through v' misses at least one of the 43 black vertices of H_0. Now, using Lemma 2, $\sigma(\mathscr{P}_6^3) \leqslant \log 42/\log 43$.

THEOREM 8.

$$\sigma(\mathscr{P}_7^3) \leqslant (\log 166)/\log 168.$$

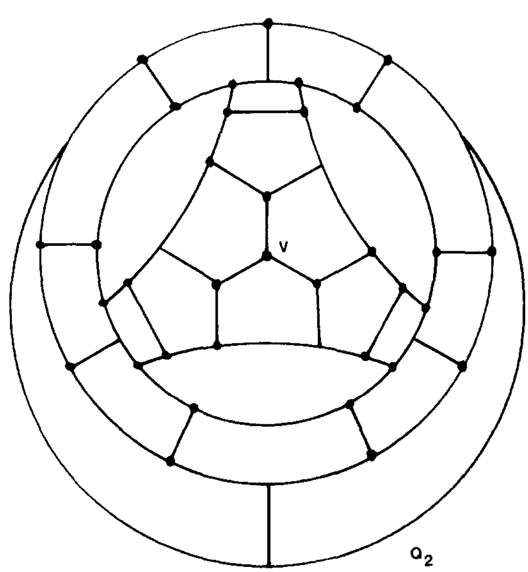

FIGURE 7

The proof combines the equality $\sigma(\mathscr{P}_7^3) = \sigma(\mathscr{P}_7^5)$ of Theorem 1 with Theorem 13(b), that we shall establish later.

We have no further classes to study, since $\mathscr{P}_n^3 = \{T\}$ for every $n \geqslant 8$. Indeed, the triangles of a graph from $\mathscr{P}_n^3 - \{T\}$ would be nonadjacent; contract them to points: every m-gon $(m > 3)$ becomes a k-gon with $k \geqslant m/2$. For $n \geqslant 8$, we have $m \geqslant 11$, hence $k \geqslant 6$, but such a planar graph does not exist.

Quadrangular Polytopes

In this section we are somewhat successful but for odd n. The difficulty in treating the case of even n is again explainable by its relationship with Barnette's conjecture.

Walther announced in [13] that he is able to prove that the shortness exponent of the family of all simple polytopes having just 4-gons and k-gons as faces is smaller than 1; here k is any fixed odd number, not smaller than 17. This yields $\sigma(\mathscr{P}_n^4) < 1$ for odd $n \geqslant 13$. Thus, we restrict ourselves to the cases $n = 5, 7, 9, 11$.

In this section w is a vertex of the cube C.

THEOREM 9.

$$\sigma(\mathscr{P}_5^4) \leqslant (\log 36)/\log 37.$$

Proof. We obtain the graph $G_0 \in \mathscr{P}_5^4$ by replacing every marked vertex of the graph Q_3 in Fig. 8 with a copy of $C - w$. In each such copy (except that which corresponds to the vertex numbered one), one vertex will be black. Also, all nonmarked vertices of Q_3 with the exception of 2, 3, 4, and v will be black. Now replace v with the fragment shown in Fig. 9. The graph obtained is H_0 and has 37 black vertices. Using the same arguments as for

