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# CONVEX CURVES IN GEAR

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## Introduction

When looking at two wheels in gear we notice the essential property that no rotation of one of them can be performed without rotating the other. But it is equally obvious that they are not too convex. Can we construct convex wheels that are in gear? This paper answers (locally) this question, but has no ambition to have any technical relevance!

Let us consider two Jordan (closed) curves $C_1$, $C_2$ in the half-plane $\mathbf{R}\times\mathbf{R}_+$, both containing the origin $\mathbf{O}=(0, 0)$. ($\mathbf{R}_+=[0, \infty)$, $\mathbf{R}_-=(-\infty, 0]$.) Suppose $C_1-\{\mathbf{O}\}$ lies in the bounded domain with frontier $C_2$. Such ordered pairs of curves $(C_1, C_2)$ will shortly be called *supporting curves* ($C_2$ *supports* $C_1$). According to intuitive evidence we say that (locally at **O**) $C_i$ *can rotate around* $(0, a)$ if there is a neighbourhood of **O** in $\mathbf{R}_-\times\mathbf{R}_+$ or in $\mathbf{R}_+\times\mathbf{R}_+$ within which $C_j$ meets the circle with centre $(0, a)$ and radius $a$ only in **O** ($\{i, j\}=\{1, 2\}$). (In fact the intuition would impose here a stronger condition, which however turns out to be superfluous for all later purposes.) Consequently, we say that $C_1$ and $C_2$ are *in gear with respect to* $(0, a_1)$ *and* $(0, a_2)$ $(a_1\leqq a_2)$ if $C_i$ cannot rotate around $(0, a_i)$ $(i\in\{1, 2\})$ and that $C_1$ and $C_2$ are *in perfect gear* if they are in gear with respect to any pair of (correctly ordered) points on the positive $y$-axis.

If the curvatures of $C_1$ and $C_2$ at **O** exist and are different, then $C_1$ and $C_2$ are not in gear with respect to any pair of points. If it happens that $C_1$ and $C_2$ have a common centre $c$ of curvature at **O**, then they possibly are in gear with respect to $c$ and $c$ only.

Our main result is that, in a certain sense, in general two supporting convex curves are in perfect gear.

## Circles tangent to convex curves

In a Baire space, "most" means "all, except those in a set of first category", i.e. "those in a residual set".

Let $\mathcal{K}$ be the space of all (closed) convex curves in $\mathbf{R}^2$ and $\mathcal{C}$ the subspace of all those lying in $\mathbf{R}\times\mathbf{R}_+$ and containing **O**. It is easily seen that both of them, endowed with Hausdorff's metric, are Baire spaces. It is not difficult to modify the proof of Klee's result [3] (see also Gruber [2]), which says that most curves in $\mathcal{K}$ are differentiable and strictly convex, to demonstrate that most curves in $\mathcal{C}$ have the same properties.

We shall consider the lower and upper radii of curvature $\varrho_i^{\pm}(p)$ and $\varrho_s^{\pm}(p)$ at an arbitrary point $p$ on $C$, in both directions (for a definition see [1], p. 14). If all four of them are equal we write $\varrho(p)$ for the common value. About $\mathscr{K}$ we know the following: On one hand [5]

*for most curves in* $\mathscr{K}$, $\varrho(p) = \infty$ *a.e.*

On the other hand [6]

*for most curves in* $\mathscr{K}$, $\varrho_i^{\pm}(p) = 0$ *and* $\varrho_s^{\pm}(p) = \infty$ *at most points* $p$.

How behaves $C$ at $\mathbf{O}$ for most curves $C \in \mathscr{C}$? From a probabilistic point of view it can be expected that the bahaviour at $\mathbf{O}$ is of the first kind, while from a topological point of view the second seems more plausible.

THEOREM 1. *For most curves in* $\mathscr{C}$, $\varrho_i^{\pm}(\mathbf{O})=0$ *and* $\varrho_s^{\pm}(\mathbf{O})=\infty$.

PROOF. Let

$$\mathscr{A} = \{C \in \mathscr{C}: \varrho_i^{+}(\mathbf{O}) = \varrho_i^{-}(\mathbf{O}) = 0\}.$$

Denote by $\mathscr{C}^*$ the set of all curves in $\mathscr{C}$ supporting some convex curve $\Gamma$ composed of a semicircle and a segment of the $y$-axis. Let $\mathscr{C}_n$ be the subset of all curves in $\mathscr{C}$ for which $\Gamma$ has diameter $2n^{-1}$.

It is rather obvious that $\mathscr{C}^*$ is the complement of $\mathscr{A}$ and that $\mathscr{C}^* = \bigcup_n \mathscr{C}_n$. We show that $\mathscr{C}_n$ is nowhere dense in $\mathscr{C}$, for every $n$. Since $\mathscr{C}_n$ is easily seen to be closed, it suffices to find in its complement a set dense in $\mathscr{C}$. This is provided by the family of all convex polygons in $\mathbf{R} \times \mathbf{R}_+$ which have $\mathbf{O}$ as a vertex and have no edge on the $x$-axis. Thus, $\mathscr{A}$ is residual.

