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The nearest point mapping is single valued nearly everywhere

By

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0. Introduction. The function $p_K: \mathbb{R}^d \to 2^{\mathbb{R}^d}$ called *metric projection* or *nearest point mapping* is well-known: For a given closed set $K \subset \mathbb{R}^d$, p_K associates to each $x \in \mathbb{R}^d$ the set of all points of K closest to x. It is known that p_K is single valued almost everywhere and at most points of $\mathbb{R}^d$ ([2]), i.e. p_K is not single valued on a set of measure zero and first Baire category. We shall prove here that p_K is single valued nearly everywhere, i.e. p_K is not single valued on a σ-porous set, which implies both preceding assertions. We also establish that, for most compact sets K, p_K is not single valued at densely many points. This will not happen, however, if the boundary of K is smooth enough, as we shall see in the last section.

1. Definitions. For $x \in \mathbb{R}^d$, $\varrho > 0$, let $B(x, \varrho)$ denote the open ball $\{y \in \mathbb{R}^d : \|x - y\| < \varrho\}$. We call a set $M \subset \mathbb{R}^d$

- *porous at* $x \in X$ if there is an $\alpha > 0$ such that, for any $\varrho > 0$, there exists $y \in B(x, \varrho)$ satisfying $B(y, \alpha \|x - y\|) \cap M = \emptyset$ ([1]),
- *porous* if it is porous at all points of $\mathbb{R}^d$,
- *σ-porous* if it is a countable union of porous sets.

If a property is shared by all elements of a metric Baire space except those in a σ-porous set, then we say that *nearly all* of them enjoy it ([3]).

2. The general case.

Theorem 1. *The nearest point mapping is single valued at nearly all points of* $\mathbb{R}^d$.

Proof. We first show that for any $\varepsilon > 0$, the set

$$D_\varepsilon = \{x \in \mathbb{R}^d : \operatorname{diam} p_K(x) \geqq \varepsilon\}$$

is porous.

Let $x \in \mathbb{R}^d$ and consider the point $y \in p_K(x)$. For any point $z \in \operatorname{int} xy$, we show that

$$B\left(z, \frac{\varepsilon \|x - z\|}{2 \|x - y\|}\right) \cap D_\varepsilon = \emptyset.$$

Indeed, suppose on the contrary u lies in the above intersection; then

$$p_K(u) \subset B(u, \|u - y\|) \backslash B(x, \|x - y\|).$$

To estimate the diameter Δ of the right side, let v be the point of the line through x and u closest to y. Then clearly

$$\|v - y\| \leqq \frac{\|u - z\| \cdot \|x - y\|}{\|x - z\|} < \frac{\varepsilon}{2}.$$

Obviously $\Delta = 2\|v - y\|$. Hence diam $p_K(u) < \varepsilon$, which contradicts $u \in D_\varepsilon$. Hence D_ε is porous.

Thus p_K is not single valued precisely at the points of $\bigcup_{n=1}^{\infty} D_{n^{-1}}$, which is a σ-porous set. This proves the theorem.

3. The typical case. It is well-known that the space $\mathscr{K}$ of all compact sets in $\mathbb{R}^d$ endowed with the Hausdorff metric δ is a Baire space. The next result describes the typical aspect of p_K.

Theorem 2. *For most compact sets $K \subset \mathbb{R}^d$, the nearest point mapping p_K is not single valued at a dense set of points.*

Proof. Let B_0 be a fixed open ball of centre b and radius r in $\mathbb{R}^d$, and let

$$\mathscr{K}_{b,r} = \{K \in \mathscr{K} : p_K \text{ is single valued at all points of } B_0\}.$$

We show that $\mathscr{K}_{b,r}$ is nowhere dense. Consider any $\varepsilon \in (0, r)$ and any compact set $K \in \mathscr{K}$. Take a point $y \in K \backslash B(b, \varepsilon/2)$ closest to b (or take any $y \in \mathbb{R}^d$ with $\|b - y\| = \varepsilon/2$ if $K \subset B(b, \varepsilon/2)$) and consider an equilateral triangle $y y_1 y_2$ such that

$$\|b - y_1\| = \|b - y_2\| < \|b - y\|.$$

If its side length is $\varepsilon/3$, then $K' = (K \backslash B(b, \varepsilon/2)) \cup \{y, y_1, y_2\}$ satisfies $\delta(K, K') \leqq \varepsilon$. Let y_1', y_2' be the points of B_0 such that $b \in y_1' y_2'$ and $y_1 y_2 y_2' y_1'$ is a rectangle, and put $v = (\|y_1' - y_2\| - \|y_1' - y_1\|)/2$.

Take now any $L \in \mathscr{K}$ with $\delta(K', L) < v$. We prove that $L \notin \mathscr{K}_{b,r}$. Let $L_i = L \cap B(y_i, v)$ $(i = 1, 2)$. Some elementary calculations show that, if $x \in y_1' y_2'$, then $p_L(x) \subset L_1 \cup L_2$. If $z_{i,j}$ is a point in L_i closest to y_j' $(i, j = 1, 2)$, then

$$\|y_1' - z_{1,1}\| < \|y_1' - y_1\| + v = \|y_1' - y_2\| - v < \|y_1' - z_{2,1}\|$$

and, analogously,

$$\|y_2' - z_{2,2}\| < \|y_2' - z_{1,2}\|.$$

Since the distance from x to L_i is a continuous function of x $(i = 1, 2)$, there is some point $y' \in y_1' y_2'$ with equal distances to L_1 and L_2. Thus p_K is not single valued at $y' \in B_0$, whence $L \notin \mathscr{K}_{b,r}$.

