NONDIFFERENTIABILITY PROPERTIES OF THE NEAREST POINT MAPPING

By

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In fact the paper is less negative than its title. Indeed, we also prove some differentiability properties of the nearest point mapping as well.

In 1973 Asplund [1] proved that the nearest point mapping p from $\mathbf{R}^d$ onto any of its closed subsets K is almost everywhere Fréchet differentiable. If K is a closed convex set in Hilbert space, then p is nonexpansive and hence Gateaux differentiable almost everywhere. On the other hand, as Fitzpatrick and Phelps have shown [3], p may be nowhere Fréchet differentiable outside K.

From the topological point of view (i.e., Baire category), the set of points of Fréchet nondifferentiability of p may be large, even in Euclidean spaces. Zajíček [5] constructed a convex body $K \subset \mathbf{R}^2$ for which p is Fréchet nondifferentiable at *most* points of $\mathbf{R}^2 \setminus K$, i.e. at all points except those in a set of first category. We shall always use the word "most" in this way. We shall also say that a *typical* element of a Baire space has a certain property if most elements of the space have that property. For results on typical convex bodies see [12].

In this paper we describe differentiability and nondifferentiability properties of the nearest point mapping p onto a typical convex body $K \subset \mathbf{R}^d$. (Recall that the space of all convex bodies in $\mathbf{R}^d$, equipped with the Hausdorff distance, is a Baire space.) The proofs will make use of results in [6], [7], [9] and [11]. The strong relationship between the differentiability properties of p and of the boundary bd K of K, known for a long time, together with the pathological differentiability properties of bd K for most K will result in a couple of strange theorems. These will reveal the (pathological) beauty of the nearest point mapping.

I thank the referee for his valuable suggestions.

Prerequisites

Throughout the paper we shall tacitly use Klee's result stating that most convex bodies are smooth [4]. (For a strengthening of Klee's result see [10].)

Let $K \subset \mathbf{R}^d$ be a convex body and τ a tangent direction at $x \in \operatorname{bd} K$. The halfplane containing $x + \tau$, whose boundary is the normal N at x to $\operatorname{bd} K$, intersects $\operatorname{bd} K$ along a curve C called normal section at x in direction τ. The point x is an endpoint of C. Consider $y \in C$ and let $z_y \in N$ be at equal distances from x and y. Following [2], we define the lower and upper radii of curvature $\rho_i^\tau(x)$ and $\rho_s^\tau(x)$ respectively, of $\operatorname{bd} K$ at x in direction τ by

$$\rho_i^\tau = \liminf_{y \to x} \| z_y - x \| ; \qquad \rho_s^\tau = \limsup_{y \to x} \| z_y - x \| .$$

We denote by $L(a, b)$ the line through a and b, by $R(a, b)$ the ray which starts at a and contains b, and by $S(a, b)$ the line-segment from a to b.

Lemma 1. *Let $x \in \operatorname{bd} K$ be such that $\rho_s^\tau(x) = \infty$ in a tangent direction τ. Then, for each point $y \in p^{-1}(x)$, there exists a sequence $\{h_n\}_{n=1}^\infty$ with $h_n \to 0+$ such that*

$$\lim_{n \to \infty} \frac{p(y + h_n\tau) - p(y)}{h_n} = \tau .$$

Proof. Let N be the normal at x to $\operatorname{bd} K$. Since $\rho_s^\tau(x) = \infty$, there exist a sequence $\{z_n\}_{n=1}^\infty$ of points on the interior normal $N_i = N \setminus p^{-1}(x)$ and a sequence $\{x_n'''\}_{n=1}^\infty$ of points in $H \cap (\operatorname{bd} K) \setminus \{x\}$, where H is the halfplane with boundary N which contains $x + \tau$, such that $\| x - z_n \| = \| x_n''' - z_n \|$, $\| x - z_n \| \to \infty$, and $x_n''' \to x$.

For each n, $\rho_s^\tau(x) = \infty$ implies that

$$\max_{v \in A_n} \| z_n - v \| = \| z_n - x_n \|$$

for some point x_n in the relative interior of A_n, where A_n is the subarc of $H \cap \operatorname{bd} K$ from x_n''' to x. Hence $L(z_n, x_n)$ is normal to $H \cap \operatorname{bd} K$ and $x_n \to x$ too. See Fig. 1.