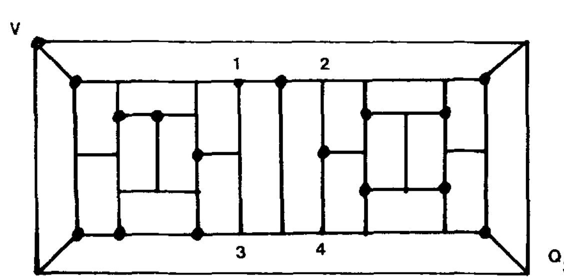

FIGURE 8

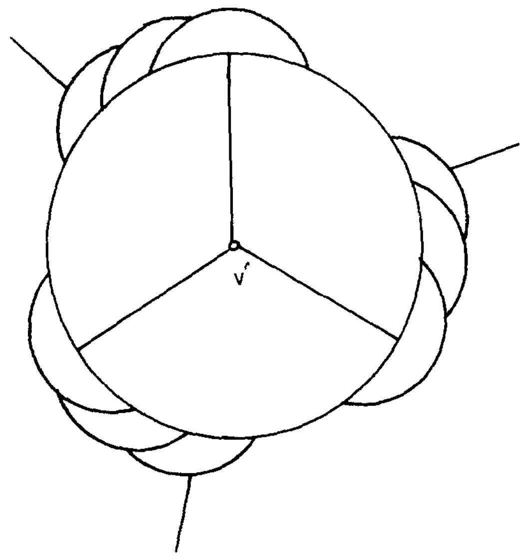

FIGURE 9

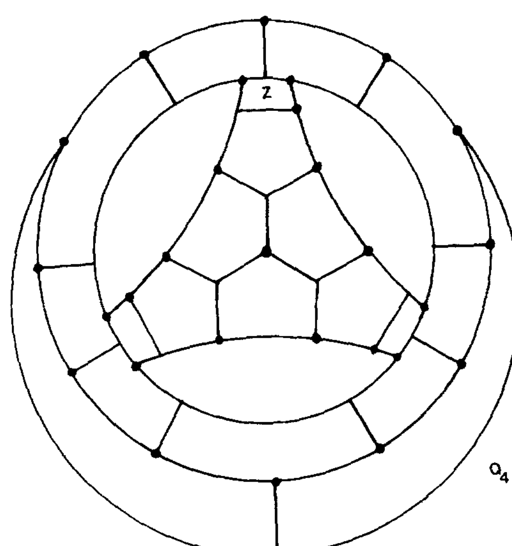

FIGURE 10

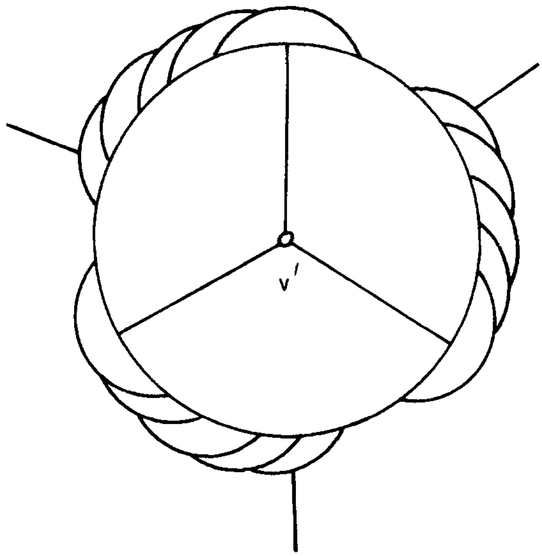

FIGURE 11

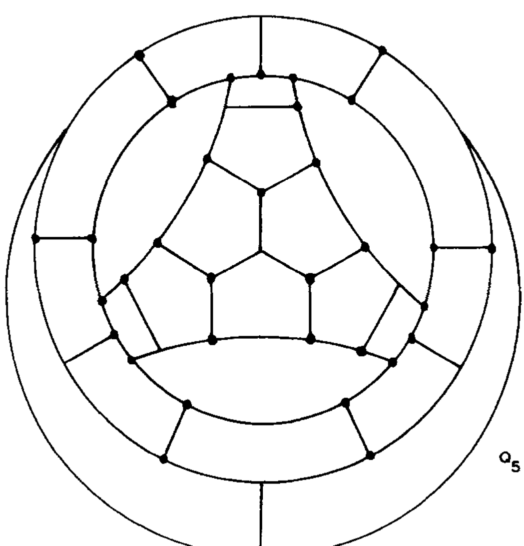

FIGURE 12

Theorem 4, we see that every circuit through v' misses at least one black vertex. Thus, by Lemma 2, the inequality of the statement holds.

