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GENERIC PROPERTIES OF COMPACT STARSHAPED SETS

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ABSTRACT. A typical compact starshaped set in $\mathbf{E}^d$ is "small" from the topological as well as from the measure theoretic viewpoint. We formulate this more explicitly in the paper by using the notions of porosity and Hausdorff dimension. Moreover, we see that the directions of the line segments in a typical compact starshaped set are many, but not too many.

1. INTRODUCTION

Our aim is to obtain deeper insight into the structure of typical compact *starshaped sets* in d-dimensional euclidean space $\mathbf{E}^d$; these are subsets S of $\mathbf{E}^d$ admitting a point $k \in S$ such that, for any $x \in S$, the line segment kx is contained in S.

A topological space is called *Baire* if the complement of any subset of first category is dense, where a *subset of first* (*Baire*) *category* is a countable union of nowhere dense sets. They are also called *meager sets.* By Baire's category theorem any complete metric space is Baire. When speaking of *most* or of *typical* elements we mean all elements except those in a meager set. A property shared by most elements of a Baire space is called *generic* [6], [7]. For results on generic properties in convexity the interested reader is referred to [4], [9]

Let St denote the space of all compact starshaped sets in E^d endowed with its common topology which is induced, for example, by the Hausdorff metric δ^H. A version of the Blaschke selection theorem implies the completeness of St with respect to δ^H.

Starshaped sets have attracted some interest in combinatorial geometry, geometry of numbers and other areas. In [10] several generic properties of compact starshaped sets were derived, among them the following: Most compact starshaped sets S are nowhere dense, have a *single-point kernel* $\{k\}$, i.e. k is the unique point such that kx is contained in S for any $x \in S$, and have a dense set of directions determined by the line segments kx.

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In §2 it will be shown that the set of directions determined by the line segments in a typical compact starshaped set S is an uncountable subset of S^{d-1} of first category.

Most compact starshaped sets are "quite dense" at their single point kernels and "quite thin" at any other point. This is expressed in §3 in a more precise way using the concept of porosity.

By means of a result on the irregularity of approximation in [3] it will be shown in §4 that most compact starshaped sets have Hausdorff dimension 1 while they are of non-σ-finite 1-dimensional Hausdorff measure.

We mention that generic properties of starshaped sets with higher dimensional kernels have been studied in [11], while the Hausdorff dimension and the corresponding Hausdorff measure for typical compacta, continua and curves are determined in [5].

2. The set of directions of a typical compact starshaped set

If a compact starshaped set S in $\mathbf{E}^d$ has a single-point kernel $\{k\}$ its *set of directions* $D(S)$ in S^{d-1} is defined by

$$D(S) = \left\{ \frac{x-k}{\|x-k\|} : x \in S \setminus \{k\} \right\}$$

where $\| \ \|$ denotes the Euclidean norm on $\mathbf{E}^d$.

Theorem 1. *Most compact starshaped sets in* $\mathbf{E}^d$ *have a single-point kernel and their set of directions is a dense subset of first category of* S^{d-1}, *of cardinality* $\mathfrak{c}$.

An elementary argument leads to the following:

$$(1) \quad \left\{ \begin{array}{l} \text{Let } X \text{ be a Baire space and assume that } Y \text{ is a subspace of } X \\ \text{containing most elements of } X. \text{ Then } Y \text{ is also Baire and if} \\ Z \subset Y \text{ contains most elements of } Y \text{ then it also contains most} \\ \text{elements of } X. \end{array} \right.$$

Let S be the subspace of St consisting of all $S \in \mathrm{St}$ with a single-point kernel. The Corollary in [10] says that

(2) S contains most $S \in \mathrm{St}$

and by [10, Theorem 2]

(3) for most $S \in \mathrm{S}$ the set of directions $D(S)$ is dense in S^{d-1}.

Proof of Theorem 1. We denote by (x, y) the line through x, $y \in \mathbf{E}^d$ $(x \neq y)$. For $S \in \mathrm{S}$ and $m = 1, 2, \ldots,$ let $\{k\}$ be the kernel of S and define

$$D_m(S) = \left\{ \frac{x-k}{\|x-k\|} : x \in S, \|x-k\| \geq \frac{1}{m} \right\}.$$

For $m = 1, 2, \ldots$ and $n = m+1, m+2, \ldots$, let

$$\mathrm{S}_{mn} = \{S \in \mathrm{S}: \exists p \in S, \|p-k\| = \frac{1}{m}, S \cap \{x: \|x-p\| < \frac{1}{n}, \|x-k\| > \frac{1}{m}\} \subset (p,k)\}.$$