Clearly, $\{x_n\} = S(y + h_n\tau, z_n) \cap \operatorname{bd} K$ for suitable numbers h_n. Let Π be the supporting hyperplane of K at x and $\{x_n'\} = S(y + h_n\tau, z_n) \cap \Pi$. Since x and z_n lie on the same side of the supporting hyperplane of K at x_n, the point x_n lies between $y + h_n\tau$ and the orthogonal projection x_n'' of x on $L(x_n, z_n)$. Let y_n' be the orthogonal projection of $y + h_n\tau$ on Π, and $\{y_n\} = L(y_n', y + h_n\tau) \cap L(x, x_n'')$. Let Γ_n be the sphere of center $y + h_n\tau$ passing through x_n''. Put $\{y_n''\} = \Gamma_n \cap S(x_n', x)$.

Now we show that

$$(*) \qquad \frac{\| y_n' - y_n'' \|}{\| y_n' - x \|} \to 0 \qquad (\text{as } n \to \infty).$$

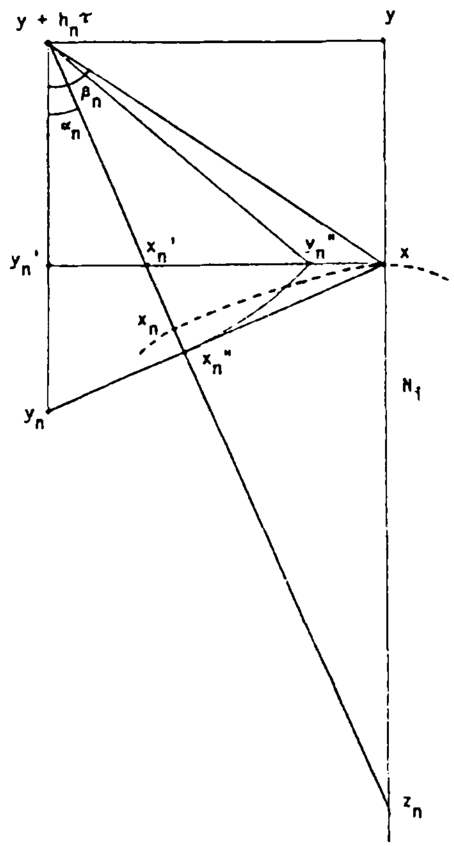

Fig. 1.

Let α_n and β_n be the measures of the angles that $R(y + h_n\tau, y_n)$ makes with $R(y + h_n\tau, x_n)$ and with $R(y + h_n\tau, x)$, respectively. Put $t = \| x - y \|$ and note that $t = \| y + h_n\tau - y'_n \|$, $h_n = \| y'_n - x \|$,

$$\| y + h_n\tau - x''_n \| = (t + \| y_n - y'_n \|)\cos \alpha_n$$

and

$$\| y_n - y'_n \| = h_n \tan \alpha_n.$$

Therefore

$$\begin{aligned} \| y'_n - y''_n \|^2 &= (t + h_n \tan \alpha_n)^2 \cos^2 \alpha_n - t^2 \\ &= h_n^2 \sin^2 \alpha_n - t^2 \sin^2 \alpha_n + t h_n \sin 2\alpha_n, \end{aligned}$$

whence

$$(**) \qquad \frac{\| y'_n - y''_n \|^2}{h_n^2} = \sin^2 \alpha_n - \frac{\sin^2 \alpha_n}{\tan^2 \beta_n} + \frac{\sin 2\alpha_n}{\tan \beta_n}.$$

Since $\| z_n - x \| \to \infty$,

$$\frac{\tan\alpha_n}{\tan\beta_n} = \frac{\| x'_n - y'_n \|}{\| x - y'_n \|} = \frac{\| y + h_n\tau - y'_n \|}{\| z_n - x \| + \| y + h_n\tau - y'_n \|} = \frac{t}{\| z_n - x \| + t} \to 0.$$

From this and (**) we get (*).

