Points on infinitely many normals to convex surfaces

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1. Introduction

Every convex surface with a spherical region has the property that infinitely many normals have a common point. How many points may lie on infinitely many normals? Countably many? Of course. Densely many? This is already less obvious. We prove here that most points may lie on infinitely many normals, in the sense that those lying on at most finitely many form a set of first Baire category. This will always be the meaning of "most". In fact we show that most convex surfaces enjoy the above property! Moreover, in our proof the normals will realise relative maxima for the distance-function from their common point.

In the planar case, the main result was already proved in [6]. However, the generalisation to higher dimensions is not straightforward.

We work in $\mathbb{R}^d$. By a *convex surface* we always understand a complete, closed one ([1], p. 3), i.e. the frontier of a bounded open convex set. The space $\mathcal{S}$ of all convex surfaces in $\mathbb{R}^d$ endowed with Hausdorff's distance is a Baire space.

2. Auxiliary results

Let x be a smooth point of the convex surface S and τ a tangent direction at x. Let $\rho_i^\tau(x)$ and $\rho_s^\tau(x)$ be the lower and upper radii of curvature of S in direction τ at x (see [1], p. 14). Also, let F be a sphere internally tangent to S at x, i.e. lying on the same side as S of the common tangent hyperplane at x. We say that F *is quasisupporting* S at x if there exists a continuum C containing x and not reduced to this point, such that

$$C \subset F \cap \operatorname{conv} S.$$

Lemma 1 ([5], [7]). *For most convex surfaces, at most points x and every tangent direction τ,*

$$\rho_i^\tau(x) = 0 \quad \textit{and} \quad \rho_s^\tau(x) = \infty.$$

Lemma 2. *For most convex surfaces, at most points x, no sphere internally tangent to S at x is quasisupporting S at x.*

Proof. Let $\mathcal{S}^\dagger$ be the space of all smooth convex surfaces and let $\mathcal{S}^* \subset \mathcal{S}^\dagger$ be the set of all surfaces having a set of second Baire category of points, at each of which there is a quasisupporting sphere of S. Then

$$\mathcal{S}^* = \left\{ S \in \mathcal{S}^\dagger : \bigcup_{n=1}^{\infty} A_n \text{ is of second category} \right\},$$

where A_n is the set of points on S at which there are quasisupporting spheres with continua corresponding to C of diameter at least n^{-1}. Setting

$$\mathcal{S}_n = \{S \in \mathcal{S}^\dagger : A_n \text{ is not nowhere dense}\},$$

we have

$$\mathcal{S}^* \subset \bigcup_{n=1}^{\infty} \mathcal{S}_n;$$

we shall prove that $\mathcal{S}_n$ is of first Baire category for every n.

On each surface $S \in \mathcal{S}^\dagger$, A_n is closed. Suppose indeed $\{x_k\}_{k=1}^{\infty}$ converges to x_∞ on S and $x_k \in A_n$. If r_k is the diameter of the quasisupporting sphere of S at x_k, we have $r_k \geqq n^{-1}$, whence

$$\liminf_{k \to \infty} r_k \geqq n^{-1}.$$

By choosing smaller quasisupporting spheres at the points of $\{x_k\}_{k=1}^{\infty}$ if necessary, we arrange that

$$\limsup_{k \to \infty} r_k < \infty.$$

Hence we can choose a subsequence $\{x_{k_m}\}_{m=1}^{\infty}$ such that $\{r_{k_m}\}_{m=1}^{\infty}$ converges to some $r_\infty \geqq n^{-1}$. All corresponding spheres lie in a large ball. Thus, using the fact that the space of all continua in a compact metric space is itself compact ([4], § 38. I, 1, p. 21 and § 42. II, 4, p. 110), the corresponding sequence of continua $\{C_{k_m}\}_{m=1}^{\infty}$ possesses a subsequence convergent to some continuum C_∞ of diameter at least n^{-1}. Then, clearly, there is a sphere of diameter r_∞ quasisupporting S at x_∞, the corresponding continuum on it being C_∞. Hence A_n is closed.

Now, if $S \in \mathcal{S}_n$, there is a disk $D \subset A_n$. Let $\mathcal{S}_{n,m}$ be the family of all $S \in \mathcal{S}_n$ with a disk $D \subset A_n$ of radius m^{-1}. Since

$$\mathcal{S}_n = \bigcup_{m=1}^{\infty} \mathcal{S}_{n,m},$$

it suffices to prove that each $\mathcal{S}_{n,m}$ is nowhere dense.

