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Typical starshaped sets

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Dedicated to Professor Otto Haupt with best wishes on his 100th birthday.

Starting with V. Klee's paper [2] from 1959, several generic results on convex sets (see the expository article [4]) have been obtained. We speak about "typical" or "most" members of a Baire space, if those members not considered form a set of first category. Results on typical elements are usually called "generic". We shall obtain here several generic results on starshaped and n-starshaped sets, which are closely related to convex sets.

The space $\mathscr{T}$ of all compact starshaped sets in $\mathbb{R}^d$, endowed with the Hausdorff metric δ, being closed in the space of all compact sets, is a Baire space.

In the first section we describe most sets in $\mathscr{T}$, in the second we investigate typical intersections of sets in $\mathscr{T}$, and in the third we introduce and investigate typical n-starshaped sets, which generalize the common notion of a starshaped set.

If we impose the condition that the kernel of the starshaped sets must include a given convex body, we get a subspace of $\mathscr{T}$ which is again a Baire space, being closed in $\mathscr{T}$. The typical members of this Baire space are studied in [5].

We shall use the following notations. Let S_{d-1} be the boundary of the unit ball $B \subset \mathbb{R}^d$. For any 2-dimensional flat F, let p_F denote the orthogonal projection on F. Also, for any set $S \subset \mathbb{R}^d$, points $u, v, w \in \mathbb{R}^d$ and number $r \in \mathbb{R}$, let conv S be the convex hull of S, diam S the diameter of S, $rS = \{rs : s \in S\}$, $u + S = \{u + s : s \in S\}$, $uv = \operatorname{conv}\{u, v\}$ and $uvw = uv \cup vw$.

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1. On most starshaped sets.

THEOREM 1. *For most compact starshaped sets, their orthogonal projection on any* 2-*dimensional flat is nowhere dense.*

Proof. Let $\mathscr{T}_n$ be the family of all sets $T \in \mathscr{T}$ such that, for some 2-dimensional flat F, $p_F T$ includes a disk of radius n^{-1}. We show that $\mathscr{T}_n$ is nowhere dense in $\mathscr{T}$. Let $\mathscr{O}$ be open in $\mathscr{T}$. Choose $T \in \mathscr{O}$ and $\varepsilon > 0$. Let G_ε be the set of all points in $\varepsilon\mathbb{Z}^d$ at distance at most ε from T. Clearly, $\delta(T, G_\varepsilon) \leqslant \varepsilon$. Choose a point x in the kernel of T. The union G'_ε of all segments joining x with points in G_ε also satisfies $\delta(T, G'_\varepsilon) \leqslant \varepsilon$. For ε small enough, $G'_\varepsilon \in \mathscr{O}$. Let

$$\xi = \sum_{y \in G_\varepsilon} \|x - y\|.$$

Consider a number $\alpha > 0$ and the set $H_\alpha = G'_\varepsilon + \alpha B$. For

$$\alpha < \pi n^{-2}(2\xi + \pi \operatorname{card} G_\varepsilon)^{-1}$$

and for every 2-dimensional flat F, the area of $p_F H_\alpha$ is less than πn^{-2}, and therefore does not contain any disk of radius n^{-1}. For every set $P \in \mathscr{T}$ with $\delta(G'_\varepsilon, P) \leqslant \alpha$, $P \subset H_\alpha$. Hence, for α small enough, every such set P lies in $\mathscr{O}$ and $p_F P$ includes no disk of radius n^{-1}, for arbitrary F.

It follows that most sets $T \in \mathscr{T}$ do not belong to any $\mathscr{T}_n$, which means that $p_F T$ includes no disk, *i.e.*, $p_F T$ is nowhere dense in F, for any 2-dimensional flat F.

COROLLARY. *Most sets belonging to $\mathscr{T}$ are nowhere dense and their kernels consist of precisely one point.*

The first assertion is a trivial consequence of Theorem 1. To prove the second, observe that the family of all segments is nowhere dense in $\mathscr{T}$. Suppose now x and y belong to the kernel of a typical $T \in \mathscr{T}$ and $x \neq y$. Then T is not a segment, so there exists $z \in T$ not on the line through x and y, $\operatorname{conv}\{x, y, z\} \subset T$ and the orthogonal projection of T on the 2-dimensional flat determined by x, y, z includes a disk, which contradicts Theorem 1. This proves the corollary.