It is easily seen that $p(y + h_n\tau)$ lies in the ball B_n of boundary Γ_n and in the halfspace P of boundary π, containing z_n. Since

$$\min_{v \in B_n \cap P} \| x - v \| = \| y'_n - x \| - \| y'_n - y''_n \|$$

and

$$\max_{v \in B_n \cap P} \| x - v \| = \| y'_n - x \| + \| y'_n - y''_n \|,$$

remembering (*), we eventually obtain

$$\frac{\| p(y + h_n\tau) - p(y) \|}{h_n} = \frac{\| p(y + h_n\tau) - x \|}{\| y'_n - x \|} \to 1.$$

Now let γ_n be the measure of the angle between $R(x, y'_n)$ and $R(x, p(y + h_n\tau))$. Clearly,

$$\sin\gamma_n \leq \frac{\| y'_n - y''_n \|}{\| y''_n - x \|} = \frac{\| y'_n - y''_n \|}{\| y'_n - x \|}\left(1 - \frac{\| y'_n - y''_n \|}{\| y'_n - x \|}\right)^{-1},$$

whence, by (*), $\gamma_n \to 0$. Therefore

$$\frac{p(y + h_n\tau) - p(y)}{h_n} \to \tau,$$

and the lemma is proved.

Lemma 2. *For a typical convex body K,*

(a) $\rho_i^\tau(x) = \rho_s^\tau(x) = \infty$ *a.e. on* bd K, *for all tangent directions* τ;

(b) $\rho_i^\tau(x) = 0$ *and* $\rho_s^\tau(x) = \infty$ *at most points of* bd K, *for all tangent directions* τ;

(c) *for most points x on* bd K, *every point on the interior normal N_i at x (which, by definition, does not contain x) lies on the normals at the points (all different from x) of a sequence converging on* bd K *to x.*

Part (a) is Theorem 2 in [6]. Part (b) is Theorem 2 in [7]. Part (c) is nowhere else explicitly stated, but its proof is part of the proof of the Theorem in [9], although the statement of (c) is not a corollary of that theorem. Hence we omit a proof here and refer the reader to [9].

Lemma 3. *For a typical planar convex body K, the set of all points $x \in$* bd K *such that*

$$\rho_i^{\pm\tau}(x) = \rho_s^{\pm\tau}(x) = 0$$

(*for both tangent directions* τ *and* $-\tau$) *is dense in* bd K.

This is contained in Theorem 3 from [11].

The d-dimensional case

Let $p'(y)$ denote the Fréchet derivative of p at y and

$$P_y : \mathbf{R}^d \rightarrow H(y)$$

denote the orthogonal projection of $\mathbf{R}^d$ onto the hyperplane

$$H(y) = \{z \in \mathbf{R}^d : \langle y - p(y), z \rangle = 0\}.$$

It is known [3] that the operators $p'(y)$ and P_y satisfy

$$p'(y) \circ P_y = p'(y) = P_y \circ p'(y).$$

Theorem 1. *For a typical convex body* K, *for almost all* $x \in$ bd K *and for any* $y \in p^{-1}(x)$, *we have*

$$p'(y) = P_y.$$

Proof. By Lemma 2(a), for a typical convex body K, $\rho_i^\tau(x) = \rho_s^\tau(x) = \infty$ a.e. on bd K, for all tangent directions τ. Let

$$E = \{x \in \text{bd } K : \rho_i^\tau(x) = \rho_s^\tau(x) = \infty \text{ for all } \tau\}.$$

Also, let L be the set of all points outside K where p is Fréchet differentiable. Almost all points outside K lie in L. Hence almost all points of bd K lie in $p(L) \cap E$. Take any $y \in p^{-1}(p(L) \cap E)$. By Lemma 1, for every unit vector τ orthogonal to $y - p(y)$ there exists a sequence $\{h_n\}_{n=1}^\infty$ such that $h_n \rightarrow 0+$ and

$$\frac{p(y + h_n\tau) - p(y)}{h_n} = \tau.$$

Then, the existence of $p'(y)$ implies that the directional derivative of p in direction τ is τ. Thus $p'(y)$ restricted to $H(y)$ is the identity and the theorem follows.