Simple arguments concerning convergence in St imply that

$$\mathrm{S}_{mn} \text{ is closed in } \mathrm{S}. \tag{4}$$

In order to prove that

$$\mathrm{S}_{mn} \text{ has empty interior in } \mathrm{S}, \tag{5}$$

assume the contrary. Since the collection of all starshaped sets consisting of finitely many line segments issuing from the single-point kernel is dense in S, there is a set of this type interior to S_{mn}. By suitably adding finitely many line segments to it, if necessary, we obtain a starshaped set $S \in \mathrm{S}_{mn}$ with the following property: The line segments in S of length at least $1/m$ appear in pairs such that for any such pair $\{s,t\}$ (i) each of the line segments s, t has length larger than $1/m$ and (ii) the endpoints of s and t distinct from the single-point kernel have distance less than $1/n$. Hence $S \notin \mathrm{S}_{mn}$. This contradiction concludes the proof of (5).

The definition of D_m and S_{mn} together with (4), (5) imply that

$$\left\{\begin{array}{l}\{S \in \mathrm{S}: D_m(S) \text{ contains an isolated point}\} \subset \bigcup_{n=m+1}^{\infty} \mathrm{S}_{mn} \text{ is of} \\ \text{first category in S for } m = 1,2,\ldots.\end{array}\right. \tag{6}$$

For $m = 1,2,\ldots, n = m+1, m+2, \ldots$, let

$$O_{mn} = \{S \in \mathrm{S}: D_m(S) \text{ contains a component of diameter } \geq \frac{1}{n}\}.$$

It is easy to show that

$$\mathrm{T}_{mn} \text{ is closed and has empty interior in } \mathrm{S}.$$

Hence

$$\left\{\begin{array}{l}\{S \in \mathrm{S}: D_m(S) \text{ is not totally disconnected}\} \subset \bigcup_{n=m+1}^{\infty} \mathrm{T}_{mn} \text{ is of} \\ \text{first category in S for } m = 1,2,\ldots.\end{array}\right. \tag{7}$$

A nonempty totally disconnected compact set without isolated points in $\mathbf{E}^d$ is homeomorphic to the Cantor discontinuum and thus has cardinality $\mathfrak{c}$, see e.g.(cardinality of continuum) [1, p. 121]. Note also that a totally disconnected compact set in $\mathbf{E}^d$ is nowhere dense. Thus (6) and (7) yield that

$$\left\{\begin{array}{l}\text{for most } S \in \mathrm{S} \text{ the set } D_m(S) \text{ either is empty or has cardinality} \\ \mathfrak{c} \text{ and is nowhere dense in } S^{d-1}.\end{array}\right. \tag{8}$$

Clearly,

$$D(S) = \bigcup_{m=1}^{\infty} D_m(S) \qquad \text{for each } S \in \mathrm{S} \tag{9}$$

and

(10) $\begin{cases} \text{the compact starshaped sets consisting of single points only form} \\ \text{a closed nowhere dense subset of } \mathrm{S}. \end{cases}$

(8), (9) and (10) together show that

(11) $\begin{cases} \text{for most } S \in \mathrm{S} \text{ the set of directions } D(S) \text{ has cardinality } \mathfrak{c} \text{ and} \\ \text{is of first category in } S^{d-1}. \end{cases}$

Now Theorem 1 follows from (3), (11), (2), and (1).

3. Porosity properties of typical compact starshaped sets

The *porosity* of a subset S of $\mathbf{E}^d$ at a point $x \in S$ is defined as

$$\limsup_{\varepsilon \to 0+} \frac{\rho(\varepsilon)}{\varepsilon}$$

where $\rho(\varepsilon)$ is the radius of the largest (solid, open, euclidean) ball disjoint from S, whose centre is at distance at most ε from x. The set S is called *strongly porous*, respectively *nonporous*, at $x \in S$ if its porosity at x is 1, respectively 0.

Theorem 2. *Most compact starshaped sets in* $\mathbf{E}^d$ *have a single-point kernel at which they are nonporous, and are strongly porous anywhere else.*

Proof. We first prove that

(12) most $S \in \mathrm{S}$ are nonporous at their kernels.

