Shortness Exponents for Polytopes Which Are $k$-Gonal Modulo $n$

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INTRODUCTION

Barnette’s famous conjecture states that the family, denoted here by $\mathcal{P}_2^0$, of all those simple 3-dimensional polytopes whose faces are even sided contains only Hamiltonian members. Malkevitch [8] raised the question whether the family $\mathcal{P}_n^0$ of all simple polytopes having only pentagons, decagons, 15-gons, etc., as faces contains only Hamiltonian members or not.

In order to generalize the preceding problems we consider the class $\mathcal{P}_n^k$ of all (always 3-dimensional) polytopes which are $k$-gonal modulo $n$. We say that a polytope is $k$-gonal modulo $n$ ($k < n$) if all vertices have the same valency and each of its faces is an $m$-gon with $m = k \pmod{n}$. The few regular polytopes are all Hamiltonian. We shall see that, for many values of $k$ and $n$, the polytopes which are $k$-gonal modulo $n$ may well be non Hamiltonian, even if they are assumed to be simple (every vertex is 3-valent). The class of all simple $k$-gonal (modulo $n$) polytopes ($k < n$) will be denoted by $\mathcal{P}_n^k$.

Grüenbaum and Walther [7] introduced the following “measure” of how short a longest circuit can be, called the shortness exponent and defined for any family $\mathcal{F}$ of graphs

$$\sigma(\mathcal{F}) = \liminf_{G \in \mathcal{F}} \frac{\log h(G)}{\log v(G)},$$

where $v(G)$ is the number of all vertices of $G$ and $h(G)$ the maximal circuit length in $G$.

We identify polytopes with the graphs of their vertices and edges.

The case of those simple polytopes which are $k$-gonal modulo 3 was treated by Zaks [14] and Walther [12]. They proved that

$$\sigma(\mathcal{P}_3^1) \leq \sigma(\mathcal{P}_3^0) \leq (\log 22)/\log 23 \quad (i = 1, 2)$$
which seems to be the best known bound for simple polytopes at all (cf. |7|),
and
\[ \sigma(P_3^1) = \sigma(P_3^2). \]

Two-gonal polytopes exist only modulo 3. It is indeed clear that each face of
a 2-gonal (modulo n) polytope would be at least a hexagon if \( n \geq 4 \), which is
impossible. The case \( n = 3 \) was mentioned above. Thus, special attention will
be paid to cases \( k = 0, 1, 3, 4, 5 \). Clearly, there are no \( k \)-gonal (modulo \( n \))
polytopes for \( k \geq 6 \).

**Preliminary Results**

We present here three lemmas which will be repeatedly, sometimes tacitly,
used.

**Lemma 1** (Bielig [1]). Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be two families of graphs and
\( \varphi: \mathcal{F}_2 \to \mathcal{F}_1 \) an injective function. If there exist constants \( a, b > 0 \) such that,
for each \( G \in \mathcal{F}_2 \),

1. there is a circuit of length at least \( ah(G) \) in \( \varphi(G) \) and
2. \( v(\varphi(G)) \leq bv(G) \),

then \( \sigma(\mathcal{F}_1) \leq \sigma(\mathcal{F}_2) \).

Before we state Lemma 2, let us consider two polytopes \( A \) and \( B \), each
having two kinds of vertices: white and black. The black vertices are all 3-
valent. Let \( v \) be a white 3-valent vertex of \( B \); it will be called the special
vertex of \( B \). Let \( B - v \) be the graph fragment obtained from \( B \) by deleting \( v \).
Let \( B \to A \) denote the graph obtained by replacing each black vertex of \( A \) by
\( B - v \), the edges adjacent to any black vertex being identified with the edges
which were adjacent to \( v \). Now let \( G_0 \) and \( H_0 \) play the role of \( B \), \( v \) and \( v' \) be
the special vertices, and \( z \) and \( z' \) the number of all black vertices in \( G_0 \) and
\( H_0 \), respectively. Suppose:

(a) each circuit of \( G_0 \) through \( v \) misses at least \( w \) black vertices
\( (z - w > 1) \), or
(b) each circuit of \( H_0 \) through \( v' \) misses at least \( w' \) black vertices
\( (z' - w' > 1) \).

Let \( G_n = H_0 \to G_{n-1} \) and \( H_n = H_{n-1} \to G_0 \). The special vertex of \( G_0 \) becomes
the special vertex of \( H_n \) \( (n = 1, 2, 3, \ldots) \).