THEOREM 10.

$$\sigma(\mathscr{P}_7^4) \leqslant (\log 42)/\log 43.$$

Proof. To construct G_0 and H_0 we again use the Grinberg–Tutte graph, now called Q_4 and shown in Fig. 10. We replace every marked vertex of Q_4 with a copy of $C - w$ in which one vertex is black, and colour the remaining vertices of Q_4 in black. The resulting graph is G_0. To get H_0 we replace the vertex z of G_0 with the fragment shown in Fig. 11. Every circuit through v' misses at least one of the 43 black vertices of H_0. By Lemma 2, this yields the theorem.

THEOREM 11.

$$\sigma(\mathscr{P}_9^4) \leqslant (\log 42)/\log 43.$$

Proof. The basic graph Q_5 is the same as in the preceding proof. We replace its vertices marked on Fig. 12 with copies of $C - w$. To obtain $G_0 \in \mathscr{P}_9^4$, the two marked vertices of the graph of Fig. 13 must be replaced with $Q_5 - v$. In G_0 we colour black, in one of the two copies of $Q_5 - v$, every vertex nonmarked on Fig. 12 and one vertex in each used copy of $C - w$. Replacing w' with the fragment shown in Fig. 14 we obtain H_0. As in Theorem 10, we get $\sigma(\mathscr{P}_9^4) \leqslant (\log 42)/\log 43$ using Lemma 2.

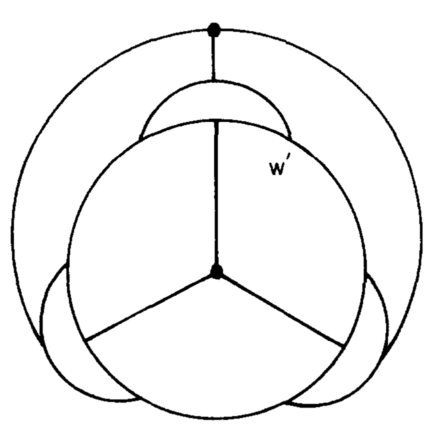

FIGURE 13

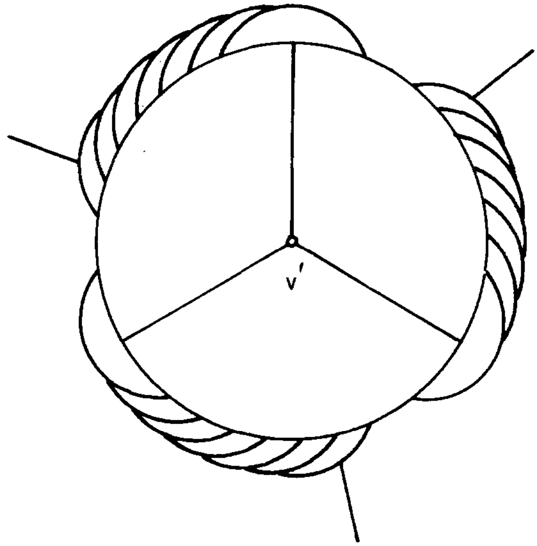

FIGURE 14

THEOREM 12.

$$\sigma(\mathscr{P}^4_{11}) \leqslant (\log 100)/\log 102.$$

Theorem 13(c), together with the equality $\sigma(\mathscr{P}^4_{11}) = \sigma(\mathscr{P}^5_{11})$ from Theorem 1, prove Theorem 12.

PENTAGONAL POLYTOPES

We are able to treat the pentagonal (modulo n) polytopes for all values of n. The next theorem shows that the shortness exponents of all classes are smaller than 1.

THEOREM 13.