Therefore $\mathscr{K}_{b,r}$ is nowhere dense. If we now let b be any point with rational coordinates and r be rational, then the union of all these $\mathscr{K}_{b,r}$ is of first category. Since every open set in $\mathbb{R}^d$ includes a ball $B(b, r)$ with b and r as above, for most $K \in \mathscr{K}$, p_K is not single valued at densely many points.

4. The case of a planar set with analytic boundary. In the preceding section we saw that p_K may not be single-valued at a dense set of points. However, we prove now that, if $\operatorname{bd} K$ is smooth enough, this can no longer happen. Although the next result is true in any dimension, for simplicity reasons we shall restrict ourselves to the planar case.

Theorem 3. *If $K \subset \mathbb{R}^2$ is closed and $\operatorname{bd} K$ is an analytic Jordan curve then p_K is not single-valued on a nowhere dense set.*

Proof. Let $x_0 \in \mathbb{R}^2$ and $\varepsilon > 0$. Choose a point $y_0 \in p_K(x_0)$. Consider the curvature $\gamma(y_0)$ of $\operatorname{bd} K$ positive if y_0 separates x_0 from the centre of curvature in y_0. Then clearly $\gamma(y_0) \geqq -\|x_0 - y_0\|^{-1}$.

Equip $\gamma(y)$ with a sign for all $y \in bdK$ in a way agreeing with the case $y = y_0$.

If $\operatorname{bd} K$ is a circle or a line, then p_K is not single valued at one point at most. So we suppose $\operatorname{bd} K$ is neither a circle, nor a line. Then, since $\operatorname{bd} K$ is analytic, there is an arc $\widehat{y_0 y^*} \subset \operatorname{bd} K$ on which γ is strictly monotone.

We may suppose, of course, that $\|y_0 - y^*\| < \|x_0 - y_0\|$ and that the whole arc $\widehat{y_0 y^*}$ lies below the line L through x_0 and y_0. Since $\operatorname{bd} K$ is an analytic Jordan curve, every arc of $\operatorname{bd} K$ reaching y_0 from the half space H below L includes or is included in $y^* y_0$. Since $\operatorname{bd} K$ is locally connected, there is a number $\alpha > 0$, such that every point of $\operatorname{bd} K$ at distance at most α from y_0 can be joined with y_0 by an arc of $\operatorname{bd} K$ of diameter less than $\|y_0 - y^*\|$.

Let $z \in \operatorname{int} x_0 y_0$ and

$$\varrho = \frac{\alpha \|x_0 - z\|}{2 \|x_0 - y_0\|}.$$

We claim that p_K is single valued on $B(z, \varrho) \cap H$. To show this, let $x \in B(z, \varrho) \cap H$. Put

$$D = B(x, \|x - y_0\|) \backslash B(x_0, \|x_0 - y_0\|).$$

Clearly, $p_K(x) \subset \bar{D}$ and $\operatorname{diam} D \leqq \alpha$. Hence $y^* \notin \bar{D}$.

We show now that $p_K(x) \subset \widehat{y_0 y^*}$. Let $u \in p_K(x)$. Any arc $\Gamma \subset \operatorname{bd} K$ joining u with y_0 either meets $\widehat{y_0 y^*} \backslash \{y_0\}$ or surrounds $\widehat{y_0 y^*}$ or surrounds $B(x_0, \|x_0 - y_0\|)$. In both latter cases, Γ would have length larger than $\|y_0 - y^*\|$, whence Γ must meet $\widehat{y_0 y^*} \backslash \{y_0\}$. Obviously, this together with the fact that $\operatorname{bd} K$ is a Jordan curve and with $y^* \notin \bar{D}$ implies $\Gamma \subset \widehat{y_0 y^*}$. Hence $u \in \widehat{y_0 y^*}$ and it is proved that $p_K(x) \subset \widehat{y_0 y^*}$.

Suppose two distinct points $y_1, y_2 \in p_K(x)$. Then, clearly,

$$\gamma(y_1) \geqq -\|x - y_1\|^{-1},$$

$$\gamma(y_2) \geqq -\|x - y_1\|^{-1}.$$

Since

$$\int_{\widehat{y_1 y_2}} \gamma(y)\, dy$$

equals the angle $y_1 x y_2$ (equipped with a sign as well, in an obvious way), but would be larger if

$$\gamma(y) > -\|x - y_1\|^{-1}$$

for all $y \in \operatorname{int} \widehat{y_1 y_2}$, we must have

$$\gamma(y_3) \leqq -\|x - y_1\|^{-1}$$

for some point y_3 between y_1 and y_2. But this contradicts the strict monotony of γ on $\widehat{y_0 y^*}$. Hence $p_K(x)$ consist of one point only. Thus p_K is single valued in $B(z, \varrho) \cap H$, and the proof is finished.

References

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