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HAMILTONIAN PROPERTIES OF GRID GRAPHS*

CHRISTINA ZAMFIRESCU† AND TUDOR ZAMFIRESCU‡

Abstract. This paper presents sufficient conditions for a grid graph to be Hamiltonian. It is proved that all finite grid graphs of positive width have Hamiltonian line graphs.

Key words. Hamiltonian graph, infinite Hamiltonian path, grid graph, line graph

AMS(MOS) subject classification. 05C45

1. Introduction. The interest in subgraphs of the graph determined by the square lattice in $\mathbb{R}^2$ is mainly explained by the importance of the lattice itself, due, in turn, to its frequent occurrence in various mathematical fields. Hamiltonian properties of such graphs have been previously examined by several authors (see, for example, [2]–[6]). Their relevance for applications (such as automatic moves in a warehouse) is obvious.

Here we use the following (more restrictive) notion of a grid graph. Let $\mathcal{G}$ be the infinite graph embedded in $\mathbb{R}^2$, with $\mathbb{Z}^2$ as vertex set and with an edge between any two vertices at Euclidean distance 1. A *grid graph* is a connected subgraph of $\mathcal{G}$, whose intersection with any infinite path of vertex set $\{\cdot\} \times \mathbb{Z}$ or $\mathbb{Z} \times \{\cdot\}$ is connected or empty. For two graphs A, B, the graph $A - B$ denotes as usual, the graph obtained from A by deleting every vertex of B that is in A and any edge of A incident to such a vertex. The union of the boundaries of all unbounded domains of $\mathbb{R}^2$ determined by a grid graph G is called *boundary* of G. Any vertex of degree 2 in a grid graph is called a *corner*. For a connected graph G we also define its *width* $w(G)$ as the smallest number n for which there exists a path P of length n in G such that $G - P$ is disconnected and the diameter of each component of $G - P$ is at least n ($n = 0$ allowed). Note that not all connected graphs and not even all grid graphs possess a width. However, all finite grid graphs except P_1, P_2, and C_4—which for our purposes present little interest and will henceforth be excluded—do have a width. A *tour* in a finite graph G is a finite sequence of not necessarily distinct vertices

$$x_1, x_2, x_3, \cdots, x_n, x_{n+1} = x_1$$

such that all $\{x_i, x_{i+1}\}$'s are distinct edges of G ($1 \leqq i \leqq n$) and every edge of G has at least one endpoint in the sequence.

In this paper, we present sufficient conditions for a grid graph to be Hamiltonian. Also, we prove that all finite grid graphs of positive width have Hamiltonian line graphs.

2. Hamiltonian finite grid graphs. Let G be a finite grid graph of positive width. This restriction on the width is natural because every graph with vanishing width has connectivity 1 and is therefore non-Hamiltonian.

Let

$$W = \min\{x : (x, y) \in G\}, \qquad E = \max\{x : (x, y) \in G\},$$

$$S = \min\{y : (x, y) \in G\}, \qquad N = \max\{y : (x, y) \in G\}.$$

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† Department of Computer Science, Hunter College of the City University of New York, 695 Park Avenue, New York, New York 10021. This author's research was supported in part by the City University of New York, PSC/CUNY Research Award Program.

‡ Fachbereich Mathematik, Universität Dortmund, Postfach 500500, 4600 Dortmund 50, Germany.

The corners of G, considered in their natural order on the boundary of G, are

$$(x_1, y_1), \cdots, (x_n, y_n),$$
$$(x_{n+1}, y_{n+1}), \cdots, (x_e, y_e),$$
$$(x_{e+1}, y_{e+1}), \cdots, (x_s, y_s),$$
$$(x_{s+1}, y_{s+1}), \cdots, (x_w, y_w),$$

where $x_1 = x_w = W$, $y_n = y_{n+1} = N$, $x_e = x_{e+1} = E$, $y_s = y_{s+1} = S$. For all indices i, except $n + 1$ and $s + 1$, the above vertices (x_i, y_i) will be called *special* and receive *auxiliary coordinates* (ξ_i, η_i) as follows:

$$\xi_i = \begin{cases} x_{i+1} - x_i & \text{for } i \leqq n - 1 \text{ or } e + 1 \leqq i \leqq s - 1, \\ x_i - x_{i-1} & \text{for } n + 2 \leqq i \leqq e \text{ or } i \geqq s + 2, \\ x_{i+1} - x_i + 1 & \text{for } i = n \text{ or } s; \end{cases}$$

