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Description of most starshaped surfaces

BY TUDOR ZAMFIRESCU
Department of Mathematics, University of Dortmund, West Germany

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Introduction

After having investigated in [7] generic properties of compact starshaped sets in $\mathbb{R}^d$, we shall restrict here our attention to compact starshaped sets whose kernels have positive dimension. While the main results in [7] are of a topological nature and concern the whole sets, the theorems presented here describe, for kernels of codimension 0 or 1, the local aspect of the boundaries and include, for kernels of positive dimension less than $d-1$, both types of statements.

Very recently, the description in [7] of typical compact starshaped sets in $\mathbb{R}^d$ was continued and completed in [2].

The space $\mathscr{T}$ of all compact starshaped sets in $\mathbb{R}^d$, endowed with the Hausdorff metric Δ, being closed in the space of all compact sets, is a Baire space. Thus it makes sense to speak about *most*, or *typical*, starshaped (always compact) sets, that is all, except those in a set of first Baire category.

If we impose the condition that the kernels of the starshaped sets must include a given convex compact set K, we get a subspace $\mathscr{T}_K$ of $\mathscr{T}$ which is again a Baire space, being closed in $\mathscr{T}$. The reason for the special interest in studying $\mathscr{T}_K$ is simple. Suppose, for instance that K has interior points. Then the boundary of every set belonging to $\mathscr{T}_K$ is a surface homeomorphic to the boundary S_{d-1} of the unit ball B; this is not guaranteed for sets belonging to $\mathscr{T}$. The space $\mathscr{S}$ of all boundaries of sets belonging to $\mathscr{T}_K$, equipped with the metric Δ, is homeomorphic to $\mathscr{T}_K$. Each member of $\mathscr{S}$ will be called a *starshaped surface*. A starshaped surface which contains no line segment will be called *strictly starshaped*.

For $A \subset \mathbb{R}^d$, we shall use the notation

$$\operatorname{diam} A = \sup_{x, y \in A} \|x-y\|, \quad cA = \{\|x\|^{-1}x : x \in A\backslash\{0\}\}.$$

The reader interested in Baire category results obtained in convexity may consult [6].

On most starshaped surfaces

Supposing, for simplicity, that $B \subset K$, every surface $S \in \mathscr{S}$ is determined in polar coordinates $(\omega, \rho(\omega))$ by an 'associated' function $\rho : S_{d-1} \to [1, \infty)$, so that $\rho(\omega)\,\omega \in S$. Since S is compact and B lies in the kernel of

$$[S] = \{\lambda x : \lambda \in [0,1], x \in S\},$$

ρ is a Lipschitz function. As such, it is differentiable a.e. This reminds us of convex surfaces. The space of all convex surfaces is also a Baire space and most convex surfaces are everywhere differentiable and contain no line segments, as Klee[3] proved. What do most starshaped surfaces look like? It is proved in this section that they also contain no line segments, but are not differentiable at most of their points. Roughly speaking, most of their points do not 'see' more than K; on the other hand, almost all points do 'see' more than K (but no βK with $\beta > 1$).

Let $S \in \mathscr{S}$ and $x \in S$. Let $\gamma(x)$ be the solid angle under which x sees K, i.e.

$$\gamma(x) = \{\|y-x\|^{-1}(y-x) : y \neq x, y \in K\}$$

and let $\delta(x)$ be the solid angle of $[S]$ at x, i.e.

$$\delta(x) = \{\|y-x\|^{-1}(y-x) : y \neq x,\ yx \subset [S]\},$$

yx denoting the line segment from y to x.

Also, for $x \in \mathbb{R}^d$, $A \subset S_{d-1}, \epsilon \geqslant 0$, put

$$\langle x, \epsilon] = \operatorname{conv}(\{x\} \cup (K+\epsilon B))$$

and, for a given function ρ, set

$$\langle A, \epsilon] = \operatorname{bd} \bigcup_{\omega \in A} \langle \rho(\omega)\,\omega, \epsilon].$$

THEOREM 1. *Most surfaces from $\mathscr{S}$ are strictly starshaped.*

Proof. Let $\mathscr{S}_n$ be the set of those surfaces in $\mathscr{S}$ which contain a line segment of length n^{-1}. We show that $\mathscr{S}_n$ is nowhere dense.