To do this, we first see that $\mathcal{S}_{n,m}$ is closed. Choose indeed the sequence $\{S_i\}_{i=1}^{\infty}$ with $S_i \in \mathcal{S}_{n,m}$ and $S_i \to S$, a smooth convex surface. The corresponding sequence of disks $\{D_i\}_{i=1}^{\infty}$ has a subsequence convergent to some disk D on S. By the preceding compactness argument concerning spaces of continua, $D \subset A_n$. To see that $\mathcal{S}^\dagger - \mathcal{S}_{n,m}$ is dense, take an arbitrary open set $\mathcal{O} \subset \mathcal{S}$, a polyhedral surface $P \in \mathcal{O}$ with faces of diameter less than m^{-1} and the frontier Q of

$$\operatorname{conv} P + \varepsilon B,$$

B being the unit ball and $\varepsilon < n^{-1}$. Then, for ε small enough,

$$Q \in \mathcal{O} \cap \mathcal{S}^\dagger$$

and $Q \notin \mathcal{S}_{n,m}$.

Thus, $\mathscr{S}_{n,m}$ is nowhere dense, $\mathscr{S}_n$ and $\mathscr{S}^*$ are of first Baire category and therefore most surfaces in $\mathscr{S}^\dagger$ have the stated property. Since most surfaces in $\mathscr{S}$ belong to $\mathscr{S}^\dagger$ by results of Klee [3] or Gruber [2], the proof of Lemma 2 is complete.

3. Main result

Theorem. *For most convex surfaces, most points of $\mathbb{R}^d$ lie on infinitely many normals.*

Proof. Let S be a smooth surface with the properties of Lemmas 1 and 2. Let M_α be the set of points in $\mathbb{R}^d$ lying on at least α normals of S. Since

$$\complement M_{\aleph_0} = \complement \bigcap_{k=1}^{\infty} M_k = \bigcup_{k=1}^{\infty} \complement M_k,$$

it suffices to prove that $\complement M_k$ is nowhere dense in $\mathbb{R}^d$, for every finite cardinal number k.

Let O be a small open set in $\mathbb{R}^d$. Let z be the furthest point of S from an arbitrarily chosen point of O. Since S is of class C^1 and the set E of all points of S satisfying the curvature conditions of Lemma 1 and the quasisupport condition of Lemma 2 is dense in S, there exists $x_0 \in E$ near z, such that the normal N at x_0 to S passes through O. Let $y \in N \cap O$.

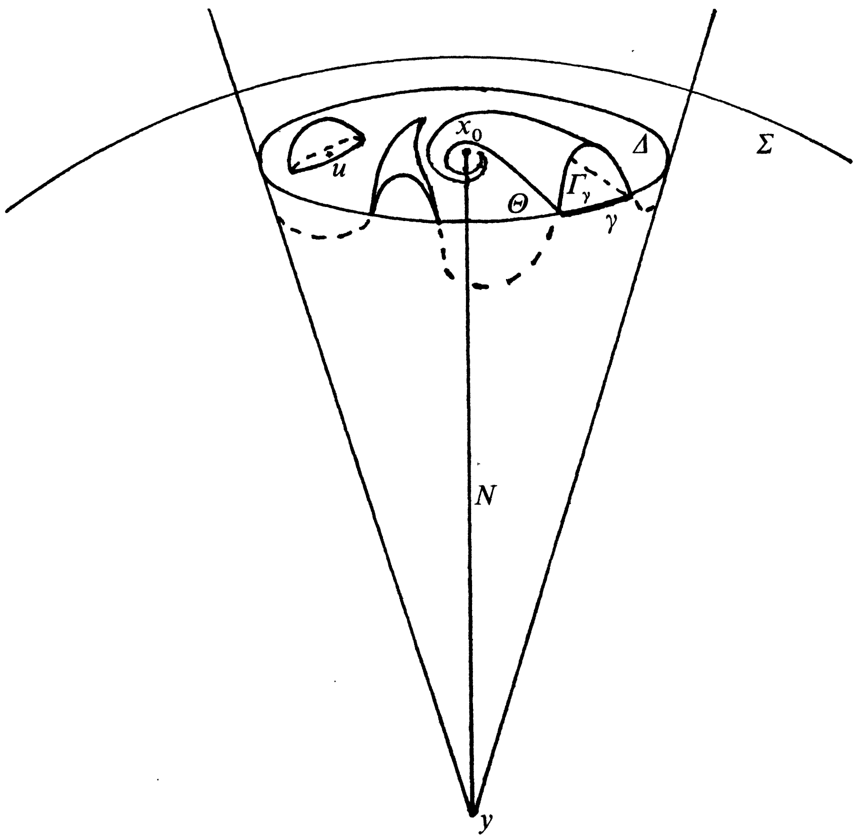

Figure 1

Let Σ be the sphere of centre y, passing through x_0. Take $\varepsilon>0$ appropriately small and consider the disk Δ of all points of Σ at distance at most ε from x_0. The cone of apex y and basis Δ cuts from S one or two disks (for ε small enough), one of which, say D, contains x_0. Consider the continuous function