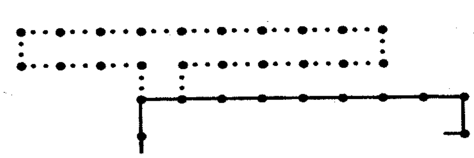
$$\eta_i = \begin{cases} y_i + 1 & \text{for } i \leqq e, \\ y_i & \text{for } i \geqq e + 1. \end{cases}$$

THEOREM 1. *Let G be a finite grid graph with $w(G) > 2$. If at every special vertex the product of its auxiliary coordinates is even, then G is Hamiltonian.*

Proof. First, we allow G to have smaller width, but still assume that $w(G) > 0$. Also, we assume that η is even at all special vertices, of auxiliary coordinates (ξ, η). Consider the height $h = N - S$. Then, clearly, h is odd. We prove that G has a Hamiltonian circuit containing the path on $y = N$ by induction on h. Indeed, the assertion is true if $h = 1$. Now suppose it is true for height h; we prove it for height $h + 2$. There are three possible situations at the left end of the path in G with $y = N$, shown below:

$(y_{n-1} < N - 2$ or $n = 1$ and $y_1 - y_w > 1)$ $(y_{n-1} = N - 2)$ $(n = 1$ and $y_1 - y_w = 1)$

There are three analogous situations at the right end of the path. Consider now the subgraph H of G spanned by the vertices (x, y) with $y \leqq N - 2$. H satisfies the requirements of our induction hypotheses, as we can easily verify. So there is a Hamiltonian circuit in H, which can be transformed into one of G—here we also use $w(G) > 0$—as the following illustration shows:



Now, to prove the theorem in its general form, let $I = \{i : \eta_i \text{ is odd}\}$ and denote by Q the set of all points (x, y_i) with $i \in I$ and

$$\begin{aligned} x_i \leqq x < x_{i+1} &\quad \text{if } i<n, \\ x_n \leqq x \leqq x_{n+1} &\quad \text{if } i=n, \\ x_{i-1} < x \leqq x_i &\quad \text{if } n+2 \leqq i \leqq e, \\ x_{i+1} < x \leqq x_i &\quad \text{if } e<i<s, \\ x_{s+1} \leqq x \leqq x_s &\quad \text{if } i=s, \\ x_i \leqq x < x_{i-1} &\quad \text{if } i \geqq s+2. \end{aligned}$$

It is readily seen that $G - Q$ is a graph G' with $w(G') > 0$ and η even at all special vertices. Thus it is Hamiltonian, and (from the proof) it is clear that the constructed Hamiltonian circuit contains all of the paths with vertex sets, respectively,

$$\begin{aligned} &\{(x, y_i - 1) : x_i \leqq x < x_{i+1}\} &&\text{if } i \in I \text{ and } i<n, \\ &\{(x, y_n - 1) : x_n \leqq x \leqq x_{n+1}\} &&\text{if } n \in I, \\ &\{(x, y_i - 1\} : x_{i-1} < x \leqq x_i\} &&\text{if } i \in I \text{ and } n+2 \leqq i \leqq e, \\ &\{(x, y_i + 1) : x_{i+1} < x \leqq x_i\} &&\text{if } i \in I \text{ and } e<i<s, \\ &\{(x, y_s + 1) : x_{s+1} \leqq x \leqq x_s\} &&\text{if } s \in I, \\ &\{(x, y_i + 1) : x_i \leqq x < x_{i-1}\} &&\text{if } i \in I \text{ and } i \geqq s+2. \end{aligned}$$