In an analogous manner one shows that the set

$$\mathscr{B} = \{C \in \mathscr{C}: \varrho_s^{+}(\mathbf{O}) = \varrho_s^{-}(\mathbf{O}) = \infty\}$$

is residual, the set of convex curves dense in $\mathscr{C}$ being this time the family of all convex polygons in $\mathbf{R} \times \mathbf{R}_+$ having $\mathbf{O}$ as interior point of an edge.

Hence $\mathscr{A} \cap \mathscr{B}$ is residual, which proves the theorem.

As a consequence we get the following result from [7]. Denote by $\mathscr{C}^0$ the subspace of $\mathscr{C}$ all the elements of which are circles. $\mathscr{C} \times \mathscr{C}^0$ is then obviously a Baire space.

COROLLARY. *Most pairs of curves in* $\mathscr{C} \times \mathscr{C}^0$ *intersect each other in every neighbourhood of* $\mathbf{O}$ *at some point different from* $\mathbf{O}$.

PROOF. Consider two topological spaces $\mathscr{X}$ and $\mathscr{Y}$, where $\mathscr{Y}$ has a countable basis, and two sets $A \subset \mathscr{X}$, $B \subset \mathscr{Y}$. We use the following known result ([4], Theorem 15.3):

$A \times B$ *is of first category in* $\mathscr{X} \times \mathscr{Y}$ *if and only if* $A$ *or* $B$ *is of first category.*

In our case, we know by Theorem 1 that most curves in $\mathscr{C}$ intersect every circle from $\mathscr{C}^0$ in a point different from $\mathbf{O}$ and as close to $\mathbf{O}$ as we want. Then, the preceding result (with $B = \mathscr{Y} = \mathscr{C}^0$) yields the Corollary.

## Convex curves in perfect gear

We easily see that the set $\mathscr{D}\subset\mathscr{C}^2$ of all pairs of supporting convex curves is a Baire space. Let $(C_1, C_2)\in\mathscr{D}$.

Let now $\varrho_i^\pm(C_k)$ and $\varrho_s^\pm(C_k)$ denote $\varrho_i^\pm(\mathbf{O})$ and $\varrho_s^\pm(\mathbf{O})$ calculated for the curve $C_k$. We observe the following: If

$$\varrho_i^-(C_2) < a_1, \quad \varrho_i^+(C_2) < a_1$$

and

$$\varrho_s^-(C_1) > a_2, \quad \varrho_s^+(C_1) > a_2,$$

then $C_1$ and $C_2$ are in gear with respect to $(0, a_1)$ and $(0, a_2)$.

Thus, if

$$\varrho_i^\pm(C_2) = 0 \quad\text{and}\quad \varrho_s^\pm(C_1) = \infty,$$

then $C_1$ and $C_2$ are in perfect gear. We also remark that the converse holds too.

THEOREM 2. *Most pairs of supporting convex curves are in perfect gear.*

PROOF. We prove that for most pairs of curves $(C_1, C_2)\in\mathscr{D}$,

$$\varrho_i^\pm(C_2) = 0 \quad\text{and}\quad \varrho_s^\pm(C_1) = \infty.$$

The argument parallels that of Theorem 1:

Let

$$\mathscr{A} = \{(C_1, C_2)\in\mathscr{D}:\ \varrho_i^+(C_2) = \varrho_i^-(C_2) = 0\}.$$

Let $\mathscr{D}^*$ be the set of all pairs $(C_1, C_2)\in\mathscr{D}$, such that $C_2$ supports some convex curve $\Gamma$ composed by a semicircle and a segment of the $y$-axis. Let $\mathscr{D}_n$ be the subset of all pairs in $\mathscr{D}^*$ for which $\Gamma$ has diameter $2n^{-1}$. $\mathscr{D}_n$ is obviously closed. The family of all pairs of supporting convex polygons, both admitting $\mathbf{O}$ as a vertex and having no edge on the $x$-axis, is dense in $\mathscr{D}$. Hence

$$\mathscr{A} = \mathscr{D} - \bigcup_{n=1}^{\infty} \mathscr{D}_n$$

is residual. Analogously,

$$\mathscr{B} = \{(C_1, C_2)\in\mathscr{D}:\ \varrho_s^+(C_1) = \varrho_s^-(C_1) = \infty\}$$

is residual and most pairs of $\mathscr{D}$ belong to $\mathscr{A}\cap\mathscr{B}$.

Theorems 1 and 2 extend in an obvious way to higher dimensions and the proofs present no difficulty. Also, one can consider pairs of supporting convex surfaces of different dimensions; it is still true that most such pairs of surfaces are in perfect gear.

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