TUDOR ZAMFIRESCU

AN INFINITESIMAL VERSION OF THE BESICOVITCH–DANZER CHARACTERIZATION OF THE CIRCLE

INTRODUCTION

Every planar convex curve such that no rectangle has exactly three vertices on it is a circle.

V. Mizel has asked whether the preceding characterization of the circle is true. A. S. Besicovitch [1] presented a not quite elementary proof and invited his audience to look for a more elementary one. Soon after, L. Danzer [2] provided such a proof and invited his readers to verify whether the characterization remains true when non-convex curves are also allowed in the competition. I am glad to accept this invitation. Moreover, we shall consider the following weaker form of the above property:

We say that a set in $\mathbb{R}^2$ has what we shall call the *infinitesimal rectangle property* if there is some $\varepsilon > 0$ such that no rectangle with sidelengths ratio at most ε has exactly three vertices in the set.

For $\varepsilon = 1$ this property becomes the one used in the above characterization, which we shall call the *global rectangle property*. Of course, decreasing ε means weakening the corresponding infinitesimal rectangle property.

Our results are:

THEOREM 1. *Every Jordan curve satisfying the infinitesimal rectangle property is convex and has constant width.*

THEOREM 2. *Every analytic curve of constant width satisfying the infinitesimal rectangle property is a circle.*

These two results enable us to present our main characterization theorems. Combining Theorem 1 with the available characterization using the global rectangular property we have at once:

THEOREM 3. *Every Jordan curve satisfying the global rectangle property is a circle.*

Combining Theorems 1 and 2 we get immediately:

THEOREM 4. *Every analytic Jordan curve satisfying the infinitesimal rectangle property is a circle.*

Thanks are due to the referee for his careful criticism.

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Proofs

We say 'locally at x' when we mean 'in any neighborhood of x'. We also say that a point y is 'close to x' if, for any neighborhood N of x, we can arrange that $y \in N$. We denote by Δ_{xy} the circular disk with xy as a diameter.

PROOF OF THEOREM 1. Let C be a Jordan curve with the infinitesimal rectangle property and suppose it is not convex. Then $C \backslash \mathrm{bd\ conv}\ C \neq \emptyset$. Let C^* be a component of $C \backslash \mathrm{bd\ conv}\ C$ (this is a Jordan arc with its endpoints removed) and let a, b be the endpoints of C^* (see figure 1).

Let S be the line with respect to which a and b are symmetrical. Let e be a furthest point of C from b. We may suppose that e and a lie in the same closed halfplane determined by S, because otherwise we have a symmetrical situation with respect to a and a furthest point of C from a.

Let L be the line through b orthogonal to be. We prove that, in some neighborhood of b, C lies in the closed halfplane B with boundary L which contains a. Suppose, on the contrary, C meets $\complement B$ locally at b. Consider the open halfspace E with $b, e \in \mathrm{bd}\, E$ and $a \notin E$. We distinguish three cases:

CASE 1. *C meets E locally at e.* Take a line parallel to L in $\complement B$. Chosen appropriately, this line meets C for the first time in a point $b' \in \complement B$, close to b. The parallel line through b' to be meets C for the last time in $e' \in E$, close to e. The halfline starting at e', parallel to L and intersecting $\complement E$, meets C again in a point e'' close to e, because C lies in the circular disk D with center b and with e on its boundary. Then the rectangle $b'\, e'\, e''\, x$ has $x \notin C$, and a contradiction is obtained.

CASE 2. *The arc $\widehat{ae} \subset C$ (not containing b) meets $\Delta_{be} \backslash E$ locally at e.* If e_1 is chosen sufficiently close to e in $\widehat{ae} \cap \Delta_{be} \backslash E$, then Δ_{be_1} meets the arc $\widehat{eb}$ (not containing a) close to e, say in e_2 (or we are in Case 1). The rectangle $e_1 e_2 b x$ has $x \notin C$, contradicting the hypothesis.

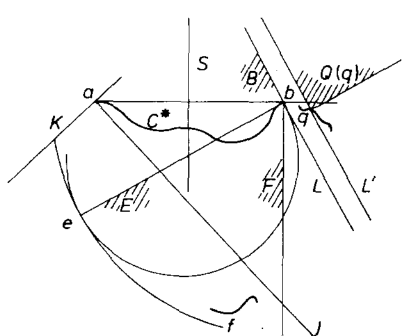

Fig. 1.

CASE 3. *The arc $\widehat{ae}$ avoids E and Δ_{be} in some neighborhood of e.* Take a line $L' \subset \complement B$ parallel to L and cutting C. Let q be a point of L' and consider the quadrant $Q(q)$ of apex q, with a boundary halfline on L', and such that $L \cap Q(q) = \emptyset$ and $S \cap Q(q)$ is unbounded. Let b_1 be the furthest point q of $L' \cap E$ from $S \cap L'$ such that int $Q(q) \cap C = \emptyset$. For appropriately chosen L', the point b_1 is close to b. The line through e orthogonal to $b_1 e$ meets $\widehat{ae}$ again in some point e_3 close to e, because $C \subset D$. Then the rectangle $b_1 e e_3 x$ has $x \notin C$; a contradiction!

Let F be the open halfplane containing a with the line through b orthogonal to ab on its boundary, and let f be a furthest point of $C \cap \bar{F}$ from the line through a and b.