Then it suffices to change each of them appropriately to obtain a Hamiltonian circuit in G. For instance, the path with consecutive vertices

$$(x_i, y_i - 1), (x_i + 1, y_i - 1), \cdots, (x_{i+1} - 1, y_i - 1),$$

where $i \in I$ and $i < n$, will be replaced by the path with consecutive vertices

$$(x_i, y_i - 1), (x_i, y_i), (x_i + 1, y_i), (x_i + 1, y_i - 1), (x_i + 2, y_i - 1), (x_i + 2, y_i), \\ \cdots, (x_{i+1} - 1, y_i), (x_{i+1} - 1, y_i - 1).$$

This concludes the proof.

Theorem 1 possibly provides a useful sufficient condition for G to be Hamiltonian, but is unfortunately far from being a characterization. For an example of a graph that is Hamiltonian but does not satisfy the hypotheses of Theorem 1, see Fig. 1 in the following section.

3. Hamiltonian infinite grid graphs. We call an infinite graph Hamiltonian if it has a Hamiltonian two-way infinite path. There are several different types of infinite grid graphs. Some of them certainly do not contain any Hamiltonian graphs. This is the case, for instance, if the boundary has more than two components. We systematically consider all possible types in a subsequent paper; here we only mention two of them.

First, we investigate those infinite grid graphs with connected boundary for which the intersection with any infinite path of vertex set $\mathbb{Z} \times \{\cdot\}$ or $\{\cdot\} \times \mathbb{Z}$ is a one-way infinite path or empty. We may suppose without loss of generality that their vertex sets include $\mathbb{Z}_+ \times \mathbb{Z}_-$; we call these graphs *SE-grid graphs.*

Let G be an SE-grid graph. Let P be the boundary two-way infinite path of G. A (finite or infinite) subpath of P is called a *step* if its endpoints, and only its endpoints, have degree 4 in G. Such a step S must then have a corner (x_c, y_c) among its vertices.

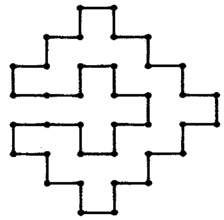


FIG. 1

Let

$$k_x = \text{card}\,\{x : (x, y_c) \text{ has degree } 3\},$$

$$k_y = \text{card}\,\{y : (x_c, y) \text{ has degree } 3\}.$$

Then we say that the step S at (x_c, y_c) is *of type* $(k_x + 1, k_y + 1)$. Clearly, a step of type (s, t) has length $s + t$. A step of type $(1, 1)$ is called a *unit step*.

Obviously, if G has only unit steps, it is not Hamiltonian. This is also the case if P only contains a one-way infinite path including just unit steps. We will prove the following theorem.

THEOREM 2. *Every SE-grid graph with just finitely many unit steps is Hamiltonian.*

Proof. We consider the *SE*-grid graph G and use all notation above. We may suppose, without loss of generality, that the natural ordering on P induces an increase of both Cartesian coordinates. If it has just one corner, then G is easily shown to be Hamiltonian. Let (v, w) be a corner of G and consider the following conditions:

(a) there is a step of type $(s, \aleph_0)$ at some corner $(x^*, y^*) \leqq (v, w)$;

(b) for infinitely many corners $(x, y) \leqq (v, w)$, the step at (x, y) is of type (s, t) with $t \geqq 2$;

(a′) there is a step of type $(\aleph_0, t)$ at some corner $(x^{**}, y^{**}) \geqq (v, w)$;

(b′) for infinitely many corners $(x, y) \geqq (v, w)$, the step at (x, y) is of type (s, t) with $s \geqq 2$.

The Hamiltonian path will be composed of a one-way infinite path A and infinitely many finite paths $B_1, C_1, B_2, C_2, \cdots$. To describe the Hamiltonian path, let us say that a path in G is *close* to another if they are disjoint but no point of the first is at Euclidean distance more than $\sqrt{2}$ from the second.