By the Corollary, a typical starshaped set T has a single point k in its kernel. Thus, T is a union of line segments meeting each other only at k and joining k with points forming a set $Q(T)$. Thus, $Q(T) = \cap\{Q^* : T = \cup\{xk : x \in Q^*\}\}$. Let $\mathscr{T}'$ be the residual space of all starshaped sets with single point kernels.

Let, for any $T \in \mathscr{T}'$,

$$V(T) = \left\{\frac{1}{\|x - k\|}(x - k) : x \in Q(T)\right\},$$

$$U(T) = \{\|x - k\| : x \in Q(T)\},$$

$$s(T) = \max U(T).$$

It is interesting that, in spite of the fact that most starshaped sets are nowhere dense, the following two theorems hold.

THEOREM 2. *For most sets $T \in \mathscr{T}'$, $V(T)$ is dense in S_{d-1}.*

Proof. Let $\mathscr{T}_n$ be the family of those sets $T \in \mathscr{T}'$ such that $S_{d-1} - V(T)$ includes an open disk of (angular) radius n^{-1}. It is obviously enough to prove that $\mathscr{T}_n$ is nowhere dense in $\mathscr{T}$. First we show that the set $\mathscr{T}_n$ is closed. Suppose, indeed, we have a sequence $\{T_i\}_{i=1}^{\infty}$ with $T_i \in \mathscr{T}_n$, convergent to some $T \in \mathscr{T}'$.

Then the kernels $\{k_i\}$ of T_i form a sequence convergent to the kernel $\{k\}$ of T. Every $S_{d-1} - V(T_i)$ includes an open disk D_i of radius n^{-1}. Suppose the sequence $\{\overline{D_i}\}_{i=1}^{\infty}$ converges to a closed disk $D \subset S_{d-1}$ of radius n^{-1} (otherwise take a subsequence). If for some $y \in Q(T)$, $\|y - k\|^{-1}(y - k) \in \operatorname{int} D$, then some sequence $\{y_i\}_{i=1}^{\infty}$ with $y_i \in Q(T_i)$ must converge to y; but this implies

$$\|y_i - k_i\|^{-1}(y_i - k_i) \to \|y - k\|^{-1}(y - k),$$

which contradicts $\overline{D_i} \to D$. Thus $T \in \mathscr{T}_n$. Hence $\mathscr{T}_n$ is closed and it only remains to be shown that $\mathscr{T}' - \mathscr{T}_n$ is dense. Let $\mathcal{O}$ be open in $\mathscr{T}$. We choose a set $T' \in \mathcal{O}$ with a single point kernel $\{k'\}$, such that $Q(T')$ is finite (as in the proof of Theorem 1). We can add to T' several segments joining k' with the points of a finite set such that

1) the new set T'' still lies in $\mathcal{O}$ – the new segments are chosen short enough – and
2) $S_{d-1} - V(T'')$ includes no disk of radius n^{-1}.

This proves that $\mathscr{T}_n$ is nowhere dense in $\mathscr{T}$ and the theorem follows.

THEOREM 3. *For most sets $T \in \mathscr{T}'$, $U(T)$ is dense in $[0, s(T)]$.*

Proof. Let $\mathscr{T}_n$ be the family of those sets $T \in \mathscr{T}'$ for which $U(T) \cap I = \emptyset$, I being some open interval of length n^{-1} contained in $[0, s(T)]$. To prove the theorem it obviously suffices to show that $\mathscr{T}_n$ is nowhere dense in $\mathscr{T}'$. We first prove that $\mathscr{T}_n$ is closed. Let, indeed, $T_i \to T$ with $T_i \in \mathscr{T}_n$ and $T \in \mathscr{T}'$. Clearly, if $\{k_i\}$ is the kernel of T_i and $\{k\}$ the kernel of T, then $k_i \to k$. Also, $s(T_i) \to s(T)$. We may suppose (otherwise we take an appropriate subsequence) that the open intervals $I_i \subset [0, s(T_i)]$ such that

