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MOST CONVEX MIRRORS ARE MAGIC

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IN FACT I am convinced that all mirrors we produce are in reality magic mirrors, although we don't intend this. In which sense are they magic? Most points see infinitely many images of any given point in the world, and are focal for the (parallel) rays from a far star, that is they also see infinitely many images of the star. Moreover, most points see infinitely many images of themselves (this assertion is not a particular case of the first one!).

Certainly, we try to produce plane, spherical, cylindrical, parabolic mirrors, but the set of such mirrors is very thin in the space of all of them, so that we most likely realise mirrors which belong to the large residual set of magic mirrors. We just don't see those infinities of images because they are too close to each other!

Now we become mathematically more explicit. The word *most* means *all, except those in a set of first Baire category*, and is used, of course, only in spaces of second Baire category. We restrict ourselves to the 2-dimensional case.

Let $\mathscr{C}$ be the space of all convex (closed) curves in the plane P. Endowed with the Hausdorff metric, $\mathscr{C}$ is of second Baire category. Some properties of most convex curves (and surfaces) have been studied by Klee[2], Gruber[1], Schneider[4] and the author[5–7].

Let C be a convex curve. We say that $x \in P$ *sees an image of* $y \in P$ if there is a broken line xzy with $z \in C$ such that C is smooth at z and the angles of xz and yz with the tangent at z to C are equal. We say that x *sees* α *images of* y (α finite or not) if there are at least α such broken lines. Also, we say that x *sees* α *images of a set* if it sees α images of every point in the set.

THEOREM 1. *For most convex curves, most points see infinitely many images of any given point in the plane.*

Proof. Let $\mathscr{S}$ be the family of all smooth convex curves C such that there is a set E dense on C, in each point u of which the lower radii of curvature in both senses $\rho_i^-(u)$ and $\rho_i^+(u)$ vanish and the upper radii of curvature in both senses $\rho_s^-(u)$ and $\rho_s^+(u)$ are infinite. That most convex curves belong to $\mathscr{S}$ was shown by R. Schneider[4] (see also[6]). We choose now arbitrarily $C \in \mathscr{S}$ and $y \in P$. We show that the complement $\complement M_{\aleph_0}$ of the set $M_{\aleph_0}$ of those points that see infinitely many images of y is of first Baire category in P.

Let M_k be the set of all points which see k images of y (k finite). We prove that $\complement M_k$ is nowhere dense.

Let O be an open set of P and $x \in O$. The smallest ellipse with focal points x, y still surrounding C will have at least one contact point z^* with C. Using the broken line xz^*y, we see that x sees an image of y.

Since E is dense on C and C is of class C^1, there is a point $x' \in O$ seeing y along a broken line $x'z'y$ with $z' \in E$ (see Fig. 1). Let p be a point on the normal at z' to C. Since $\rho_i^{\pm}(z') = 0$ and $\rho_s^{\pm}(z') = \infty$, there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of points on C converging to z', such that the normals at z_n pass through p. Let x_n be the point on the

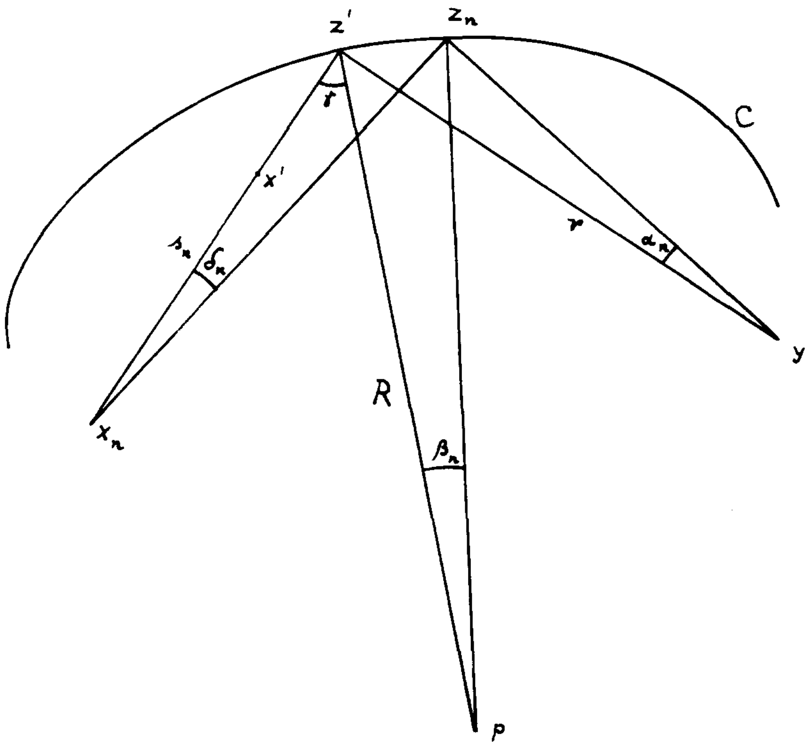

Fig. 1.