$\mathscr{S}_n$ is obviously closed in $\mathscr{S}$. Let $\mathscr{O}$ be an open set in $\mathscr{S}$ and consider $S \in \mathscr{O}$ with associated function ρ. Let $\epsilon > 0$ be chosen such that $\Delta(S, S') \leqslant 2\epsilon$ implies $S' \in \mathscr{O}$.

Consider $\nu \in (0, \epsilon)$ and

$$S_\nu = \{x \in \mathbb{R}^d : \min_{y \in [S]} \|x-y\| = \epsilon + \nu\}.$$

Let $C_x = \operatorname{int} \langle x, 0]$. It is clearly possible to find a ν such that, for any $x \in S_\nu$,

$$\operatorname{diam} C_x \backslash ([S]+\epsilon B) < n^{-1}.$$

The family of open sets $\{C_x\}_{x \in S_\nu}$ covers the compact set $[S]+\epsilon B$ (see Figure 1). Hence there is a finite set $A_\nu \subset S_\nu$ such that $\{C_x\}_{x \in A_\nu}$ covers $[S]+\epsilon B$. We claim that, for ν small enough, the starshaped surface

$$S'_\nu = \operatorname{bd} \bigcup_{x \in A_\nu} C_x$$

contains no line segment of length n^{-1}. Indeed, every line segment in S'_ν belongs to the boundary of some C_x and is disjoint from $[S]+\epsilon B$; thus it belongs to the boundary of $C_x \backslash ([S]+\epsilon B)$ and therefore has length less than n^{-1}. Hence $S'_\nu \in \mathscr{O} \backslash \mathscr{S}_n$. Thus $\mathscr{S} \backslash \mathscr{S}_n$ is dense and $\mathscr{S}_n$ nowhere dense in $\mathscr{S}$. Since any surface in $\mathscr{S}$ containing a line segment lies in $\bigcup_{n=1}^{\infty} \mathscr{S}_n$, the conclusion of the theorem follows.

THEOREM 2. *On most $S \in \mathscr{S}$, for most points $x \in S$, $\gamma(x) = \delta(x)$.*

Proof. Let n be a natural number and

$$\delta_n(x) = \{\|y-x\|^{-1}(y-x) : \|y-x\| = n^{-1}, yx \subset [S]\}.$$

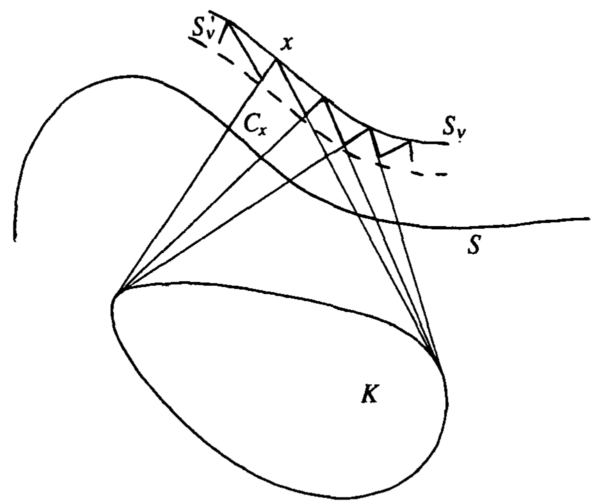

Fig. 1

Clearly $\delta(x) = \bigcup_{n=1}^{\infty} \delta_n(x)$ and it is easily seen that δ_n is upper semicontinuous.

First we prove that

$$E_n = \{x \in S : \delta_n(x)\backslash\gamma(x) \text{ includes no disk of radius } n^{-1}\}$$

is dense in S, for most $S \in \mathscr{S}$.