**Lemma 2** (Bielig [1], see also [9]). If (b) holds, then
\[ \sigma(\{G_n: n = 1, 2, 3, \ldots\}) \leq (\log(z' - w'))/\log z'. \]
We shall also use the following related lemma:

**Lemma 3.** If (a) holds, then

\[ \sigma(|H_n; n = 1, 2, 3, \ldots|) \leq (\log(z - w))/\log z. \]

**Proof.** Consider the family \( \{H'_n: n = 1, 2, 3, \ldots\} \) defined as follows:

\[ H'_1 = G_0; \quad H'_{n+1} = H'_n \rightarrow G_0 = G_0 \rightarrow H'_n. \]

By Lemma 2,

\[ \sigma(|H'_n; n = 1, 2, 3, \ldots|) \leq (\log(z - w))/\log z. \]

Now let \( \varphi \) be defined as \( \varphi(H'_n) = H_n. \) The relationship between \( H'_n \) and \( H_n \) is

\[ H_n = H_0 \rightarrow H'_n. \]

Obviously, for each \( n \) there exists a circuit of length at least \( h(H'_n) \) in \( H_n; \)
also \( v(H_n) \leq (v(H_0) - 1) v(H'_n). \) Thus, by Lemma 1,

\[ \sigma(|H_n; n = 1, 2, 3, \ldots|) \leq (\log(z - w))/\log z. \]

Besides these lemmas, Theorem 1 will be useful. In analogy with the equality \( \sigma(\mathcal{P}_3) = \sigma(\mathcal{P}_2) \), we have some further equalities among shortness exponents for classes of \( k \)-gonal (modulo \( n \)) polytopes.

**Theorem 1.** \( \sigma(\mathcal{P}_3) = \sigma(\mathcal{P}_4), \quad \sigma(\mathcal{P}_3^2) = \sigma(\mathcal{P}_3^5), \quad \sigma(\mathcal{P}_1^4) = \sigma(\mathcal{P}_1^5). \)

**Proof.** Throughout the paper \( T, C, \) and \( D \) will be the tetrahedron, the cube, and the dodecahedron, respectively. In each of them let some vertex be special. Let \( P \) be a polytope with all vertices black. We have the implications:

\[ P \in \mathcal{P}_3 \Rightarrow C \rightarrow P \in \mathcal{P}_4, \]
\[ P \in \mathcal{P}_3 \Rightarrow T \rightarrow P \in \mathcal{P}_5, \]
\[ P \in \mathcal{P}_3 \Rightarrow T \rightarrow P \in \mathcal{P}_7, \]
\[ P \in \mathcal{P}_3 \Rightarrow D \rightarrow P \in \mathcal{P}_8, \]
\[ P \in \mathcal{P}_1 \Rightarrow D \rightarrow P \in \mathcal{P}_1^5, \]
\[ P \in \mathcal{P}_1 \Rightarrow C \rightarrow P \in \mathcal{P}_1^4. \]

Now, by Lemma 1, the three equalities hold.
Barnette's conjecture, for which only partial answers are known (see [2, 4]), suggests the following weaker conjecture (Malkevitch [8, Problem 1]):

**Conjecture.** The family $\mathcal{P}_4^0$ consists only of Hamiltonian polytopes.

The case $n = 3$ (i.e., of $\mathcal{P}_3^0$), mentioned in the Introduction, was settled by Walther [12] and also considered before by Grünbaum (private communication to Zaks).

Malkevitch's question about $\mathcal{P}_3^0$ was answered in the negative by Zaks [15], but the problem of estimating the shortness exponent of $\mathcal{P}_3^0$ remained open.

**Theorem 2.**

$$\sigma(\mathcal{P}_3^0) \leq (\log 78)/\log 79.$$ 

**Proof.** To prove this we use the graph $Q_1$, of Fig. 1. This non-Hamiltonian graph, which has only pentagons and 15-gons as faces, was constructed in [16]. It was shown there that every circuit in $Q_1$ misses at least one vertex. Let $H_0 = D$, with an arbitrarily chosen special vertex, and $G_0$ be the graph obtained from $Q_1$ by replacing $w$ by $D - z$, where $z$ is again some vertex of $D$. Now let $v$, some vertex of $D$ which is not $z$ or one of its neighbours, be the special vertex of $G_0$. Colour in black all the 79 vertices of $Q_1 - w$ in $G_0$. Then, by Lemma 3, we get a sequence $\{H_n : n = 1, 2, 3, \ldots\}$ with shortness exponent at most $(\log 78)/\log 79$ and the graphs of which have just penta-, 20-, and 60-gons as faces. Thus the theorem is proved.
Monogonal Polytopes

Bielig and Schulz [2] observed that $\sigma(\mathcal{P}_2) < 1$. Their construction leads more precisely to $\sigma(\mathcal{P}_2) \leq (\log 42)/\log 43$.