$U(T_i) \cap I_i = \emptyset$ satisfy $\overline{I_i} \to I$ for some closed interval $I \subset [0, s(T)]$. If, for some $x \in Q(T)$, $\|x - k\| \in \text{int}\, I$, then some sequence $\{x_i\}_{i=1}^{\infty}$ with $x_i \in Q(T_i)$ must converge to x. This yields

$$\|x_i - k_i\| \to \|x - k\|,$$

which contradicts $\overline{I_i} \to I$. Thus $T \in \mathcal{T}_n$, whence $\mathcal{T}_n$ is closed and we only have to show that $\mathcal{T}' - \mathcal{T}_n$ is dense. Let $\mathcal{O}$ be open in $\mathcal{T}'$. We choose a set $T' \in \mathcal{O}$, such that $V(T')$ is finite. Let j be the middle point of an arbitrary interval I' of length n^{-1} included in $[0, s(T')]$ and let $x_0 \in Q(T')$ be such that $\|x_0 - k\| = s(T')$. We consider a point x_1 at distance $\alpha > 0$ from the segment joining k to x_0, such that $\|x_1 - k\| = j$ and the line through k and x_1 does not meet $Q(T')$. Let T'' be the starshaped set with kernel $\{k\}$ and $Q(T'') = Q(T') \cup \{x_1\}$. For α small enough, $T'' \in \mathcal{O}$. Since there exist finitely many intervals of length n^{-1} covering $[0, s(T')]$, we may repeat the procedure finitely many times and eventually get a set $T \in \mathcal{O} - \mathcal{T}_n$.

THEOREM 4. *Most starshaped sets are not locally connected at any point outside the kernel.*

Proof. Let $\mathcal{T}''$ be the family of all starshaped sets $T \in \mathcal{T}'$ which are locally connected at some point $x_T \neq k_T$, $\{k_T\}$ being the kernel of T. Thus $\mathcal{T}'' = \bigcup_{n=2}^{\infty} \mathcal{T}''_n$, where

$$\mathcal{T}''_n = \{T \in \mathcal{T}'' : \exists x_T \text{ with } \|x_T - k_T\| \geq n^{-1}\}.$$

To prove the theorem it suffices to show, for fixed but arbitrary n, that $\mathcal{T}''_n$ is of first category.

Each $T \in \mathcal{T}''_n$ being locally connected at x_T, there exists an $\varepsilon > 0$ such that for any pair of points

$$y, z \in (x_T + \varepsilon B) \cap T,$$

there exists a continuum $C \subset x_T + n^{-2}B$ containing y and z. Thus $\mathcal{T}''_n \subset \bigcup_{m=1}^{\infty} \mathcal{T}_m$, where

$$\mathcal{T}_m = \{T \in \mathcal{T}''_n : \forall\, y, z \in (x_T + m^{-1}B) \cap T, \exists C \subset x_T + n^{-2}B \text{ with } y, z \in C\}.$$

We show that $\mathcal{T}_m$ is nowhere dense in $\mathcal{T}'$. First let us prove that it is closed in $\mathcal{T}'$. Let $T_i \to T$ with $T_i \in \mathcal{T}_m$ and $T \in \mathcal{T}'$. Then $k_{T_i} \to k_T$. We may also suppose that, if x_{T_i} is the point at distance at least n^{-1} from k_{T_i} where T_i is locally connected, then $\{x_{T_i}\}_{i=1}^{\infty}$

converges (otherwise consider a subsequence) to some point $x \in T$, for which, then, $\|x - k\| \geqslant n^{-1}$ as well. For any choice of y, z in $(x + m^{-1}B) \cap T$, there are two sequences $\{y_i\}_{i=1}^{\infty}$, $\{z_i\}_{i=1}^{\infty}$ converging to y, z such that

$$y_i, z_i \in (x_{T_i} + m^{-1}B) \cap T_i,$$

because

$$(x_{T_i} + m^{-1}B) \cap T_i \to (x + m^{-1}B) \cap T.$$

Now, $T_i \in \mathscr{T}_m$ implies the existence of the continuum $C_i \subset x_{T_i} + n^{-2}B$ containing y_i and z_i. Again we may suppose (see [3], §38. I, 1, p. 21 and §42. II, 4, p. 110) that $\{C_i\}_{i=1}^{\infty}$ converges (otherwise take a subsequence) to some continuum C.

Clearly $y, z \in C$ and $C \subset x + n^{-2}B$. Thus $\mathscr{T}_m$ is closed.

To finish the proof it suffices to show that $\mathscr{T}' - \mathscr{T}_m$ is dense.