Let $\mathscr{S}_n$ be the family of all those surfaces $S \in \mathscr{S}$ such that there exist disks D_1 and D_2 on S_{d-1}, both of (angular) radius n^{-1}, with the property that, for all $\omega \in D_1$,

$$D_2 \subset \delta_n(\rho(\omega)\,\omega)\backslash\gamma(\rho(\omega)\,\omega).$$

It is an easy exercise to check that $\mathscr{S}_n$ is closed in $\mathscr{S}$. Let $\mathscr{O}$ be an open set in $\mathscr{S}$ and let $S \in \mathscr{O}$ have associated function ρ. Let $\epsilon > 0$ be chosen such that any element of $\mathscr{S}$ at Hausdorff distance at most ϵ from S lies in $\mathscr{O}$. Let A be a finite set on S_{d-1} such that $S_1 = \langle A, 0]$ satisfies $\Delta(S, S_1) < \epsilon$ and $S_{d-1}\backslash A$ includes no disk of radius n^{-1}.

Thus we have found a surface $S_1 \in \mathscr{O}$ such that for every disk $D \subset S_{d-1}$ with radius n^{-1} there exists a point $\omega \in A \cap D$, at which obviously

$$\delta_n(\rho_1(\omega)\,\omega) \subset \gamma(\rho_1(\omega)\,\omega),$$

ρ_1 being the function associated with S_1. This proves that the complement of $\mathscr{S}_n$ is dense and $\mathscr{S}_n$ is nowhere dense in $\mathscr{S}$.

Let $\mathscr{S}^n$ be the family of all $S \in \mathscr{S}$ such that E_n is dense in S. For any surface $S \in \mathscr{S}\backslash\mathscr{S}^n$ there are disks D, $D' \subset S_{d-1}$ of radii m^{-1}, n^{-1} ($m \in \mathbb{N}$) such that, for all $\omega \in D$,

$$D' \subset \delta_n(\rho(\omega)\,\omega)\backslash\gamma(\rho(\omega)\,\omega).$$

Then $S \in \mathscr{S}_{\max\{m,n\}}$. It follows that $\mathscr{S}\backslash\mathscr{S}^n \subset \bigcup_{m=1}^{\infty} \mathscr{S}_m$, whence most surfaces in $\mathscr{S}$ belong to $\mathscr{S}^n$.

Now let $S \in \mathscr{S}^n$. Since γ is continuous and δ_n upper semicontinuous on S, E_n is open and $S\backslash E_n$ nowhere dense in S. Since most surfaces of $\mathscr{S}$ belong to $\mathscr{S}^n$ for every $n \in \mathbb{N}$ and

$$\{x \in S : \gamma(x) = \delta(x)\} = \bigcap_{k=1}^{\infty} E_n = S\backslash \bigcup_{k=1}^{\infty} (S\backslash E_n),$$

at most points of each such surface, γ equals δ. This concludes the proof.

At each point x with $\gamma(x) = \delta(x)$, ρ is not differentiable or $x \in K$. Clearly the family of all surfaces having a common point with K is nowhere dense in $\mathscr{S}$. Thus most surfaces in $\mathscr{S}$ are not differentiable at most of their points. In contrast to this, we know that each surface of $\mathscr{S}$ is differentiable a.e. The position of the tangent hyperplanes for most surfaces in $\mathscr{S}$ is described by the next theorem.

THEOREM 3. *On most surfaces in $\mathscr{S}$ there is a.e. a tangent hyperplane which also supports K.*

Proof. Denote by μ the Lebesgue outer measure on S_{d-1}. Let $\mathscr{S}^\dagger$ be the family of all surfaces $S \in \mathscr{S}$ such that there exists a set $A \subset S_{d-1}$ of positive outer measure with the property that, for all $x \in S$ with $\|x\|^{-1} x \in A$,

$$\langle x, \epsilon] \subset [S]$$

for some $\epsilon > 0$ depending on x. Let $\mathscr{S}_n$ be the set of all those surfaces in $\mathscr{S}^\dagger$ satisfying the additional conditions

$$\mu A = \epsilon = n^{-1}.$$

Clearly $\mathscr{S}^\dagger = \bigcup_{n=1}^{\infty} \mathscr{S}_n$. We show that $\mathscr{S}_n$ is nowhere dense in $\mathscr{S}$ for every n.