**Theorem 3.**

$$\sigma(\mathcal{P}_2) \leq (\log 26)/\log 27.$$ 

*Proof.* Let $G_0 = T$ with one vertex black, and $H_0$ be the graph of Fig. 2. This $H_0$ is a slight modification of the famous non-Hamiltonian graph of Tutte [10]. The fact that no path from $b$ to $c$ contains all black vertices of $abc$, observed and used by Walther [12] and Grünbaum–Walther [7], implies that the hypotheses of Lemma 2 are fulfilled. The theorem now follows.

Case $n = 3$ was mentioned in the Introduction.

**Theorem 4.**

$$\sigma(\mathcal{P}_3) \leq (\log 100)/\log 102.$$ 

*Proof.* Let $G_0$ be the graph of Fig. 3, where each square vertex is replaced with the graph $F$ of Fig. 4. Now, $H_0$ is obtained from $G_0$ by replacing the vertex $v$ with the graph fragment of Fig. 5. The black vertices

*Figure 2*
Figure 3

Figure 4

Figure 5
of $G_0$ and $H_0$ are in each copy of $F$ those marked on Fig. 4. In a graph containing $F$ as a subgraph, every circuit traversing $F$ and passing through all black vertices of $F$ must go through $F$ from $a$ to $c$ or from $b$ to $d$ (see [3, 15, 16]). It follows that every circuit of $H_0$ through $v'$ misses in at least two $F$-graphs, at least one black vertex, or misses completely one copy of $F$. There are 102 black vertices in $H_0$. It is easy to see that each $G_n$ is a member of $\mathcal{P}_4$. Thus, by Lemma 2, $\sigma(\mathcal{P}_4) \leq (\log 100)/\log 102$.

Theorem 4 concludes this section, because no polytopes are monogononal modulo $n$ for $n \geq 5$.

**Triangular Polytopes**

The first class of polytopes to be considered is $\mathcal{P}_4$.

**Theorem 5.**

$$\sigma(\mathcal{P}_4) \leq (\log 26)/\log 27.$$  

Since the proof is completely analogous to that of Theorem 3, we only mention that the graph $H_0$ will be taken from Fig. 6 instead of Fig. 2.

![Figure 6](image-url)
Theorem 6.

\[ \sigma(\mathcal{P}_3^1) \leq (\log 36)/\log 37. \]

The proof uses the equality \( \sigma(\mathcal{P}_3^1) = \sigma(\mathcal{P}_2^4) \) of Theorem 1 and also Theorem 9 from the following section.

Theorem 7.

\[ \sigma(\mathcal{P}_6^2) \leq (\log 42)/\log 43. \]

Proof. To construct \( G_0 \) and \( H_0 \) we use the graph \( Q_2 \) of Fig. 7, which allows no Hamiltonian circuit (see Grinberg [5] and Tutte [11], or [6, p. 1145]). We replace every marked vertex of \( Q_2 \) with \( T - w \), in which one vertex is black, and obtain in this way \( G_0 \in \mathcal{P}_6^2 \), in which the nonmarked vertices of \( Q_2 \) have also to be black. To get \( H_0 \) we contract in \( G_0 \) the triangle at \( v \) to the special vertex \( v' \). Every circuit through \( v' \) misses at least one of the 43 black vertices of \( H_0 \). Now, using Lemma 2, \( \sigma(\mathcal{P}_6^2) \leq \log 42/\log 43. \)

Theorem 8.

\[ \sigma(\mathcal{P}_7^1) \leq (\log 166)/\log 168. \]
The proof combines the equality $\sigma(\mathcal{P}_3) = \sigma(\mathcal{P}_5)$ of Theorem 1 with Theorem 13(b), that we shall establish later.