(a) $\sigma(\mathscr{P}^5_6) \leqslant (\log 42)/\log 43$
(b) $\sigma(\mathscr{P}^5_7) \leqslant (\log 166)/\log 168$
(c) $\sigma(\mathscr{P}^5_9) \leqslant (\log 42)/\log 43$
(d) $\sigma(\mathscr{P}^5_{5m+1}) \leqslant (\log(54m-8))/\log(54m-6) \quad (m \geqslant 2)$
(e) $\sigma(\mathscr{P}^5_{5m+2}) \leqslant (\log(54m+4))/\log(54m+6) \quad (m \geqslant 2)$
(f) $\sigma(\mathscr{P}^5_{5m+3}) \leqslant (\log(54m+46))/\log(54m+48) \quad (m \geqslant 1)$

(g) $\sigma(\mathscr{P}^5_{5m+4}) \leqslant (\log(54m+58))/\log(54m+60) \qquad (m \geqslant 2)$

(h) $\sigma(\mathscr{P}^5_{5m+5}) \leqslant (\log(90m+78))/\log(90m+80) \qquad (m \geqslant 1)$.

Proof. We first prove inequalities (d)–(h). Here we consider graphs in $\mathscr{P}^5_{5m+r}$ with $m, r \in \mathbb{N}$ and $r \leqslant 5$. Let G^m_0, G^m_n, and H^m_0 play the roles of G_0, G_n, and H_0. The graphs used to construct G^m_0 and H^m_0 are those of Figs. 3, 17, and 18. Each square vertex must be replaced with a graph fragment F_m or F'_m (Fig. 15), in the way indicated by the position of the edges a, b, c, d. These graph fragments were found and used by Zaks [15], F_1 by Faulkner and Younger [3]. The vertices marked in Fig. 15 are black. By the results in [15] and the arguments in the proof of Theorem 4, every circuit through v misses at least two black vertices in each of the preceding graphs. In this proof we also make repeated use of the graphs L_n shown in Fig. 16. They have only pentagons and $(n+2)$-gons as faces. In the following, the vertex v' will play the role of the special vertex.