Theorem 2. *For a typical convex body* K, *at most points* $y \notin K$, *the directional derivative of* p *in some direction does not exist* (*hence, at most points* $y \notin K$, $p'(y)$ *does not exist*).

Proof. For a typical convex body K, at most points x on bd K the following happens:

(i) $\rho_s^\tau(x) = \infty$ for every tangent direction τ, by Lemma 2(b);

(ii) every point on the interior normal N_t at x lies on the normals at the points (different from x) of a sequence converging on bd K to x, by Lemma 2(c).

Obviously,

$$M = (\mathbf{R}^d \setminus K) \setminus \bigcup_{x \in F} p^{-1}(x),$$

where F is the set of all points x verifying (i) and (ii), is a set of first category. Choose arbitrarily $y \notin K \cup M$ and let $p(y) = x$.

From (ii) it follows that there exist a sequence $\{z_n\}_{n=1}^{\infty}$ of points on N_t and a sequence $\{x_n\}_{n=1}^{\infty}$ of points on bd K such that $x_n \to x$, $x_n \neq x$, $z_n \to x$, and all $L(x_n, z_n)$ are normals of bd K, as one can easily verify. Let $\{y_n\} = L(x_n, z_n) \cap \Xi$, where Ξ is the hyperplane through y orthogonal to $y - x$. Clearly, $x_n = p(y_n)$.

We may suppose the sequence of rays $\{R(y, y_n)\}_{n=1}^{\infty}$ to be convergent to some ray $Y = \{y + h\tau_0 : h \geqq 0\}$, where $\|\tau_0\| = 1$, otherwise consider an appropriate subsequence. One verifies immediately that $z_n \to x$ yields

$$\frac{\| x_n - x \|}{\| y_n - y \|} \to 0.$$

Let y'_n be the orthogonal projection of y_n on Y and set $k_n = \| y - y'_n \|$. We have

$$\frac{\| p(y'_n) - p(y) \|}{k_n} \leqq \frac{\| x - x_n \| + \| x_n - p(y'_n) \|}{\| y - y_n \| - \| y_n - y'_n \|} \leqq \frac{\| x - x_n \| + \| y_n - y'_n \|}{\| y - y_n \| - \| y_n - y'_n \|}$$

$$= \left(\frac{\| x - x_n \|}{\| y - y_n \|} + \frac{\| y_n - y'_n \|}{\| y - y_n \|} \right) \left(1 - \frac{\| y_n - y'_n \|}{\| y - y_n \|} \right)^{-1} \to 0.$$

Thus,

$$\lim_{n \to \infty} \frac{p(y + k_n \tau_0) - p(y)}{k_n} = \mathbf{0}.$$

But condition (i) and Lemma 1 imply that we can also find $\{h_n\}_{n=1}^{\infty}$ such that $h_n \to 0+$ and

$$\lim_{n \to \infty} \frac{p(y + h_n \tau_0) - p(y)}{h_n} = \tau_0.$$

Hence the directional derivative of p at y in direction τ_0 does not exist, which proves the theorem.

The planar case

In the plane we can provide additional information on the aspect of p for typical K.

Theorem 3. *For a typical planar convex body K, at most points $y \notin K$, p has no directional derivative in any nonnormal direction.*

Proof. For a typical convex body $K \subset \mathbf{R}^2$, at most points $x \in \operatorname{bd} K$ we have $\rho_i^{\pm\tau}(x) = 0$ and $\rho_s^{\pm\tau}(x) = \infty$, where τ and $-\tau$ are the two tangent directions at x, by Lemma 2(b). Thus, as before, the set of all points $y \in \mathbf{R}^2$ whose images through p are such points x is residual in $\mathbf{R}^2 \setminus K$. To prove the theorem it suffices to show that, in each such y, p has no directional derivatives in both directions τ and $-\tau$.

Choose arbitrarily one of the directions τ and $-\tau$, say τ. By Lemma 1, we can find $h_n \to 0+$ such that

$$\lim_{n\to\infty} \frac{p(y + h_n\tau) - p(y)}{h_n} = \tau.$$

In the proof of Theorem 2, property (ii) guaranteed by Lemma 2(c) enabled us to find a tangent direction τ_0 at x such that

$$\lim_{n\to\infty} \frac{p(y + k_n\tau_0) - p(y)}{k_n} = \mathbf{0}$$

for suitable $k_n \to 0+$. But, in our case, τ_0 could happen to be $-\tau$.