Let $\mathcal{O}$ be open in $\mathscr{S}$ and let $S \in \mathcal{O}$ have associated function ρ. For some $\epsilon > 0$, $\Delta(S, S') \leqslant \epsilon$ implies $S' \in \mathcal{O}$. Let A be a finite set on S_{d-1} such that $S' = \langle A, 0]$ satisfies $\Delta(S, S') \leqslant \epsilon$. Clearly the set T of all points of S' belonging to more than one 'cone' $\langle \rho(\omega)\,\omega, 0]$ satisfies $\mu c T = 0$. Let

$$U = S' \cap (T + \alpha B), \quad V = S' \backslash U.$$

Evidently, for α small enough, $\mu c U < n^{-1}$ and $A \cap cU = \emptyset$. For every point $x \in V$, there exists a point $y \in T$ such that the line through x and y touches K. Choose $\beta > 0$ and let $S'' \in \mathcal{O}$ satisfy $\Delta(S', S'') < \beta$. Elementary arguments show that, for β small enough,

$$\langle z, n^{-1}] \subset [S'']$$

holds for no $z \in S''$ with $\|z\|^{-1} z \in cV$. Hence the above inclusion is true for points $z \in S''$ with $\|z\|^{-1} z \in cU$ at most. Since $\mu c U < n^{-1}$, $S'' \notin \mathscr{S}_n$. Thus $\mathscr{S}_n$ is nowhere dense in $\mathscr{S}$.

Consequently most surfaces in $\mathscr{S}$ do not belong to $\mathscr{S}^\dagger$, which implies that, for most $S \in \mathscr{S}$ and almost all points $x \in S$, the inclusion $\langle x, \epsilon] \subset [S]$ holds for no $\epsilon > 0$. Let $F' \subset S$ be this set of points x and let F be the set of all points in F' where S is differentiable. Clearly $S \backslash F$ has measure zero. Let $x \in F$. There exists, for every $n \in \mathbb{N}$, a segment s_n joining x with a point of $K + n^{-1} B$ and meeting $\complement[S]$. If we exclude now those surfaces S meeting K, which form a nowhere dense family, we can say that, for n large enough, there is a point $y_n \in s_n \cap S$ such that $xy_n \backslash [S] \neq \emptyset$.

For $n \to \infty$, some subsequence of $\{s_n\}_{n=1}^{\infty}$ converges to a segment s_∞ joining x with a point of K. The corresponding subsequence of $\{y_n\}_{n=1}^{\infty}$ or a subsequence of it converges to a point $y \in s_\infty$. If $y \neq x$, then $xy \subset S$. Thus, by Theorem 1, $x = y$ for most $S \in \mathscr{S}$. This implies that s_∞ lies in the tangent hyperplane H of S at x. It follows that H meets K. Since obviously $H \cap \operatorname{int} K = \emptyset$, H supports K. The theorem is proved.

Thus, roughly speaking, Theorems 2 and 3 read as follows: on most surfaces $S \in \mathscr{S}$, most points on S see via $[S]$ nothing more than K and there exists a.e. on S a tangent hyperplane which supports K.

At least twenty-five years ago Fejes-Tóth asked for a characterization of those plane convex bodies which can be realized as convex kernels of non-convex plane sets. De Bruijn and Post (see [5]) proved that every plane convex body can be so realized. Klee [4] extended this result to separable Banach spaces (the 2-dimensional case was also treated in [5]). Obviously not aware of the work previously done, Breen [1] also proved that in a Euclidean space each convex body is the kernel of some non-convex compact set, thus answering a more recent question of Lay, identical with the old one of Fejes-Tóth.

Now the stated result follows immediately from the next theorem.

THEOREM 4. *For most $S \in \mathscr{S}$, K is the kernel of* $[S]$.

Proof. Choose S in the residual set in $\mathscr{S}$ revealed by Theorem 2. Let $x \in [S] \backslash K$. Consider two hyperplanes H_1, H_2 separating x from K and disjoint from $K \cup \{x\}$. Suppose H_1 also separates x from H_2. Then, for any point y of S between H_1 and H_2, the hyperplane H_1 separates the cone $\bigcup_{\lambda \geqslant 0}(y + \lambda(K - y))$ from x. By Theorem 2, we find a point $y \in S$ between H_1 and H_2 such that $\gamma(y) = \delta(y)$. Thus $yx \not\subset [S]$ and x does not belong to the kernel of $[S]$, which proves the theorem.