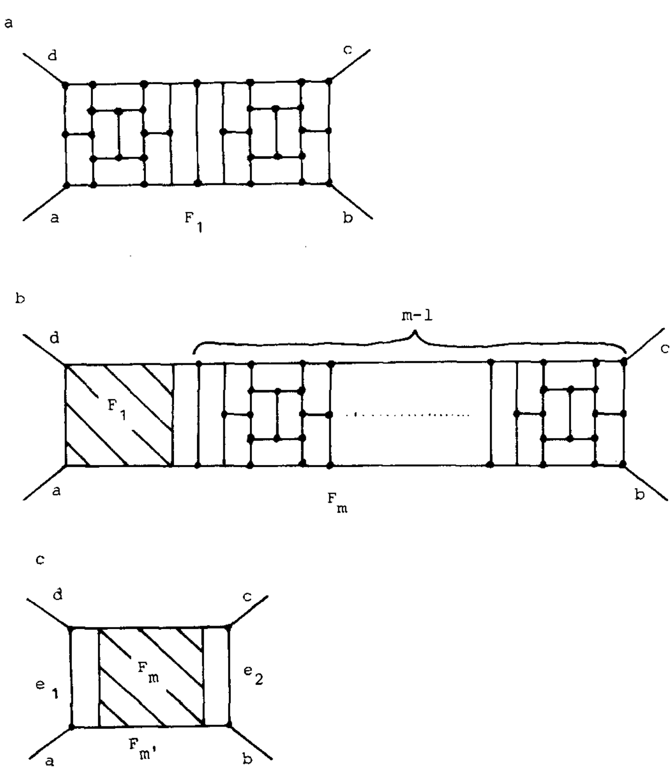

FIGURE 15

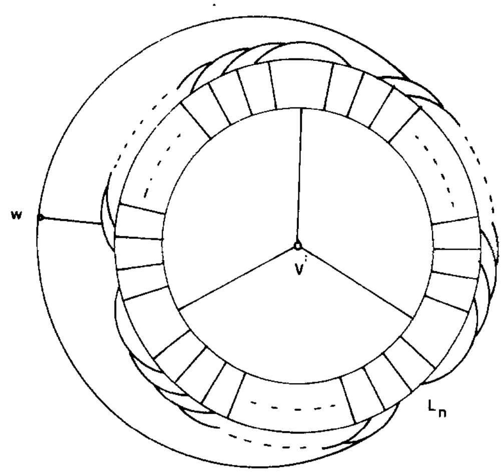

FIGURE 16

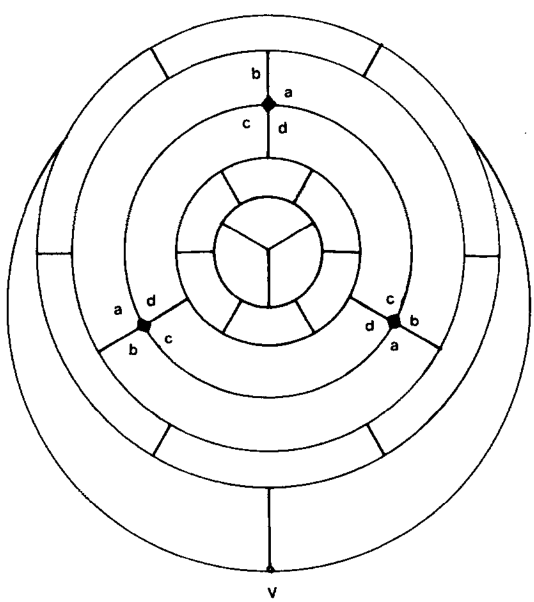

FIGURE 17

(d) Let G_0^2 be the graph of Fig. 17. The square vertices must be replaced with copies of F_1 (Fig. 15a)). Let H_0^2 be obtained by replacing v with $L_{11} - w$ (see Fig. 16). Now H_0^2 has 102 black vertices and every circuit through v' misses at least two of them. Each G_n^2 is in $\mathscr{P}_{11}^5$. Thus, by Lemma 2, $\sigma(\mathscr{P}_{11}^5) \leqslant (\log 100)/\log 102$. Now let G_0^m be the graph of Fig. 17, where the square vertices are replaced with copies of F_{m-1}. To get H_0^m, replace v by $L_{5m+1} - w$. Using the preceding considerations and Lemma 2, we get

$$\sigma(\mathscr{P}_{5m+1}^5) \leqslant (\log[3(16 + (m-1)\,18) - 2])/\log[3(16 + (m-1)\,18)]$$
$$= (\log(54m - 8))/\log(54m - 6),$$

for all $m \geqslant 2$.

(e) In analogy to (d), we prove (e) using F'_{m-1} instead of F_{m-1} and L_{5m+2} instead of L_{5m+1}. This yields

$$\sigma(\mathscr{P}_{5m+2}^5) \leqslant (\log[3(20 + (m-1)\,18) - 2])/\log[3(20 + (m-1)\,18)]$$
$$= (\log(54m + 4))/\log(54m + 6),$$

for all $m \geqslant 2$.

(f) Now let G_0^m be the graph of Fig. 3, where the square vertices are replaced with F_m (see Fig. 15b)). Now H_0^m is obtained from G_0^m by replacing v with $L_{5m+3} - w$. Then, analogously,

$$\sigma(\mathscr{P}_{5m+3}^5) \leqslant (\log[3(16 + 18m) - 2])/\log[3(16 + 18m)]$$
$$= (\log(54m + 46))/\log(54m + 48),$$

for all $m \geqslant 1$.

(g) Using F'_m instead of F_m and L_{5m+4} instead of L_{5m+3}, we get

$$\sigma(\mathscr{P}_{5m+4}^5) \leqslant (\log[3(20 + 18m) - 2])/\log[3(20 + 18m)]$$
$$= (\log(54m + 58))/\log(54m + 60),$$

for all $m \geqslant 2$.

(h) To prove the assertion for $n = 5m + 5$, we use the graph of Fig. 18. In analogy to (c) we employ F_m and L_{5m+5} to construct G_0^m and H_0^m. This yields

$$\sigma(\mathscr{P}_{5m+5}^5) \leqslant (\log[5(16 + 18m) - 2])/\log[5(16 + 18m)]$$
$$= (\log(90m + 78))/\log(90m + 80),$$

for all $m \geqslant 1$.