We have no further classes to study, since $\mathcal{P}_n^3 = \{T\}$ for every $n \geq 8$. Indeed, the triangles of a graph from $\mathcal{P}_n^3 - \{T\}$ would be nonadjacent; contract them to points: every $m$-gon ($m > 3$) becomes a $k$-gon with $k \geq m/2$. For $n \geq 8$, we have $m \geq 11$, hence $k \geq 6$, but such a planar graph does not exist.

**Quadrangular Polytopes**

In this section we are somewhat successful but for odd $n$. The difficulty in treating the case of even $n$ is again explainable by its relationship with Barnette's conjecture.

Walther announced in [13] that he is able to prove that the shortness exponent of the family of all simple polytopes having just 4-gons and $k$-gons as faces is smaller than 1; here $k$ is any fixed odd number, not smaller than 17. This yields $\sigma(\mathcal{P}_4^z) < 1$ for odd $n \geq 13$. Thus, we restrict ourselves to the cases $n = 5, 7, 9, 11$.

In this section $w$ is a vertex of the cube $C$.

**Theorem 9.**

$$\sigma(\mathcal{P}_4^z) \leq (\log 36)/\log 37.$$  

**Proof.** We obtain the graph $G_0 \in \mathcal{P}_z^4$ by replacing every marked vertex of the graph $Q_3$ in Fig. 8 with a copy of $C - w$. In each such copy (except that which corresponds to the vertex numbered one), one vertex will be black. Also, all nonmarked vertices of $Q_3$, with the exception of 2, 3, 4, and $v$ will be black. Now replace $v$ with the fragment shown in Fig. 9. The graph obtained is $H_0$ and has 37 black vertices. Using the same arguments as for
Figure 9

Figure 10
Theorem 4, we see that every circuit through \( v' \) misses at least one black vertex. Thus, by Lemma 2, the inequality of the statement holds.

**Theorem 10.**

\[ \sigma(\mathcal{F}_4^4) \leq (\log 42)/\log 43. \]

**Proof.** To construct \( G_0 \) and \( H_0 \) we again use the Grinberg–Tutte graph, now called \( Q_4 \) and shown in Fig. 10. We replace every marked vertex of \( Q_4 \) with a copy of \( C - w \) in which one vertex is black, and colour the remaining vertices of \( Q_4 \) in black. The resulting graph is \( G_0 \). To get \( H_0 \) we replace the vertex \( z \) of \( G_0 \) with the fragment shown in Fig. 11. Every circuit through \( v' \) misses at least one of the 43 black vertices of \( H_0 \). By Lemma 2, this yields the theorem.

**Theorem 11.**

\[ \sigma(\mathcal{F}_9^4) \leq (\log 42)/\log 43. \]

**Proof.** The basic graph \( Q_5 \) is the same as in the preceding proof. We replace its vertices marked on Fig. 12 with copies of \( C - w \). To obtain \( G_0 \in \mathcal{F}_9^4 \), the two marked vertices of the graph of Fig. 13 must be replaced with \( Q_5 - v \). In \( G_0 \) we colour black, in one of the two copies of \( Q_5 - v \), every vertex nonmarked on Fig. 12 and one vertex in each used copy of \( C - w \). Replacing \( w' \) with the fragment shown in Fig. 14 we obtain \( H_0 \). As in Theorem 10, we get \( \sigma(\mathcal{F}_9^4) \leq (\log 42)/\log 43 \) using Lemma 2.
Theorem 12.

\[ \sigma(\mathcal{P}_{11}^4) \leq \frac{\log 100}{\log 102}. \]

Theorem 13(c), together with the equality \( \sigma(\mathcal{P}_{11}^4) = \sigma(\mathcal{P}_{11}^5) \) from Theorem 1, prove Theorem 12.

Pentagonal Polytopes

We are able to treat the pentagonal (modulo \( n \)) polytopes for all values of \( n \). The next theorem shows that the shortness exponents of all classes are smaller than 1.

Theorem 13.