Construction of A. We simply take A to be a one-way infinite subpath of P ending (supposing that we come from infinity) at a point (x_0, y_0) chosen as follows:

If (a) is satisfied, let $(x_0, y_0) = (x^*, y^*)$. If not, let (x_0, y_0) be a corner of type (s_0, t_0) with $t_0 \geqq 2$ lying below all unit steps of G if (b) is satisfied, and a corner of type $(s_0, 1)$ lying below all unit steps and below all corners of type (s, t) with $t \geqq 2$ if (b) is not satisfied.

Construction of B_n. B_n begins at (x_{n-1}, y_{n-1}), goes through $(x_{n-1}, y_{n-1} - 1)$, then goes close to $A \cup \bigcup_{i<n} (B_i \cup C_i)$ up to a point (x'_n, y'_n) such that if (a′) is satisfied, $y'_n + 1 = y^{**}$; if not, $(x'_n - 1, y'_n + 1)$ is a corner above all unit steps of G, of type (s'_n, t'_n) with $s'_n \geqq 2$ if (b′) is satisfied, and $s'_n = 1$ if (b′) is not satisfied.

Construction of C_n. C_n begins at (x'_n, y'_n), goes through $(x'_n + 1, y'_n)$, then turns down and goes close to B_n until a point (x_{n-1}, y) is reached. Then if (a) is true, C_n ends at $(x_n, y_n) = (x_{n-1}, y) = (x_{n-1}, y_{n-1} - 2)$. Otherwise, C_n again follows P downward until the next corner (x_n, y_n) of type (s_n, t_n) with $t_n \geqq 2$ if (b) is fulfilled, and $t_n = 1$ if (b) is not fulfilled.

The proof that the mechanism works presents an interest only if (a) and (a′) are not satisfied, and essentially differs between having (b) (or (b′)) satisfied or unsatisfied.

While, in the case where (b) and (b′) are satisfied, all B_n and C_n are coordinate-monotone, this is not the case when (b) or (b′) is not satisfied. It works close to the previous path, essentially because G contains, together with a vertex (x, y), all vertices $(x+z, y-z)$, $z \in \mathbb{N}$. This is enough when (b) and (b′) are fulfilled. If not, the algorithm still works, essentially because, in that situation, for some corner (x_-, y_-) and any corner $(x, y) \leqq (x_-, y_-)$, $(x-z, y-z) \in G$ $(z \in \mathbb{N})$ if (b) is not satisfied, and for some corner (x_+, y_+) and any corner $(x, y) \geqq (x_+, y_+)$, $(x+z, y+z) \in G$ $(z \in \mathbb{N})$ if (b′) is not satisfied. Hence the theorem is proved.

Although Theorem 2 provides an excitingly weak sufficient condition for an *SE*-grid graph to be Hamiltonian, it fails, however, to be a characterization.

Now we mention a sufficient condition for a grid graph G to be Hamiltonian, in the case where its intersection with any infinite path of vertex set $\mathbb{Z} \times \{\cdot\}$ is a finite path or empty and its intersection with any path of vertex set $\{\cdot\} \times \mathbb{Z}$ is a one-way infinite path or empty. In such a case, G is called an *N-grid graph*. The easy proof is left to the reader.

THEOREM 3. *If, for every corner (x, y) of an N-grid graph G, y is even, then G is Hamiltonian.*

In §§ 1 and 2 we made little progress toward a satisfactory answer to the following problem.

Problem. Provide a characterization of Hamiltonian grid graphs.

Among the cases treated here, we consider that of an *SE*-grid graph as the most hopeful, since our Theorem 2 already comes rather close to being a characterization.

4. Hamiltonian line graphs of grid graphs. Clearly, a finite grid graph G is Eulerian if and only if it has no vertices of degrees 1 or 3; see Fig. 2. The line-graph $L(G)$ of G is Eulerian if and only if all vertex degrees in G are of the same parity, which also means that vertices of degree 1 or 3 must fail, with the remarkable exception illustrated in Fig. 3. In short, G is Eulerian precisely when it is as shown in Fig. 2, and $L(G)$ is Eulerian if and only if G is as shown in Fig. 2 or 3. Much more often, $L(G)$ is Hamiltonian, as already demonstrated by the following corollary.