line $x'z'$ seeing y along $x_n z_n y$, if any. Let α_n, β_n, δ_n be the measures of the angles under which y, p, x_n see the segment with endpoints z', z_n. Also, let $\gamma = \angle x'z'p$. From

$$\gamma + \delta_n = \angle x_n z_n p + \beta_n$$

and

$$\angle pz'y + \beta_n = \angle pz_n y + \alpha_n,$$

we get

$$\delta_n = 2\beta_n - \alpha_n.$$

(Thus, x_n is well defined if $2\beta_n > \alpha_n$.) Let s_n, R, r denote the distances from z' to x_n, p, y respectively. We have

$$\frac{\sin \beta_n}{\|z' - z_n\|} = \frac{\sin \angle pz'z_n}{\|p - z_n\|}; \qquad \frac{\sin \alpha_n}{\|z' - z_n\|} = \frac{\sin \angle yz'z_n}{\|y - z_n\|}.$$

Hence, for $n \to \infty$,

$$\frac{\sin \beta_n}{\sin \alpha_n} \to \frac{r}{R \cdot \cos \gamma}.$$

We also have

$$\frac{\sin \delta_n}{\|z' - z_n\|} = \frac{\sin \angle x_n z_n z'}{s_n},$$

yielding

$$s_n = \frac{\sin \angle x_n z_n z' \cdot \|y - z_n\| \cdot \sin \alpha_n}{\sin \angle yz'z_n \cdot \sin (2\beta_n - \alpha_n)}.$$

Here

$$\frac{\sin (2\beta_n - \alpha_n)}{\sin \alpha_n} = 2\frac{\sin \beta_n}{\sin \alpha_n} \cos \beta_n \cos \alpha_n - \cos 2\beta_n,$$

whence

$$\frac{\sin(2\beta_n - \alpha_n)}{\sin \alpha_n} \to \frac{2r}{R \cdot \cos \gamma} - 1$$

and $s_n \to f(R)$, where

$$f(R) = \frac{Rr \cdot \cos \gamma}{2r - R \cdot \cos \gamma}.$$

Now move p. If R converges from above to 0, then $f(R) \to 0$; if R converges from below to $2r/\cos \gamma$, then $f(R) \to \infty$. For $R < 2r/\cos \gamma$ and n large enough, x_n and s_n are well defined.

We choose now a sequence $\{p_n\}_{n=1}^{\infty}$ of points on the interior normal at z' to C (p_n and C lie on the same side of the tangent at z' to C), such that the corresponding sequence $\{R_n\}_{n=1}^{\infty}$ verifies

$$0 < R_n < \frac{2r}{\cos \gamma}, \qquad \liminf_{n\to\infty} R_n = 0, \qquad \limsup_{n\to\infty} R_n = \frac{2r}{\cos \gamma}.$$

Since $\liminf_{n\to\infty} f(R_n) = O$ and $\limsup_{n\to\infty} f(R_n) = \infty$, we can obviously choose a sequence $\{z_n^+\}_{n=1}^{\infty}$ of points on C such that $z_n^+ \to z'$ from one side, p_n lies on the normal at z_n^+ to C and the corresponding sequence $\{s_n^+\}_{n=1}^{\infty}$ satisfies

$$\liminf_{n\to\infty} s_n^+ = 0, \qquad \limsup_{n\to\infty} s_n^+ = \infty.$$

This together with the fact that C is of class C^1 yields $x' \in M_{\aleph_0}$.

Let $z_1, \ldots, z_{2k+1}$ be in this order on C, such that y sees an image of x' via z_i ($i = 1, \ldots, 2k+1$). Since C contains no arc of ellipse (E is dense!), there is a point z_i' between z_i and z_{i+1} ($i = 1, \ldots, 2k$) such that y does not see x', but another point x_i' near x' on the line $z_i x'$, via z_i'. Let in general $[L_1, L_2, L_3]$ denote the triangular domain with sides on the lines L_1, L_2 and L_3. In particular, let $D_i = [x'z_i, x'z_{i+1}, x_i'z_i']$ ($i = 1, \ldots, 2k$). Again because C is of class C^1, every point of any domain D_i sees y via some point between z_i and z_{i+1}. Among these $2k$ domains, at least k must have a nonempty intersection. Thus we get a small polygonal domain $D \subset M_k$ with the point x' on its boundary. Hence $D \cap O \subset M_k \cap O$. This shows that $\complement M_k$ is nowhere dense.

Now it is an easy matter to derive that $\mathfrak{C}M_{\aleph_0}$ is of first Baire category:

$$\mathfrak{C}M_{\aleph_0} = \mathfrak{C} \bigcap_k M_k = \bigcup_k \mathfrak{C}M_k$$

and the proof is complete.