(a) \( \sigma(\mathcal{P}_{6}^5) \leq \frac{\log 42}{\log 43} \)
(b) \( \sigma(\mathcal{P}_{7}^5) \leq \frac{\log 166}{\log 168} \)
(c) \( \sigma(\mathcal{P}_{9}^5) \leq \frac{\log 42}{\log 43} \)
(d) \( \sigma(\mathcal{P}_{5m+1}^5) \leq \frac{\log(54m - 8)}{\log(54m - 6)} \quad (m \geq 2) \)
(e) \( \sigma(\mathcal{P}_{5m+2}^5) \leq \frac{\log(54m + 4)}{\log(54m + 6)} \quad (m \geq 2) \)
(f) \( \sigma(\mathcal{P}_{5m+3}^5) \leq \frac{\log(54m + 46)}{\log(54m + 48)} \quad (m \geq 1) \)
(g) \( \sigma(\mathcal{P}_{5m+4}^5) \leq \frac{\log(54m + 58)}{\log(54m + 60)} \) \( (m \geq 2) \)

(h) \( \sigma(\mathcal{P}_{5m+2}^5) \leq \frac{\log(90m + 78)}{\log(90m + 80)} \) \( (m \geq 1) \).

Proof. We first prove inequalities (d)-(h). Here we consider graphs in \( \mathcal{P}_{5m+1}^r \) with \( m, r \in \mathbb{N} \) and \( r \leq 5 \). Let \( G_0^m, G_n^m, \) and \( H_0^m \) play the roles of \( G_0, G_n, \) and \( H_0 \). The graphs used to construct \( G_0^m \) and \( H_0^m \) are those of Figs. 3, 17, and 18. Each square vertex must be replaced with a graph fragment \( F_m \) or \( F'_m \) (Fig. 15), in the way indicated by the position of the edges \( a, b, c, d \). These graph fragments were found and used by Zaks [15], \( F_1 \) by Faulkner and Younger [3]. The vertices marked in Fig. 15 are black. By the results in [15] and the arguments in the proof of Theorem 4, every circuit through \( v \) misses at least two black vertices in each of the preceding graphs. In this proof we also make repeated use of the graphs \( L_n \) shown in Fig. 16. They have only pentagons and \( (n + 2) \)-gons as faces. In the following, the vertex \( v' \) will play the role of the special vertex.
(d) Let $G_0^2$ be the graph of Fig. 17. The square vertices must be replaced with copies of $F_1$ (Fig. 15(a)). Let $H_0^2$ be obtained by replacing $v$ with $L_{11-w}$ (see Fig. 16). Now $H_0^2$ has 102 black vertices and every circuit through $v'$ misses at least two of them. Each $G_n^2$ is in $\mathcal{S}_{11}^2$. Thus, by Lemma 2, $\sigma(\mathcal{S}_{11}^2) \leq \log 100) / \log 102$. Now let $G_0^m$ be the graph of Fig. 17, where the square vertices are replaced with copies of $F_{m-1}$. To get $H_0^m$, replace $v$ by $L_{5m+1-w}$. Using the preceding considerations and Lemma 2, we get

$$\sigma(\mathcal{S}_{5m+1}^2) \leq \frac{\log[3(16 + (m - 1) 18) - 2]}{\log[3(16 + (m - 1) 18)]}$$

$$= \frac{\log(54m - 8)}{\log(54m - 6)},$$

for all $m \geq 2$.

(e) In analogy to (d), we prove (e) using $F_{m-1}'$ instead of $F_{m-1}$ and $L_{5m+2}$ instead of $L_{5m+1}$. This yields

$$\sigma(\mathcal{S}_{5m+2}^2) \leq \frac{\log[3(20 + (m - 1) 18) - 2]}{\log[3(20 + (m - 1) 18)]}$$

$$= \frac{\log(54m + 4)}{\log(54m + 6)},$$

for all $m \geq 2$.

(f) Now let $G_0^m$ be the graph of Fig. 3, where the square vertices are replaced with $F_m$ (see Fig. 15(b)). Now $H_0^m$ is obtained from $G_0^m$ by replacing $v$ with $L_{5m+1-w}$. Then, analogously,

$$\sigma(\mathcal{S}_{5m+3}^2) \leq \frac{\log[3(16 + 18m) - 2]}{\log[3(16 + 18m)]}$$

$$= \frac{\log(54m + 46)}{\log(54m + 48)},$$

for all $m \geq 1$.

(g) Using $F_m'$ instead of $F_m$ and $L_{5m+4}$ instead of $L_{5m+3}$, we get

$$\sigma(\mathcal{S}_{5m+4}^2) \leq \frac{\log[3(20 + 18m) - 2]}{\log[3(20 + 18m)]}$$

$$= \frac{\log(54m + 58)}{\log(54m + 60)},$$

for all $m \geq 2$.