Let $\mathcal{O} \subset \mathscr{T}'$ and $T' \in \mathcal{O}$ be such that $V(T')$ is finite. If $s(T') < n^{-1}$, then $T' \notin \mathscr{T}_m$. If $s(T') \geqslant n^{-1}$, for every $x \in Q(T')$, consider a point x' satisfying $\|x - x'\| < m^{-1}$ and $\|x - k_{T'}\| = \|x' - k_{T'}\|$ (see Figure 1). Let Q' be the set of all these points x'. We claim that the starshaped set T'' with $k_{T''} = k_{T'}$ and $Q(T'') = Q(T') \cup Q'$, which lies in $\mathcal{O}$ if all $\|x - x'\|$ are small enough, does not belong to $\mathscr{T}_m$. Indeed, let $x_0 \in T''$ with $\|x_0 - k_{T'}\| \geqslant n^{-1}$. Then there are $x, x' \in Q(T'')$ such that $x_0 \in xk_{T'}x'$. Choose $y \in (x_0 + m^{-1}B) \cap xk_{T'}$ and $z \in (x_0 + m^{-1}B) \cap x'k_{T'}$. Clearly, every continuum containing y and z must pass through $k_{T'} \notin x_0 + n^{-2}B$.

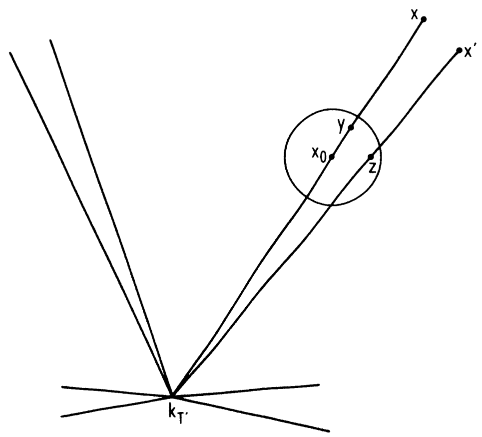

Figure 1

Thus, $T'' \notin \mathscr{T}_m$, $\mathscr{T}' - \mathscr{T}_m$ is dense, and the proof is finished.

2. Typical intersections of starshaped sets.

We consider in this section the Baire space ϑ of all pairs of starshaped sets in $\mathbb{R}^d$ which meet in at least one point. It is easily seen that most pairs in ϑ consist of starshaped sets having the same properties as most sets in $\mathscr{T}$. The reason for including this short section is the following not difficult but interesting result.

THEOREM 5. *For most pairs* $(T_1, T_2) \in \vartheta$, $T_1 \cap T_2$ *is infinite for* $d = 2$ *and a single point for* $d \geqslant 3$.

Proof. Suppose first $d \geqslant 3$. Let ϑ_n be the set of all pairs $(T_1, T_2) \in \vartheta$ such that diam $(T_1 \cap T_2) \geqslant n^{-1}$. We prove that ϑ_n is nowhere dense. Let $\mathcal{O}$ be open in ϑ. We construct, as in the proof of Theorem 1, the starshaped sets T_1' and T_2' with $(T_1', T_2') \in \mathcal{O}$, both finite unions of segments. Of course we can arrange that

1) the kernels do not coincide,
2) precisely one (by inclusion) maximal segment in T_1' and one maximal segment in T_2' meet,
3) their intersection is a single point.

Thus, if

$$U_i(\varepsilon) = T_i' + \varepsilon B \quad (i = 1, 2),$$

then, for ε small enough, $U_1(\varepsilon) \cap U_2(\varepsilon)$ is connected and

$$\lim_{\varepsilon \to 0} \operatorname{diam} (U_1(\varepsilon) \cap U_2(\varepsilon)) = 0.$$

Let ε be such that

1) diam $(U_1(\varepsilon) \cap U_2(\varepsilon)) < n^{-1}$ and
2) $(V_1, V_2) \in \vartheta$, $\delta(V_1, T_1') < \varepsilon$ and $\delta(V_2, T_2') < \varepsilon$ imply $(V_1, V_2) \in \mathcal{O}$.