$$f\colon \Delta \to \mathbb{R}$$

defined by

$$f(x)=\|x'-y\|-\|x_0-y\|,$$

x' being the intersection of the line through x and y with D. If

$$f(\operatorname{bd}\Delta)\leqq 0,$$

where $\operatorname{bd}\Delta$ means the frontier of Δ, define $\Theta=\Delta$. Otherwise, let γ be a component of

$$\partial^+\Delta=\{x\in \operatorname{bd}\Delta\colon f(x)>0\}.$$

For each such component γ, consider the component Γ_γ of

$$\{x\in\Delta\colon f(x)>0\}$$

containing γ. Put $\Gamma=\bigcup_\gamma \Gamma_\gamma$. If $x_0\in\bar\Gamma$, then there exists a sequence of points $\{x_i\}_{i=1}^\infty$ convergent to x_0 and a (possibly constant) sequence $\{\gamma_i\}_{i=1}^\infty$ of components of $\partial^+\Delta$ such that

$$x_i\in\Gamma_{\gamma_i}.$$

Then, the sequence of continua $\{\bar\Gamma_{\gamma_i}\}_{i=1}^\infty$ contains a subsequence convergent to some continuum Γ^* containing x_0 and intersecting $\operatorname{bd}\Delta$. Since $f(\bar\Gamma_{\gamma_i})\geqq 0$ for all i,

$$f(\Gamma^*)\geqq 0$$

holds too. Thus Σ is a quasisupporting sphere of S at x_0, which contradicts the choice of x_0 (Lemma 2). Hence $x_0\notin\bar\Gamma$ and we put $\Theta=\Delta-\Gamma$. Since $x_0\in\operatorname{int}\Theta$, where $\operatorname{int}\Theta$ means the relative interior of Θ, the choice of x_0 (Lemma 1) implies

$$f(\Theta)\not\subset(-\infty,0].$$

Let u be a point of Θ, where $g=f|_\Theta$ attains its absolute maximum. Then

$$g(x_0)=0,\quad g(\operatorname{bd}\Theta)\leqq 0$$

yield $u\neq x_0$, $u\notin\operatorname{bd}\Theta$. Hence the line through y and u is a normal of S, different from N.

We prove now that y is an interior point of M_k for $k<\aleph_0$.

Consider again the point u found above. The set $g^{-1}(g(u))$ is compact and does not intersect $\{x_0\}\cup\operatorname{bd}\Theta$. Let δ be the minimal distance (not Hausdorff!) from $\{x_0\}\cup\operatorname{bd}\Theta$ to $g^{-1}(g(u))$. Let Φ be the compact set of all points of Σ at (minimal) distance $\frac{\delta}{2}$ from $g^{-1}(g(u))$. Clearly $g(\Phi)<g(u)$; put

$$\eta=g(u)-\max g(\Phi).$$

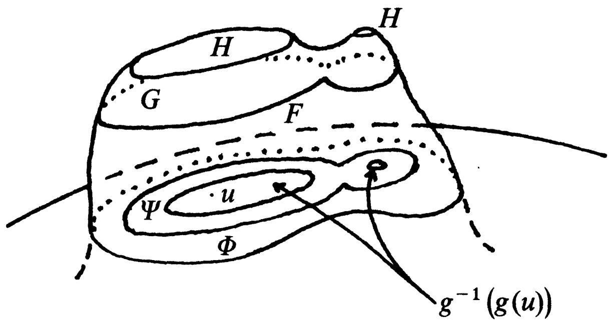

Figure 2

If Ψ denotes the compact set of all points of Σ at distance at most $\frac{\delta}{2}$ from $g^{-1}(g(u))$, then

$$\operatorname{bd}\Psi \subset \Phi.$$

Let G (respectively F, H) be the compact set of all points of D which are collinear with y and some point in Ψ (respectively Φ, $g^{-1}(g(u))$). Every point of the ball

$$B^* = \left\{ y' : \|y' - y\| \leqq \frac{\eta}{3} \right\}$$

is closer to any point of F than to any point of H. Hence, for each point y' of B^*, any furthest point v of G from y' lies outside F and thus belongs to $\operatorname{int} G$. It follows that the line through y' and v is normal to S.

By keeping the point y, but taking a new ε smaller than the above $\frac{\delta}{2}$, we get a new set Δ which is disjoint from the old set Ψ, hence a new G which is disjoint from the old G. This means that every point belonging both to the old and to the new B^* lies on two distinct normals to S. Repeating this procedure, we get a ball around y all points of which lie on k distinct normals of S.

Hence $y \in \operatorname{int} M_k$, $\complement M_k$ is nowhere dense, and the proof is complete.

References

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