Klee [4] also proved that, given a convex body K and positive numbers σ and τ with $\sigma < \tau$, there exists a starshaped set with K as kernel containing the σ-neighbourhood and contained in the τ-neighbourhood of K. Theorem 4 shows that near any compact starshaped set whose kernel includes K, there is another one, the kernel of which equals K.

Typical starshaped sets with lower dimensional kernels

Let $\mathscr{P}_k$ be the space of all compact starshaped sets the kernels of which include a given k-dimensional compact convex set C lying in a k-dimensional linear subspace L of $\mathbb{R}^d$. We suppose again for simplicity that $S_{k-1} = S_{d-1} \cap L \subset C$.

First let $k = d-1$. The boundary of $P \in \mathscr{P}_{d-1}$ is a union of three sets: two of them, above and below L, are homeomorphic to $\mathbb{R}^{d-1}$ and the third is a ring-shaped set $R \subset L$ around C, the exterior part of its boundary with respect to L being a starshaped $(d-2)$-dimensional surface S. In fact the boundary of P may be quite ugly at points of R. However most $P \in \mathscr{P}_{d-1}$ look nice:

THEOREM 5. *For most $P \in \mathscr{P}_{d-1}$, $\operatorname{bd} P$ is homeomorphic to S_{d-1} and includes the boundary of C with respect to L.*

Proof. Let $\omega \in S_{d-2}$ and consider the half-line

$$h_\omega = \{\lambda\omega : \lambda \geqslant 0\}.$$

We only have to prove that $h_\omega \cap P \subset C$ for most $P \in \mathscr{P}_{d-1}$ and every $\omega \in S_{d-2}$.

Let $\mathscr{P}_{(n)}$ be the family of all sets $P \in \mathscr{P}_{d-1}$ such that, for some $\omega \in S_{d-2}$,

$$\operatorname{diam}(h_\omega \cap P \backslash C) \geqslant n^{-1}.$$

We show that $\mathscr{P}_{(n)}$ is nowhere dense in $\mathscr{P}_{d-1}$.

Let $\mathcal{O}$ be open in $\mathscr{P}_{d-1}$ and consider $P \in \mathcal{O}$ and $\epsilon > 0$, such that $\Delta(P, P') < \epsilon$ implies $P' \in \mathcal{O}$. Let

$$P_1 = P \cup \bigcup_{x \in S} \operatorname{conv}(\epsilon B \cup \{x\}),$$

S being the surface mentioned prior to the statement of Theorem 5 (see Figure 2). Clearly $\Delta(P, P_1) < \epsilon$. Let $\eta > 0$. Let a_1, a_2 be the two points of ηS_{d-1} which are furthest

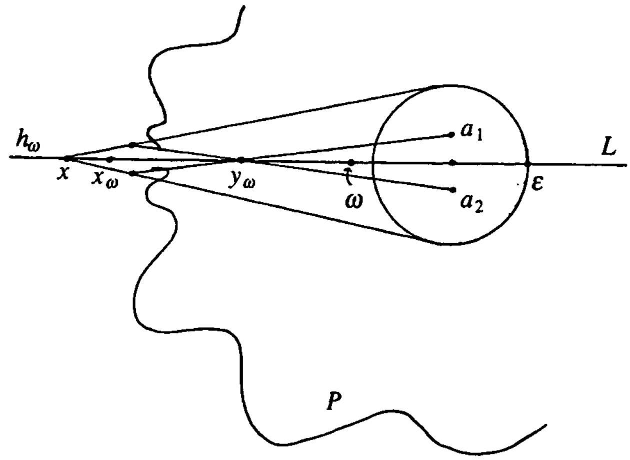

Fig. 2

from L. The (unbounded) closed convex cone with apex a_i and containing bd C on its boundary intersects P_1 in Q_i $(i = 1, 2)$. *If* η is small enough, $P_2 = Q_1 \cup Q_2$ still belongs to $\mathcal{O}$. Let y_ω be the end-point of $h_\omega \cap C$ different from the origin and define $x_\omega \in h_\omega \setminus C$ by $\|x_\omega - y_\omega\| = n^{-1}$. Also, let