Now choose $(V_1, V_2) \in \vartheta$ such that $\delta(V_i, T_i') < \varepsilon$ $(i = 1, 2)$. Then

$$V_1 \cap V_2 \subset U_1(\varepsilon) \cap U_2(\varepsilon),$$

whence

$$\operatorname{diam}(V_1 \cap V_2) < n^{-1}.$$

It follows that ϑ_n is nowhere dense in ϑ; therefore, for most pairs $(T_1, T_2) \in \vartheta$,

$$\operatorname{diam}(T_1 \cap T_2) = 0.$$

Let now $d = 2$. This time we define ϑ_n in the following way: it consists of all pairs $(T_1, T_2) \in \vartheta$ such that card $(T_1 \cap T_2) \leqslant n$. Let $\mathcal{O}$ be open in ϑ. We construct as before the finite unions of segments T_1' and T_2' such that $(T_1', T_2') \in \mathcal{O}$. At least one segment s_1 of T_1' meets at least one segment s_2 of T_2' and it can be arranged that s_1 and s_2 do not lie on a line. In any neighbourhood of s_1 we can add $n + 1$ segments to T_1' originating at the kernel of T_1' and crossing s_2, such that the new starshaped set T_1'' still satisfies $(T_1'', T_2') \in \mathcal{O}$. Now it is easily seen that, for ε small enough, each pair (T_1, T_2) with $\delta(T_1, T_1'') < \varepsilon$ and $\delta(T_2, T_2') < \varepsilon$ has the property that

$$\operatorname{card}(T_1 \cap T_2) \geqslant n + 1.$$

By choosing ε such that all above pairs (T_1, T_2) belong to $\mathcal{O}$, we see that ϑ_n is nowhere dense. It follows that most pairs in ϑ do not belong to $\bigcup_{n=1}^{\infty} \vartheta_n$, *i.e.* they meet at infinitely many points.

This Theorem 5 was already used in Section 3 of [1].

3. On most n-starshaped sets

A set $M \subset \mathbb{R}^d$ is said to be *n-starshaped* if its kernel

$$K(M) = \{x \in M : \forall\, y \in M, \exists\, W \in \mathscr{P}_{x,y} \text{ with } W \subset M\}$$

is not empty, $\mathscr{P}_{x,y}$ being the family of all polygonal lines homeomorphic to [0, 1], joining x with y and consisting of at most n line segments.

It can be easily seen that, equipped with the Hausdorff metric δ, the space $\mathscr{T}^{(n)}$ of all n-starshaped compact sets in $\mathbb{R}^d$ is a Baire space.

In contrast to starshaped sets, the n-starshaped sets with $n \geqslant 2$ need not be simply connected. The following question naturally arises: Are typical n-starshaped sets simply connected or not? From now on, $n \geqslant 2$.

THEOREM 6. *For $d \geqslant 3$, most sets belonging to $\mathscr{T}^{(n)}$ include no Jordan closed curve.*

Proof. Let

$$x_0, x_1, \ldots, x_r \in \mathbb{R}^d, \quad W_i \in \mathscr{P}_{x_0, x_i} \quad (i = 1, \ldots, r)$$

such that $W_i \cap W_j = \{x_0\}$ for $i \neq j$ and let $\varepsilon > 0$ be such that

$$A = \bigcup_{i=1}^{r} W_i + \varepsilon B$$

is simply connected, the natural number $r \geqslant 2$ being not fixed. We call A a *spider* of *head* x_0 and *breadth*

$$\operatorname{br} A := \inf\{\xi : (W_i + \varepsilon B) \cap (W_j + \varepsilon B) \subset x_0 + \frac{1}{2}\xi B \quad (i \neq j)\}.$$

Notice that different spiders may have different numbers r of "arms".

For each bounded set $T \subset \mathbb{R}^d$, let

$$\lambda(T) = \inf\{\operatorname{br} A : \text{the spider } A \text{ includes } T\}.$$

Obviously, λ is an increasing mapping, *i.e.* $T_1 \subset T_2$ yields $\lambda(T_1) \leqslant \lambda(T_2)$.