COROLLARY TO THEOREM 1. *If G is a finite grid graph with $w(G) > 2$, then $L(G)$ is Hamiltonian.*

Proof. We use the terminology of § 1. Let k be the number of all special vertices of G having at least one of its auxiliary coordinates 1. We prove by induction on k the following assertion: G has a tour visiting all special vertices (ξ_i, η_i) with $\xi_i\eta_i$ even.

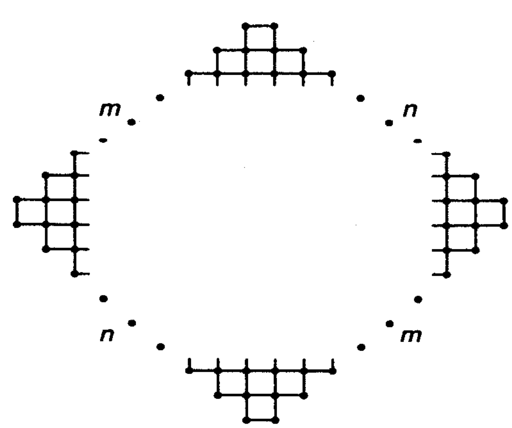


FIG. 2

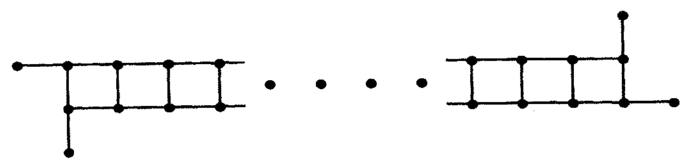


FIG. 3

For $k = 0$, delete all special vertices (ξ_i, η_i) with $\xi_i \eta_i$ odd. The resulting graph has, by Theorem 1, a Hamiltonian path, which is a tour in G visiting all special vertices of G that were not deleted.

Suppose now that the assertion is true for at most k special vertices with at least one auxiliary coordinate 1; we prove it for $k + 1$ such vertices.

Case 1. There exists a special vertex (x_i, y_i) with auxiliary coordinates $(1, 1)$. Suppose that $i \leqq n$. Delete (x_i, y_i). If the previous corner (x_{i-1}, y_{i-1}) had ξ_{i-1} even and η_{i-1} odd, it also must be deleted. If, moreover, $\eta_{i-1} = 1$, we must look at the corner (x_{i-2}, y_{i-2}) and delete it, also, in the case where ξ_{i-2} is even and η_{i-2} odd, and so forth. If the corner (x_{i+1}, y_{i+1}) had ξ_{i+1} odd and η_{i+1} even, delete it. If, moreover, $\xi_{i+1} = 1$, we look at (x_{i+2}, y_{i+2}), and delete it, provided that ξ_{i+2} is odd and η_{i+2} even, and so forth. If $n + 2 \leqq i \leqq e$ or $e + 1 \leqq i \leqq s$ or $i \geqq s + 2$, proceed similarly.

Case 2. There is a special vertex (x_i, y_i) with auxiliary coordinates $(\xi_i, 1)$ $(\xi_i \neq 1)$. If $i \leqq n$, delete that vertex. If the previous corner (x_{i-1}, y_{i-1}) had ξ_{i-1} even and η_{i-1} odd, delete it, also. If, moreover, $\eta_{i-1} = 1$, look at (x_{i-2}, y_{i-2}) and delete it if ξ_{i-2} is even and η_{i-2} odd, and so forth. Proceed similarly if $n + 2 \leqq i \leqq e$ or $e + 1 \leqq i \leqq s$ or $i \geqq s + 2$, instead of $i \leqq n$.

Case 3. There is a special vertex (x_i, y_i) with auxiliary coordinates $(1, \eta_i)$ $(\eta_i \neq 1)$ (analogous to Case 2).