By (3) it suffices for the proof of (12) to verify the following proposition:

(13) $\begin{cases} \text{Let } S \in \mathrm{S} \text{ be such that } D(S) \text{ is dense in } S^{d-1}. \text{ Then } S \text{ is non-} \\ \text{porous at its kernel } \{k\}. \end{cases}$

Assume this is not true. Then there is a $\rho > 0$ such that there are balls with centres y arbitrarily close to k and radii $\rho\|k - y\|$ disjoint from S. Since $D(S)$ is dense in S^{d-1} we may choose $d_1, \dots, d_n \in D(S)$ such that any ball with centre in S^{d-1} and radius ρ intersects $\{d_1, \dots, d_n\}$. Let $kp_1, \dots, kp_n$ be segments in S having directions $d_1, \dots, d_n$ respectively and let σ be the smallest length of such a segment. Then for any point y with $\|y - k\| \le \sigma$ the ball $B(y, \rho\|y - k\|)$ with centre y and radius $\rho\|y - k\|$ intersects at least one of these segments. This contradiction proves (13) and thus settles (12).

For l, m, $n = 1, 2, \dots$, let

$$\mathrm{S}_{lmn} = \{S \in \mathrm{S} \colon \exists x \in S \text{ with } \|x - k\| \ge \frac{1}{l}, \text{such that } \forall y \text{ with } \|x - y\| \le \frac{1}{m}, \; B(y, (1 - \frac{1}{n})\|x - k\|) \cap S \ne \varnothing\}.$$

It is a simple matter to show that

$$\mathrm{S}_{lmn} \text{ is closed and has empty interior in } \mathrm{S}.$$

Hence

$$(14)\qquad \begin{cases} \{S \in \mathrm{S}\colon \exists x \in S \setminus \{k\} \text{ such that } S \text{ is not strongly porous at } x\} \\ \subset \bigcup_{l,m,n=1}^{\infty} \mathrm{S}_{lmn} \text{ is of first category in } \mathrm{S}. \end{cases}$$

Propositions (12), (14), (2), and (1) together imply Theorem 2.

4. Hausdorff measure and Hausdorff dimension of typical compact starshaped sets

Let $\alpha \geq 0$. The *α-dimensional Hausdorff measure* $\mu_\alpha(S)$ of a subset S of $\mathbf{E}^d$ is defined by

$$\mu_\alpha(S) = \lim_{\varepsilon \to +0} \inf \left\{ \sum_{k=1}^{\infty} (\operatorname{diam} U_k)^\alpha \colon U_k \subset \mathbf{E}^d,\ \operatorname{diam} U_k \leq \varepsilon,\ S \subset \bigcup_{k=1}^{\infty} U_k \right\},$$

where diam denotes diameter. A set S has *σ-finite μ_α-measure* if it can be represented as a countable union of sets of finite μ_α-measure. For any $S \subset \mathbf{E}^d$ there is a unique number δ, $0 \leq \delta \leq d$, called the *Hausdorff dimension* of S such that $\mu_\alpha(S) = +\infty$ for $\alpha < \delta$ and $\mu_\alpha(S) = 0$ for $\alpha > \delta$ [2], [8].

Theorem 3. *Most compact starshaped sets in $\mathbf{E}^d$ have non-σ-finite 1-dimensional Hausdorff measure but are still of Hausdorff dimension 1.*

Proof. We first derive the following simple lemma:

(15) Suppose that in a measure space with measure μ a measurable set M is an uncountable union of disjoint measurable sets of positive measure, say $M = \bigcup\{M_i\colon i \in I\}$. Then M cannot be represented as a countable union of measurable sets of finite measure.

Assume this is not true. Then there are measurable sets N_n with

$$(16)\qquad \mu(N_n) < \infty \qquad \text{for } n = 1, 2, \ldots$$

and

$$M = \bigcup_{n=1}^{\infty} N_n.$$

For each $i \in I$ we have that

$$M_i = \bigcup_{n=1}^{\infty} (M_i \cap N_n)$$

and thus

$$\sum_{n=1}^{\infty} \mu(M_i \cap N_n) \geq \mu(M_i) > 0.$$

Hence for each $i \in I$ there is an n with $\mu(M_i \cap N_n) > 0$. The uncountability of I then implies that $\mu(M_i \cap N_{n_0}) > 0$ for uncountably many i's and a fixed

n_0. Thus there is an $\alpha > 0$ such that $\mu(M_i \cap N_{n_0}) \geq \alpha$ for uncountably many i's. This shows that N_{n_0} contains countably many disjoint measurable sets of measure $\geq \alpha$. Therefore $\mu(N_{n_0}) = \infty$, in contradiction to (16), which concludes the proof of (15).

In fact μ_1 is not a measure but a metric outer measure of $\mathbf{E}^d$. Hence all Borel sets and thus in particular all line segments are μ_1-measurable, see [2, Theorem 1.5]. These remarks together with (15) show that

(17) any $S \in \mathrm{St}$ which can be represented as disjoint union of uncountably many line segments of positive length has non-σ-finite μ_1-measure.