We show that for each Jordan (closed) curve J, $\lambda(J) > 0$. Suppose, on the contrary, there exists a Jordan curve J with $\lambda(J) = 0$. Then we get a sequence $\{A_u\}_{u=1}^{\infty}$ of spiders including J, such that $\operatorname{br} A_u \to 0$. By taking a subsequence if necessary, we arrange that the sequence $\{c_u\}_{u=1}^{\infty}$ of heads of spiders converges to some point c of the projective space $\mathbb{P}^d \supset \mathbb{R}^d$ (see Figure 2). Let a, b be furthest points of J. Suppose, for example, $a \neq c$. For each index u, a belongs to a certain set $W_u + (\operatorname{br} A_u)B$, W_u being a polygonal line with at most n line-segments lying in $\mathbb{R}^d$ and having an endpoint at c_u. A subsequence of $\{W_u\}_{u=1}^{\infty}$ converges to a polygonal line $W \subset \mathbb{P}^d$ containing a and c and having possibly self-intersections. Let w be in $W \cap \mathbb{R}^d$ between a and c if $b \notin W$ and between a and b if $b \in W$; the order a, w, c on W means that, for large u, there are points $a_u, w_u, c_u \in W_u$ near a, w, c, in this order on W_u. Then, clearly, for this large u, there must exist two distinct points $e_u{}^1, e_u{}^2$ on the two arcs J^1, J^2 of J joining a to b, both near w, such that

$$\|e_u{}^i - w_u\| \leqslant \operatorname{br} A_u \quad (i = 1, 2).$$

This implies $e_u{}^i \to w$, whence w belongs to both J^1 and J^2, which is impossible.

Let $\mathscr{T}_m^{(n)}$ be the set of all $P \in \mathscr{T}^{(n)}$ with $\lambda(P) \geqslant m^{-1}$. We show that $\mathscr{T}_m^{(n)}$ is nowhere dense in $\mathscr{T}^{(n)}$. Let $\mathscr{O}$ be open in $\mathscr{T}^{(n)}$, consider $P \in \mathscr{O}$ and let $\varepsilon \in (0, m^{-1})$ be such that $P' \in \mathscr{T}^{(n)}$ and $\delta(P, P') < 4\varepsilon d$ imply $P' \in \mathscr{O}$. Let

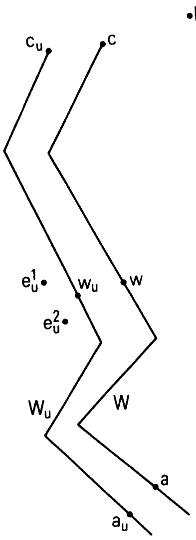

Figure 2

$$Z \subset \varepsilon\mathbb{Z}^d$$

be so that $\delta(P, Z) < \varepsilon d$. Fix $q \in K(P)$ and let $q(Z) \in Z$ have distance at most εd from q. For each point $p \in P$ there exists $W \in \mathscr{P}_{p,q}$ with $W \subset P$. Let $p, p_1, p_2, \ldots, p_j, q$ be the segment-endpoints on W ($j \leqslant n - 1$). Let

$$p(Z), p_1(Z), p_2(Z), \ldots, p_j(Z) \in Z$$

have distance at most εd from $p, p_1, p_2, \ldots, p_j$. Thus, $p(Z)$, $p_1(Z)$, $p_2(Z), \ldots, p_j(Z)$, $q(Z)$ determine a polygonal line W_p in $\mathscr{P}_{p(Z),q(Z)}$. Consider, for every $p(Z)$, the polygonal line

$$W(p) \in \mathscr{P}_{p(Z),q(Z)}$$

having exactly n pairwise non-collinear segments, such that

$$\delta(W_p, W(p)) \leqslant \varepsilon d$$

and such that for every pair of distinct points $p'(Z)$, $p''(Z)$,

$$W(p') \cap W(p'') = \{q(Z)\}.$$

This is possible if $d \geqslant 3$. Then

$$Z' = \bigcup_{p \in P} W(p)$$

satisfies

$$\delta(Z, Z') < 2\varepsilon d.$$

The set Z' is the union of, say, k polygonal lines having pairwise only the point $q(Z)$ in common. Let $v > 0$ be such that $Z' + vB$ is a spider of breadth less than ε. Clearly, for each set $J \subset Z' + vB$, $\lambda(J) < m^{-1}$. If $P' \in \mathscr{T}^n$ is such that

$$\delta(Z', P') < v,$$

then, on one hand,

$$\delta(P, P') \leqslant \delta(P, Z) + \delta(Z, Z') + \delta(Z', P') < 4\varepsilon d$$

and thus $P' \in \mathcal{O}$, and on the other hand $P' \subset Z' + vB$, whence $\lambda(P') < m^{-1}$ and $P' \notin \mathscr{T}_m^{(n)}$. It follows that $\mathscr{T}_m^{(n)}$ is nowhere dense in $\mathscr{T}^{(n)}$. Hence, for most $P \in \mathscr{T}^{(n)}$, $\lambda(P) = 0$.