Remark. In Theorem 1 we can consider the given point at infinity. Nothing changes in statement or proof, except that $f(R) = (1/2)R \cdot \cos \gamma$, x_n is always defined, and $\{p_n\}_{n=1}^{\infty}$ is chosen so that

$$\liminf_{n\to\infty} R_n = 0, \qquad \limsup_{n\to\infty} R_n = \infty.$$

COROLLARY. *For most convex curves, most points see infinitely many images of any given countable set in the plane (even extended with a line at infinity).*

Proof. Because any countable intersection of residual sets is residual.

It also happens that, for most convex curves, most points see infinitely many images of themselves. This is equivalent to the following.

THEOREM 2. *For most convex curves, most points lie on infinitely many normals.*

Proof. Let $\mathcal{S}$ be the same family of convex curves as in the proof of the preceding theorem and consider $C \in \mathcal{S}$. Let O be an open set, $x \in O$ and z be a point on C such that x lies on the interior normal at z to C. The already considered set E has a point z' such that the normal at z' intersects O. Let x' be a point of this intersection. This point x' lies on infinitely many normals and it can be seen (like we already did in the proof of Theorem 1, in order to show that x' is a boundary point of a domain the points of which see k images of y) that x' is on the boundary of a domain the points of which lie on at least k normals of C. Thus the set of points not belonging to infinitely many normals is of first Baire category, and the theorem is proved.

Theorems 1 and 2 show in fact that the set $M^*_{\aleph_0}$ of all pairs $(x, y) \in P \times P$ such that x sees $\aleph_0$ images of y, intersects certain 2-planes in residual sets. A converse of the Kuratowski–Ulam theorem (Theorem 15.4 in [3]) suggests that $M_{\aleph_0}$ might also be quite large—residual—in $P \times P$. The next theorem confirms this.

THEOREM 3. *For most convex curves and most pairs of points (x, y) in $P \times P$, x sees infinitely many images of y.*

Proof. $\mathcal{S}$ is as before; let $C \in \mathcal{S}$. Let M^*_k denote the set of all pairs $(x, y) \in P \times P$ such that x sees k images of y (k finite). We show that $\mathfrak{C}M_k$ is nowhere dense in $P \times P$.

Let O^* be an open set in $P \times P$, $(x, y) \in O^*$ and O_x, O_y be two open sets in P such that $x \in O_x$, $y \in O_y$ and $O_x \times O_y \subset O^*$. Like in the proof of Theorem 1 we can find a point $x' \in O_x$ such that $(x', y) \in M^*_{\aleph_0}$.

Let $z_1, \ldots, z_{4k+1}$ be points in this order on C, such that y sees an image of x' via z_i $(i = 1, \ldots, 4k + 1)$. Between z_i and z_{i+1} there always exists a point z'_i such that y does not see x', but sees another point x'_i near x' on the line $z_i x'$, via z'_i. Now, another point x''_i between x' and x_i' on $x'x'_i$ sees an image of another point y'_i near y on $z_i y$. Let $D_i = [x'z_i, x'z_{i+1}, x'_i z'_i]$, $D'_i = [x'z_i, x'z_{i+1}, x''_i z'_i]$ and $D''_i = [yz_i, yz_{i+1}, y'_i z'_i]$ $(i = 1, \ldots, 4k)$. Let v be arbitrarily chosen in D''_i. The ray A'_i of those points seeing v via z'_i passes through $D_i - D'_i$. The ray A_i of those points seeing v via z_i misses D_i in such a way that D'_i lies between A_i and A'_i. Thus each point of D'_i sees an image of v (whence of any point in D''_i) via a point of C between z_i and z_{i+1}. Among these $4k$ domains D'_i at least $2k$ have a nonempty intersection D'. Among the $2k$ domains D''_i with the above $2k$ indices, at least k have a nonempty intersection D''. These domains D', D'' are polygonal and have x', y respectively on their boundaries. Clearly $D' \times D'' \subset M^*_k$. Hence

$$(D' \cap O_x) \times (D'' \cap O_y) \subset M^*_k \cap O^*,$$

which proves that $\mathfrak{C} M^*_k$ is nowhere dense indeed. Now the theorem follows immediately.

One may ask whether $M^*_{\aleph_0}$ equals $P \times P$ for some (or most?) convex curves. The answer is: no. Let C be any convex curve and Γ its circumscribed circle. The intersection $C \cap \Gamma$ contains at least two points (and, for most convex curves C, precisely three points [7]). Let $a \in C \cap \Gamma$ and b be a point on the interior normal at a, outside Γ. It is easily seen that b lies on at most two normals of C, namely those at the nearest point and at the furthest point of C from b. Thus $(b, b) \notin M^*_3$. (Also $(b, b') \notin M^*_3$ for any point b' chosen like b.) Hence not even M^*_3 can equal $P \times P$.

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