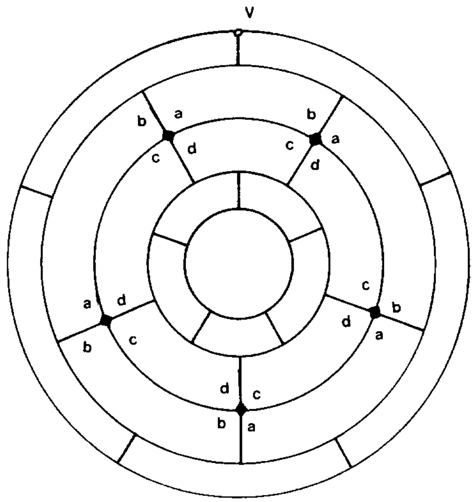

FIGURE 18

(b) We obviously have (by (g)),

$$\sigma(\mathscr{P}^5_7) \leqslant \sigma(\mathscr{P}^5_{14}) \leqslant (\log 166)/\log 168.$$

(a) The construction of G_0 again uses the Grinberg–Tutte graph. The marked vertices of the graph Q_6 in Fig. 19 must be replaced by $D-w''$, w'' being an arbitrary vertex of D. Then we replace two of the vertices of the cube C not belonging to the same face, with a copy of Q_6-w'. We colour black every nonmarked vertex and one of the vertices of each copy of $D-w''$ in one of the copies of Q_6-w'. Now H_0 is obtained from G_0 by replacing one vertex in the other (uncoloured) copy of Q_6-w' with L_6-w. Thus, by Lemma 2, we get $\sigma(\mathscr{P}^5_6) \leqslant (\log 42)/\log 43$.

(c) The proof is completely analogous to the proof of (a) and uses Q_7 (Fig. 20) and L_9 (Fig. 16). This completes the proof.

By using the graph Q_8 of Fig. 21 instead of the graph in Fig. 17, and L_{10k+i}, we can improve three sequences of estimates in Theorem 13:

$$\sigma(\mathscr{P}^5_{10k+i}) \leqslant (\log(54k-7))/\log(54k-6) \quad (i=-1, 0, 1)\ (k \geqslant 2).$$

For $i=0$, we drop the edge f_j and for $i=-1$ both edges e_j and f_j in Q_8 $(j=1, 2, 3)$.

Thus, we get

$$\sigma(\mathscr{P}^5_{19}), \sigma(\mathscr{P}^5_{20}), \sigma(\mathscr{P}^5_{21}) \leqslant (\log 101)/\log 102 = 0.9978698...,$$