Since, for each Jordan curve J, $\lambda(J) > 0$, most sets in $\mathscr{T}^{(n)}$ do not include any Jordan curve. The theorem is proved.

COROLLARY. *For $d \geqslant 3$, most compact n-starshaped sets are nowhere dense and simply connected.*

THEOREM 7. *For $d = 2$, most compact n-starshaped sets are nowhere dense, but not simply connected. In fact only those in a nowhere dense subset of $\mathscr{T}^{(n)}$ are simply connected.*

Proof. The proof of the fact that most n-starshaped sets are nowhere dense is similar to (and simpler than) the proof of Theorem 8 in the next section.

We shall show that the family $\mathscr{T}'$ of all simply connected n-starshaped sets is nowhere dense in $\mathscr{T}^{(n)}$.

We saw in the second part of the proof of Theorem 6 that in any open set $\mathcal{O} \subset \mathscr{T}^{(n)}$ we find an element T which is a finite union of polygonal lines with at most n line-segments and with a common endpoint $q \in K(T)$ (see Figure 3). Let $\varepsilon > 0$ be such that $U \in \mathscr{T}^{(n)}$ and $\delta(T, U) < \varepsilon$ imply $U \in \mathcal{O}$. Now let $q_1, q_2 \in \mathbb{R}^2$ and $W_i \in \mathscr{P}_{q,q_i}$ $(i = 1, 2)$ be such that

(i) diam $W_i < \varepsilon/2$ $(i = 1, 2)$,
(ii) W_i is not a union of $n - 1$ line segments $(i = 1, 2)$,
(iii) $W_i \cap T = \{q\}$ $(i = 1, 2)$,

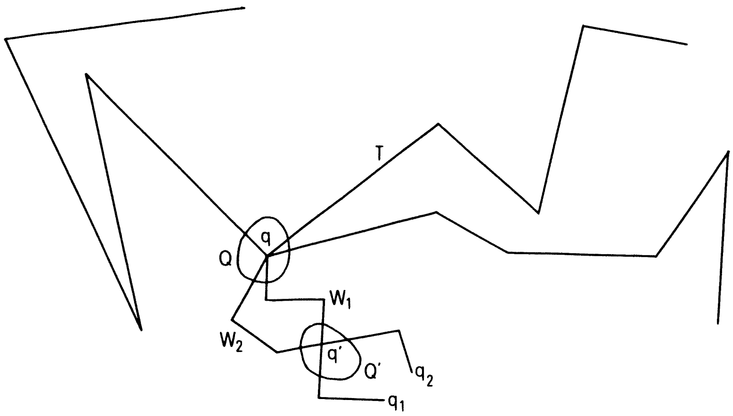

Figure 3

(iv) $\exists q' \neq q$ such that $W_1 \cap W_2 = \{q, q'\}$,
(v) q' is not a vertex of W_1 or W_2.

Let Q, Q' be disjoint neighbourhoods of q, q' respectively and let $\nu \in (0, \varepsilon/2)$ be such that for any polygonal lines W_i' with at most n line-segments satisfying $\delta(W_i, W_i') \leqslant \nu$ $(i = 1, 2)$, we have

$$W_1' \cap W_2' \subset Q \cup Q',$$
$$W_1' \cap W_2' \cap Q' \neq \emptyset,$$

and

$$W_i' \cap T \subset Q \quad (i = 1, 2).$$

Put $T' = T \cup W_1 \cup W_2$. Clearly $K(T') = \{q\}$. Let $T'' \in \mathcal{T}$ have distance at most ν from T'. Then T'' must include two polygonal lines meeting exactly twice: once in Q and a second time in Q'. This shows that $T'' \notin \mathcal{T}'$. But $T'' \in \mathcal{O}$ since

$$\delta(T'', T) \leqslant \delta(T'', T') + \delta(T', T) < \nu + \varepsilon/2 < \varepsilon.$$

This proves the theorem.

4. Typical n-starshaped sets with thick kernels.

If we impose the condition that $K(M)$ contains a given convex body, for instance the unit ball B, then the space $\mathcal{T}_B^{(n)}$ of all such n-starshaped sets $M \subset \mathbb{R}^d$ $(d \geqslant 2)$ is again a Baire space. While every starshaped set ($n = 1$) is topologically a ball, we shall prove here that most n-starshaped sets ($n \geqslant 2$) in $\mathscr{T}_B^{(n)}$ have a quite different shape, for any dimension $d \geqslant 2$.