Thus, a different argument is needed here.

We see that $\rho_i^{\pm\tau}(x) = 0$ implies the existence of a sequence $\{z_n\}_{n=1}^{\infty}$ of points on N_i converging to x and of a sequence $\{x_n'\}_{n=1}^{\infty}$ of points on bd K, converging from both sides to x ($x_n' \neq x$), such that $\| x - z_n \| = \| x_n' - z_n \|$.

For every n, $\rho_i^{\pm\tau}(x) = 0$ yields

$$\min_{v\in A_n} \| z_n - v \| = \| z_n - x_n \|$$

for some point x_n in the relative interior of A_n, where A_n is the subarc of bd K from x_n' to x. Hence $L(z_n, x_n)$ is normal to bd K, and $x_n \to x$ from both sides.

Now, as in the proof of Theorem 2, let $\{y_n\} = L(x_n, y_n) \cap \Xi$, where Ξ is the line through y orthogonal to $y - x$. Since $x_n \to x$ from both sides, we are able to find a subsequence of $\{x_n\}_{n=1}^{\infty}$ such that, for all corresponding indices n,

$$R(y, y_n) = \{y + h\tau : h \geqq 0\}.$$

Thus τ_0 from the proof of Theorem 2 equals τ and the rest of the argument follows the proof of Theorem 2.

For reflection properties of typical convex curves see [8].

Theorem 4. *For a typical planar convex body K, $p' = \mathbf{0}$ at a set of points dense in $\mathbf{R}^2 \setminus K$.*

Proof. The set G of all points $x \in \text{bd}\, K$ such that $\rho_i^{\pm\tau}(x) = \rho_s^{\pm\tau}(x) = 0$ (for both tangent directions τ and $-\tau$) is dense on bd K, by Lemma 3. Clearly, the set of all $y \in \mathbf{R}^2$ with $p(y) \in G$ is dense in $\mathbf{R}^2 \setminus K$.

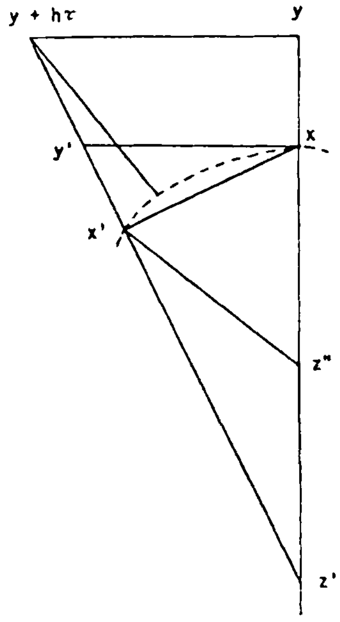

Fig. 2.

Consider such a point y, $x = p(y)$, a unit vector τ orthogonal to $x - y$, the point $y + h\tau$ for some $h > 0$, and the point $x' \in \text{bd}\, K$ such that $y + h\tau - x'$ and $x - x'$ are orthogonal (see Fig. 2). Obviously, $p(y + h\tau)$ lies between x' and x on bd K and $\| x - p(y + h\tau) \| < \| x - x' \|$. Also, $\| x - x' \| < \| x - y' \|$, where y' is the intersection of $L(x', y + h\tau)$ with the supporting line of K at x. Let

$$\{z'\} = L(x', y + h\tau) \cap L(x, y)$$

and $z'' \in L(x, y)$ be such that $\| z'' - x \| = \| z'' - x' \|$. Since $\rho_i^\tau(x) = \rho_s^\tau(x) = 0$, we have $z'' \to x$ for $h \to 0$. Then $z' \to x$ too, whence

$$\frac{\| p(y + h\tau) - p(y) \|}{h} < \frac{\| x - y' \|}{\| y - (y + h\tau) \|} = \frac{\| x - z' \|}{\| y - z' \|} \to 0.$$

It follows that $p'(y) = \mathbf{0}$.

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