(h) To prove the assertion for $n = 5m + 5$, we use the graph of Fig. 18. In analogy to (c) we employ $F_m$ and $L_{5m+5}$ to construct $G_0^m$ and $H_0^m$. This yields

$$\sigma(\mathcal{S}_{5m+5}^2) \leq \frac{\log[5(16 + 18m) - 2]}{\log[5(16 + 18m)]}$$

$$= \frac{\log(90m + 78)}{\log(90m + 80)},$$

for all $m \geq 1$. 


(b) We obviously have (by (g)),

$$
\sigma(\mathcal{P}_5^2) \leq \sigma(\mathcal{P}_14^5) \leq (\log 166)/\log 168.
$$

(a) The construction of $G_0$ again uses the Grinberg–Tutte graph. The marked vertices of the graph $Q_6$ in Fig. 19 must be replaced by $D - w''$, $w''$ being an arbitrary vertex of $D$. Then we replace two of the vertices of the cube $C$ not belonging to the same face, with a copy of $Q_6 - w'$. We colour black every nonmarked vertex and one of the vertices of each copy of $D - w''$ in one of the copies of $Q_6 - w'$. Now $H_6$ is obtained from $G_0$ by replacing one vertex in the other (uncoloured) copy of $Q_6 - w'$ with $L_6 - w$. Thus, by Lemma 2, we get $\sigma(\mathcal{P}_6^5) \leq (\log 42)/\log 43$.

(c) The proof is completely analogous to the proof of (a) and uses $Q$, (Fig. 20) and $L$, (Fig. 16). This completes the proof.

By using the graph $Q_8$ of Fig. 21 instead of the graph in Fig. 17, and $L_{10k+1}$, we can improve three sequences of estimates in Theorem 13:

$$\sigma(\mathcal{P}_{10k+1}^5) \leq (\log(54k - 7))/\log(54k - 6) \quad (i = -1, 0, 1) \quad (k \geq 2).$$

For $i = 0$, we drop the edge $f_j$ and for $i = -1$ both edges $e_j$ and $f_j$ in $Q_8$ ($j = 1, 2, 3$).

Thus, we get

$$\sigma(\mathcal{P}_{19}^5), \sigma(\mathcal{P}_{20}^5), \sigma(\mathcal{P}_{21}^5) \leq (\log 101)/\log 102 = 0.9978698...,$$
while Theorem 13 only gives

\[ \sigma(\mathcal{P}_1^{40}) \leq (\log 220)/\log 222 = 0.9983249..., \]
\[ \sigma(\mathcal{P}_2^{40}) \leq (\log 348)/\log 350 = 0.9990217..., \]
\[ \sigma(\mathcal{P}_3^{41}) \leq (\log 208)/\log 210 = 0.9982103... . \]

**Final Remarks**

Let \( \mathcal{P} \) denote a family of simple polytopes and \( M(\mathcal{P}) \) the family of the corresponding medial graphs. Bielig proved in [1] that \( \sigma(\mathcal{P}) = \sigma(M(\mathcal{P})) \). Let \( \mathcal{R} \) be the family of all 4-valent polytopes. Take \( \mathcal{P} \) to be the family of all simple polytopes. Then, since \( M(\mathcal{P}) \subset \mathcal{R} \), we have

\[ \sigma(\mathcal{R}) \leq \sigma(\mathcal{P}) \leq (\log 22)/\log 23. \]

Now let \( \mathcal{R}_n^k = \mathcal{P}_n^k \cap \mathcal{R} \); we obtain the inequalities

\[ \sigma(\mathcal{P}_1^2) \leq \sigma(\mathcal{P}_1^2); \quad \sigma(\mathcal{P}_0^3) \leq \sigma(\mathcal{P}_0^3); \]
\[ \sigma(\mathcal{R}_n^3) \leq \sigma(\mathcal{P}_n^3) \quad (n = 4, 5, 6, 7). \]

(For numerical estimates see the Introduction and Theorems 3, 5–8.)

Many, if not all, of the bounds found in this paper for the various shortness exponents are probably not the best possible ones. It seems to be more difficult to find nontrivial lower bounds, or even to determine the
shortness exponents. The most interesting open questions, however, are (in our opinion) those strongly related to Barnette’s conjecture and regarding $\mathcal{P}_4$ and $\mathcal{P}_n$ for even $n$, namely, whether they possess non-Hamiltonian graphs or not.

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