THEOREM 8. *Most sets belonging to* $\mathscr{T}_B^{(n)}$ *are nowhere dense outside* B.

Proof. Let $b \notin B$. We show that the set $\mathscr{B}$ of those $P \in \mathscr{T}_B^{(n)}$ which contain b is nowhere dense.

Let $\mathscr{O} \subset \mathscr{T}_B^{(n)}$ be open. Take $P \in \mathscr{O}$. Let $\varepsilon > 0$ be such that $\delta(P, P') < 4\varepsilon d$ and $P' \in \mathscr{T}_B^{(n)}$ imply $P' \in \mathscr{O}$. Let $y \in B$ and

$$Z \subset y + \varepsilon \mathbb{Z}^d$$

be such that $\delta(P, Z) < \varepsilon d$ and no pair of points in Z are collinear with b.

For each pair of points $p \in P$, $q \in B$, there exists $W \in \mathscr{P}_{p,q}$ with $W \subset P$. Let $p, p_1, p_2, \ldots, p_j, q$ be the segment-endpoints on W ($j \leqslant n - 1$). Let $p(Z)$, $p_1(Z)$, $p_2(Z), \ldots, p_j(Z) \in Z$, $q(z) \in Z \cap B$ have distance at most εd from $p, p_1, p_2, \ldots, p_j, q$. Thus we get a polygonal line $W(p) \in \mathscr{P}_{p(Z),q(Z)}$. For each $z \in B \cap Z$ consider the "ball" B_z consisting of all points in B at distance at most εd from z. Let

$$W'(p) = W(p) \cup \operatorname{conv}(\{p_j(Z)\} \cup B_{q(Z)})$$

and

$$Z' = \bigcup_{p \in P} W'(p).$$

It is easily verified that $\delta(Z, Z') < \varepsilon d$.

If $b \in Z'$, then there are (finitely many) "cones" like

$$\operatorname{conv}(\{p_j(Z)\} \cap B_{q(Z)})$$

containing b. Let $p_1', \ldots, p_m'$ be the apexes of these "cones", let $p_1'', \ldots, p_m''$ be the points playing the role of $p_{j-1}(Z)$, and let $B_1', \ldots, B_m'$ be the "balls" playing the role of $B_{q(Z)}$. Denote by s_i the intersection of the line through b and p_i' with B_i'. Let a_i be a point with $\|a_i - p_i'\| < \varepsilon$, such that the angle $a_i\, p_i' b$ is right and no pair of points in $Z \cup \{b\}$ are collinear with a_i. Let C_i be the set of all points $x \in B_i'$ such that the measure of the angle $x p_i' b$ is at most $\theta > 0$, θ being chosen so that $\delta(s_i, C_i) < \varepsilon$. It can be easily arranged so that

$$b \notin C_i' = \text{conv}(\{a_i\} \cup C_i).$$

Let

$$D_i = (\text{conv}(\{p_i'\} \cup B_i') - \text{conv}(\{p_i'\} \cup C_i)) \cup C_i'.$$

The set

$$Z'' = \left(Z' - \bigcup_{i=1}^{m} \text{int conv}(\{p_i'\} \cup C_i)\right) \cup \bigcup_{i=1}^{m} (C_i' \cup a_i, p_i'')$$

belongs to $\mathscr{P}_B^{(n)}$ and $\delta(Z', Z'') < \varepsilon$. Clearly, $b \notin Z''$.

In case $b \notin Z'$, let $Z'' = Z'$.

In both cases let γ be the distance from b to Z''. If $P' \in \mathscr{T}_B^{(n)}$ and $\delta(Z'', P') < \min\{\varepsilon d, \gamma\}$, then $b \notin P'$. Since

$$\delta(P, P') \leqslant \delta(P, Z) + \delta(Z, Z') + \delta(Z', Z'') + \delta(Z'', P') < 4\varepsilon d,$$

$P' \in \mathscr{O}$. Hence $\mathscr{B}$ is nowhere dense in $\mathscr{T}_B^{(n)}$.

Let Q be a countable dense set in $\mathbb{R}^d$. Then most sets in $\mathscr{T}_B^{(n)}$ are disjoint from $Q - B$. This proves the theorem.

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