Case I. $f \notin \operatorname{bd} F$. Let $b' \in C \backslash \overline{C^*}$, close to b. Then the halfcircle $\widehat{b'f} \subset \operatorname{bd} \Delta_{b'f}$ which meets ab also meets C^* in a point b'', close to b. Then the rectangle $b'b''fx$ has $x \notin C$, which is impossible.

Case II, $f \in \operatorname{bd} F$. In this case, let d be a furthest point of $C \backslash F$ from a. Let K be the line through a orthogonal to ad. Exactly as in the case of b and L, one shows that in some neighborhood of a, C lies in the closed halfplane with boundary K which contains b.

Now, if g is a furthest point of $C \backslash F$ from the line through a and b, we obtain a contradiction with respect to a and g, exactly as in Case I above where a contradiction was obtained with respect to b and f.

Thus it is proved that C is convex.

To finish the proof it remains to show that C has constant width.

It is true that the proof in [2] carefully avoids using the infinitesimal rectangle property even in showing that a convex curve with the global rectangle property has exclusively double normals. However, a few lines suffice to give the desired version.

Let mn be a chord of C and M be a supporting line at m orthogonal to mn, but suppose that the line through n orthogonal to mn does not support C. If $\{m\} = M \cap C$, take a line M' parallel to M such that $M' \cap C = \{m', m''\}$, the two points being close to m for suitable M'. One of them, say m', does not belong to mn. The line tnrough m' parallel to mn intersects C again in a point n' close to n. Then the rectangle $m''m'n'x$ has $x \notin C$. If $\{m\} \neq M \cap C$, then we simply take $m' \in M \cap C \backslash \{m\}$ and see that the rectangle $m'mnx$ has $x \notin C$ for m' sufficiently close to m. The contradiction obtained shows that every normal is a double normal.

Now it is known that a convex curve whose normals are all double normals is of constant width (see, for example, [1]), and the proof is finished.

PROOF OF THEOREM 2. Let $\gamma(x)$ be the curvature at x of the analytic curve C of constant width w. Let x^* be the point on C diametrically opposite to x. It is well known that $\gamma(x)^{-1} + \gamma(x^*)^{-1} = w$ for all $x \in C$. Suppose C is not a circle. Then $\gamma(x_0) \neq 2/w$ for some $x_0 \in C$, say $\gamma(x_0) < 2/w$. Then $\gamma(x_0^*) > 2/w$ and there exists a point $y \in C$ with $\gamma(y) = \gamma(y^*) = 2/w$ such that, in any neighborhood of y, there are points x to the left of y with $\gamma(x) > 2/w$ and points x' to the right of y with $\gamma(x') < 2/w$. The analyticity of C excludes the possibility that $\gamma'(y_n) = 0$ for some sequence $\{y_n\}_{n=1}^{\infty}$ convergent to y, except for the case $\gamma'(y) \equiv 0$ which is, however, impossible. Thus γ is monotone on each one of two arcs to the left and right of y: more precisely, there are an arc $\widehat{y_1 y}$ and an arc $\widehat{y y_2}$ on C such that γ is decreasing on both arcs, so in fact γ is decreasing on $\widehat{y_1 y_2}$. For any $x \in \widehat{y_1 y_2}$, consider the disk $D_x = \Delta_{xx^*}$. The inequality $\gamma(x) > 2/w$ valid on $\widehat{y_1 y} \setminus \{y\}$ yields $y_1 \in \operatorname{int} D_y$ and the inequality $\gamma(x) < 2/w$ valid on $\widehat{y y_2} \setminus \{y\}$ yields $y \notin D_x$. Since $D_x \to D_y$ as $x \to y$ in $\widehat{y y_2}$, there is some point $z \in \widehat{y y_2}$ such that $y_1 \in D_z$ and yet $y \notin D_z$. Let $t \in \widehat{y_1 y} \cap \operatorname{bd} D_z$. We claim that $\widehat{y_1 y}$ and bd D_z are not tangent at t. Indeed, if they were tangent, $\gamma(x) > 2/w$ for all $x \in \widehat{ty}$ implies $\widehat{ty} \subset D_z$, which contradicts $y \notin D_z$.

For appropriately chosen y_1 and y_2, z and t are close to y, whence the vertex u of the rectangle ztz^*u must lie on C and, since the diagonals are equally long, tu must be a diameter of C. It follows that this diameter is not orthogonal to C at t, and a contradiction is obtained.

Open problems

I invite the readers to verify whether indeed the infinitesimal rectangle property characterizes the circle among all convex curves of constant width; in other words, to reprove Theorem 2 without the analyticity condition. I conjecture this to be true. One should also try further to extend the characterization (global or infinitesimal). So, for example, is the circle characterized by the above properties among all planar continua different from a Jordan arc?

Finally, I conjecture that the same infinitesimal or global rectangle property characterizes in $\mathbb{R}^d$ the spheres of dimension less than d.

References

1. Besicovitch, A. S., 'A Problem on a Circle', *J. London Math. Soc.* **36** (1961), 241–244.
2. Danzer, L. W., 'A Characterization of the Circle' in *Convexity, Proc. Symposia in Pure Math.*, Vol. VII, Amer. Math. Soc., Providence, R.I., 1963, pp. 99–100.

Author's address:
Tudor Zamfirescu,
Universität Dortmund,
Fachbereich Mathematik,
Postfach 50 05 00,
D-4600 Dortmund 50,
F.R.G.