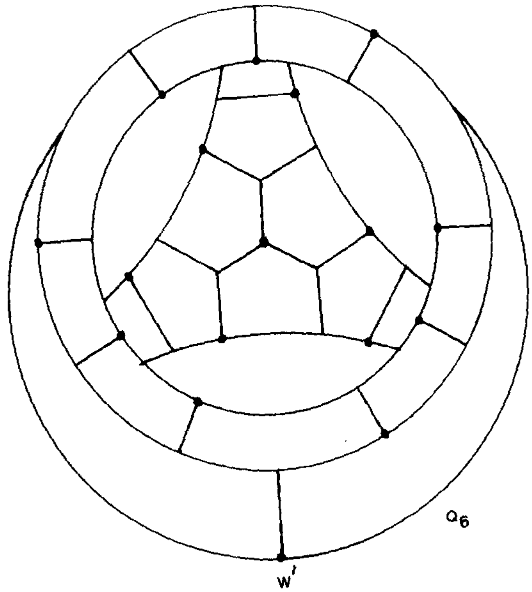

FIGURE 19

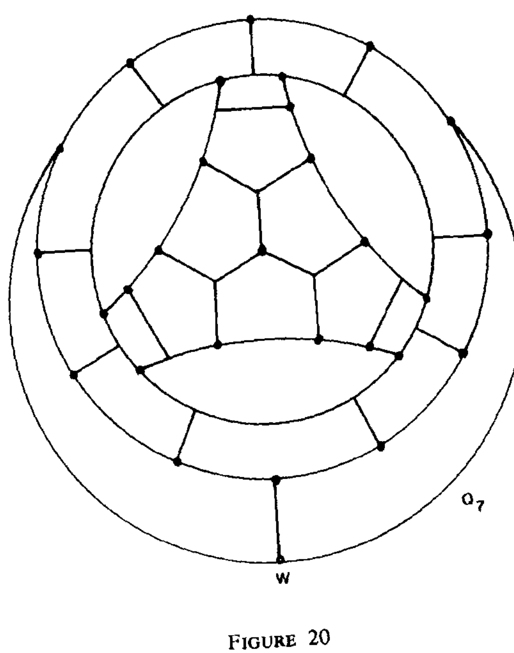

FIGURE 20

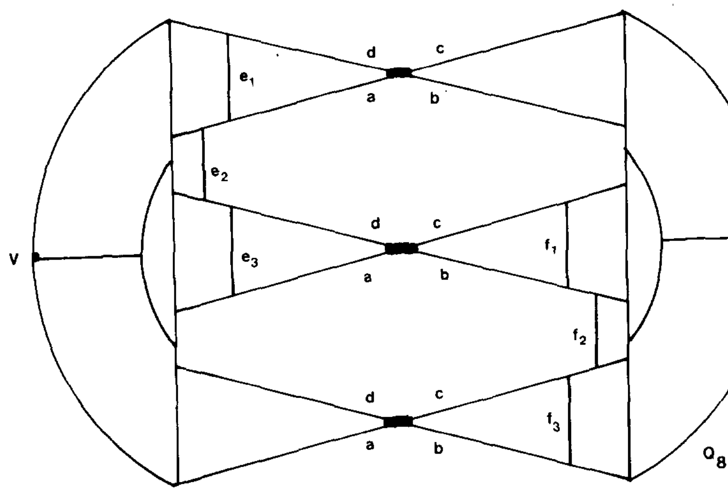

FIGURE 21

while Theorem 13 only gives

$$\sigma(\mathscr{P}^5_{19}) \leqslant (\log 220)/\log 222 = 0.9983249...,$$

$$\sigma(\mathscr{P}^5_{20}) \leqslant (\log 348)/\log 350 = 0.9990217...,$$

$$\sigma(\mathscr{P}^5_{21}) \leqslant (\log 208)/\log 210 = 0.9982103....$$

FINAL REMARKS

Let $\mathscr{P}$ denote a family of simple polytopes and $M(\mathscr{P})$ the family of the corresponding medial graphs. Bielig proved in [1] that $\sigma(\mathscr{P}) = \sigma(M(\mathscr{P}))$. Let $\mathscr{R}$ be the family of all 4-valent polytopes. Take $\mathscr{P}$ to be the family of all simple polytopes. Then, since $M(\mathscr{P}) \subset \mathscr{R}$, we have

$$\sigma(\mathscr{R}) \leqslant \sigma(\mathscr{P}) \leqslant (\log 22)/\log 23.$$

Now let $\mathscr{R}^k_n = \mathscr{Q}^k_n \cap \mathscr{R}$; we obtain the inequalities

$$\sigma(\mathscr{R}^1_2) \leqslant \sigma(\mathscr{P}^1_2); \qquad \sigma(\mathscr{R}^0_3) \leqslant \sigma(\mathscr{P}^0_3);$$

$$\sigma(\mathscr{R}^3_n) \leqslant \sigma(\mathscr{P}^3_n) \qquad (n = 4, 5, 6, 7).$$

(For numerical estimates see the Introduction and Theorems 3, 5–8.)

Many, if not all, of the bounds found in this paper for the various shortness exponents are probably not the best possible ones. It seems to be more difficult to find nontrivial lower bounds, or even to determine the shortness exponents. The most interesting open questions, however, are (in our opinion) those strongly related to Barnette's conjecture and regarding $\mathscr{P}_4^0$ and $\mathscr{P}_n^4$ for even n, namely, whether they possess non-Hamiltonian graphs or not.

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