By Theorem 1 most $S \in \mathrm{St}$ consist of uncountably many line segments of positive lengths. This combined with (17) settles the first part of Theorem 3.

The proof of the second part is similar to the proof of [3, Theorem 2]. It is based on two propositions. The first one is taken from [3, Theorem 1].

(18) Let $\alpha_1, \alpha_2, \dots,$ be positive reals and $\varphi_1, \varphi_2, \dots,$ nonnegative upper semicontinuous real functions on a Baire space X such that

$$\{x \in X : \varphi_n(x) = o(\alpha_n) \text{ as } n \to \infty\}$$

is dense in X. Then for most $x \in X$ the inequality $\varphi_n(x) < \alpha_n$ holds for infinitely many indices n.

For $\varepsilon > 0$ and $S \in \mathrm{St}$, let $M_\varepsilon(S)$ be the maximum number of points in S with pairwise distances at least ε. Using the fact that (St, δ^H) is a closed subspace of the space of all compact subsets of $\mathbf{E}^d$ endowed with the metric δ^H, the proof of [3, Theorem 2] then yields the following proposition:

(19) For $\varepsilon > 0$ fixed, the function M_ε is upper semicontinuous on St.

This is the second tool needed.

Fix $\tau > 0$ and choose a sequence $0 < \alpha_1 < \alpha_2 < \cdots$, for which

$$n = o(\alpha_n), \qquad \alpha_n = o(n^{1+\tau}) \qquad \text{as } n \to \infty. \tag{20}$$

We now show that

(21) for most $S \in \mathrm{St}$ the inequality $M_{1/n}(S) < \alpha_n$ holds for infinitely many n.

The compact starshaped sets consisting of finitely many line segments form a set dense in St and for any such starshaped set S we clearly have $M_{1/n}(S) = O(n)$. Hence by (20)

$$\{S \in \mathrm{St} : M_{1/n}(S) = o(\alpha_n) \text{ as } n \to \infty\} \text{ is dense in St.}$$

Since St is Baire, this combined with (19) and (18) yields (21).

The next proposition required is the following:

$$(22) \qquad \begin{cases} \text{Let } S \in \text{St satisfy } M_{1/n}(S) < \alpha_n \text{ for infinitely many } n. \text{ Then} \\ \mu_{1+\tau}(S) = 0. \end{cases}$$

Choose ε_1, $\varepsilon_2 > 0$. Since $M_{1/n}(S) < \alpha_n$ for infinitely many n, by (20) we may choose an n for which

$$(23) \qquad 2/n < \varepsilon_1, \qquad \alpha_n (2/n)^{1+\tau} < \varepsilon_2, \qquad M = M_{1/n}(S) < \alpha_n.$$

By the definition of M there is a maximal system of points in S with mutual distances not less than $1/n$ and consisting of precisely M points. The balls $B_1, \ldots, B_M$ of radius $1/n$ with centres at the points of our maximal system cover S. (Otherwise there is a point in S having distance larger than $1/n$ from each point of the maximal system. Hence the latter cannot be maximal.) Then (23) implies that

$$\begin{aligned} \inf &\left\{ \sum_{k=1}^{\infty} (\operatorname{diam} U_k)^{1+\tau} : U_k \subset \mathbf{E}^d, \operatorname{diam} U_k \leq \varepsilon_1, S \subset \bigcup_{k=1}^{\infty} U_k \right\} \\ &\leq \sum_{k=1}^{M} (\operatorname{diam} B_k)^{1+\tau} \\ &= M_{1/n}(S)\,(2/n)^{1+\tau} \\ &< \alpha_n (2/n)^{1+\tau} < \varepsilon_2. \end{aligned}$$

Since ε_1, $\varepsilon_2 > 0$ were arbitrary, the definition of $\mu_{1+\tau}$ shows that $\mu_{1+\tau}(S) = 0$, concluding the proof of (22).

From (21) and (22) the following assertion results.

If $\tau > 0$ then $\mu_{1+\tau}(S) = 0$ for most $S \in \text{St}$.

Applying this for $\tau = 1, 1/2, 1/3, \ldots,$ we see that, for most $S \in \text{St}$, $\mu_{1+1/n}(S)$ vanishes for any n. In other words, the Hausdorff dimension of most compact starshaped $S \in \text{St}$ is at most 1. Since, by the first part of Theorem 3, most $S \in \text{St}$ have Hausdorff dimension at least 1, this concludes the proof of the second part of Theorem 3.

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