After the deletions, there will be at most k special vertices with at least one auxiliary coordinate 1. The induction hypothesis guarantees the existence in the graph obtained after deletions of a tour T containing all special vertices whose (new) auxiliary coordinates have even products. This ensures that T is a tour of the same kind in the original graph, also.

Now the following known result finishes the proof.

LEMMA (see [1]). *For any finite graph G, $L(G)$ is Hamiltonian if and only if G has a tour.*

The restriction $w(G) > 2$ is not natural, merely imposed by the proof method. This is shown by the following strengthening, which has, however, a less elegant proof.

THEOREM 4. *For every finite grid graph G of positive width, $L(G)$ is Hamiltonian.*

Proof. Let $P_k(G)$ be the path of all vertices (x, k) in G. We prove that G has a tour T such that $P_N(G) - T$ consists (if not empty) of isolated vertices. We proceed again by induction on the height $h = N - S$. Indeed, the assertion is true for $h = 1$. We suppose it to be true for height h and prove it for height $h + 1$.

Consider the subgraph H of the grid graph G of height $h + 1$, spanned by the vertices (x, y) with $y \leqq N - 1$. If H has vertices of degree 1, they must lie on the line $y = N - 1$, and will be deleted. If the resulting graph has vertices of degree 1, delete them also. Repeat the procedure until the remaining graph H' has no vertices of degree 1. Then H' is a grid graph of height h and therefore has a tour T' such that $P_{N-1}(H) - T'$ consists of isolated vertices, at most.

Let us consider a connected component P' of $P_{N-1}(H') \cap T'$. Observe that P' must exist because $w(G) > 0$, and that P' must be a path. Let

$$P'' = \{x : (x, N-1) \in P' \text{ and } (x, N) \in G\}.$$

If $P'' \neq \emptyset$ and $u = \min P''$ is different from $v = \max P''$, then we replace the subpath of P' spanned by P'' by the path ex (P') spanned by

$$(u,N-1),(u,N),(u+1,N),(u+1,N-1),(u+2,N-1),(u+2,N),$$
$$(u+3,N),(u+3,N-1),\cdots,(v-1,N-1),(v-1,N),(v,N),(v,N-1)$$

for $v-u$ odd, or by

$$(u,N-1),(u,N),(u+1,N),(u+1,N-1),(u+2,N-1),(u+3,N-1),(u+3,N),$$
$$(u+4,N),(u+4,N-1),\cdots,(v-1,N-1),(v-1,N),(v,N),(v,N-1)$$

for $v-u$ even. Let

$$Q=\bigcup_{P'}\text{ex}(P').$$

Now, if $(x_n, y_n) \notin Q$ (remember that $y_n = N$), we extend Q by replacing its edge $\{(x, N-1), (x, N)\}$ with minimal x by the path spanned by

$$(x,N-1),(x-1,N-1),\cdots,(x_n,N-1),(x_n,N),(x_n+1,N),\cdots,(x,N).$$

Similarly, if $(x_{n+1}, y_{n+1}) \notin Q$ (remember also that $y_{n+1} = N$), we extend Q by replacing its edge $\{(x, N-1), (x, N)\}$ with maximal x by the path spanned by

$$(x,N-1),(x+1,N-1),\cdots,(x_{n+1},N-1),(x_{n+1},N),(x_{x+1}-1,N),\cdots,(x,N).$$

Suppose now that $(x_0, N-1) \in P_{N-1}(H') - T'$. Then, since

$$(x_0,N-1),(x_0,N)\notin Q,$$

we replace for every such x_0 the edge $\{(x_0+1, N-1), (x_0+1, N)\}$ by the path spanned by

$$(x_0+1,N-1),(x_0,N-1),(x_0,N),(x_0+1,N).$$

Thus we transformed T' into a tour of G. By the lemma, $L(G)$ is Hamiltonian.

Remark. Concerning the width, Theorem 4 is the best possible, because finite grid graphs of vanishing width may have non-Hamiltonian line graphs; P_3 is a suitable example.

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