$$\alpha = \min\{\|x_\omega - y\| : \omega \in S_{d-2}, y \in P_2\}.$$

Since the sets $\{x_\omega : \omega \in S_{d-2}\}$ and P_2 are compact and disjoint, $\alpha > 0$. Then, for every set $P_3 \in \mathcal{P}_{d-1}$ with $\Delta(P_2, P_3) < \alpha$, for any $\omega \in S_{d-2}$ and for any $u \in h_\omega \cap P_3$, the distance from u to C is less than n^{-1}. Hence $\mathcal{P}_{(n)}$ is nowhere dense and consequently most members of $\mathcal{P}_{d-1}$ are d-dimensional topological disks.

Theorems 2 and 3 are also – suitably modified – valid for sets belonging to $\mathcal{P}_{d-1}$: for most $P \in \mathcal{P}_{d-1}$, most points on bd$P$ see not more than C via P and there exists a.e. on bd$P$ a tangent hyperplane which intersects L along a supporting $(d-2)$-plane of C.

The situation changes if $k \leqslant d-2$. Then it is not difficult to see that the orthogonal projection of a typical set in $\mathcal{P}_k$ on $L^\perp$ looks like a typical $(d-k)$-dimensional compact starshaped set (see [2] and [7]). Let L_x be the subspace generated by x and L and, for $P \in \mathcal{P}_k$, let

$$\Omega(P) = \{\omega \in L^\perp \cap S_{d-1} : L_\omega \cap P \not\subset L\}.$$

THEOREM 6. *For most* $P \in \mathcal{P}_k$ *the following holds*: P *is nowhere dense*, $\Omega(P)$ *is dense, uncountable and of first category in* $L^\perp \cap S_{d-1}$, *for each* $\omega \in \Omega(P)$ *the set* $L_\omega \cap P$ *is a* $(k+1)$-*dimensional topological disk* D_ω, *every such disk* D_ω *intersects* L *along* C, *and the boundary of* D_ω *with respect to* L_ω *contains* C *and enjoys the properties of Theorems* 2 *and* 3 *with respect to* C.

Proof. The proof is basically an adaptation of those of theorems 1 and 2 in [7], theorem 1 in [2] and Theorems 5, 2 and 3 of the present paper. The only thing which should be explained is why C lies in the boundary of every topological disk D_ω with $\omega \in \Omega(P)$.

Let $\mathcal{P}_{(n)}$ be the family of all $P \in \mathcal{P}_k$ such that there exists a $(k+1)$-dimensional linear subspace $J \supset L$ for which the measure (volume) of the smaller component of $P \cap J \backslash L$ is at least n^{-1}. Let $\mathcal{O}$ be open in $\mathcal{P}_k$ and let $P \in \mathcal{O}$ be a finite union of $(k+1)$-dimensional topological disks lying in $(k+1)$-dimensional linear half-subspaces with (relative) boundary L. We can easily arrange that no union of two of these half-subspaces is a linear subspace of $\mathbb{R}^d$, i.e. no two points of $\Omega(P)$ are opposite. Then, for $\epsilon > 0$ small enough, every element of $\mathcal{O}$ at distance at most ϵ from P does not belong to $\mathcal{P}_{(n)}$. This proves that $\mathcal{P}_{(n)}$ is nowhere dense.

Hence, for most sets belonging to $\mathcal{P}_k$, the mentioned topological disks contain C in their boundaries. The proof of the fact that, for most $P \in \mathcal{P}_k$, no point of $L \backslash C$ belongs to P parallels the proof of the assertion for $k = d-1$ (see Theorem 5).

In a Euclidean space, Theorem 6 extends Klee's result mentioned at the end of the preceding section, which also applies for sets in $\mathcal{P}_k$ for $k < d$.

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