On Some Questions about Convex Surfaces

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(Received March 23, 1993)
(Revised Version January 14, 1994)

Introduction

Chapter A35 in the very enjoyable book [4] of CROFT, FALCONER and GUY treats geodesics on more or less smooth convex surfaces in $\mathbb{R}^3$. Our aim here is to answer a few questions mentioned there.

So for example the 43 years old problem of GÖTZ and RYBARSKI ([6], p. 301–302): Is the sphere the only surface for which whenever points can be joined by two distinct segments (i.e., shortest paths between two points), then they can be joined by an infinity of segments? We shall show here that every such surface is a "Wiedersehensfläche", i.e., a surface on which all geodesics from an arbitrary point meet again, the lengths of the geodesic arcs up to that meeting point being all equal (the term was invented by BLASCHKE). GREEN proved BLASCHKE'S conjecture that all $C^3$ Wiedersehensflächen are spheres.

STEINHAUS [10] showed that there always exist at least two distinct segments from a point to any farthest point on a $C^3$ surface homeomorphic to $S^2$. This is also an easy consequence of the more recent well-known Aleksandrov-Toponogov theorem, which works for any $C^3$ variety (and in any dimension). We shall prove it for any $C^1$ convex surface. For arbitrary convex surfaces this is no longer true, as the example of a long thin pyramid shows.

STEINHAUS also asked what can be said qualitatively about the set of all farthest points from a given point on a convex surface, observing that it may not be connected. It has been pointed out by ALEKSANDROV that an (intrinsic) circle may be homeomorphic to any compact subset of the Euclidean circle (among other possibilities). The set of farthest points, a special circle on the surface, may be homeomorphic to any compact subset of the line, as well shall see. Each of its components must be a point or a Jordan arc. Its Hausdorff dimension is as expected at most 1 and its 1-dimensional Hausdorff measure at most $\pi r_x$, where $r_x$ is the distance from the given point $x$.

The last section is devoted to the well-known conjecture of HILBERT and COHN-VOSSEN [8] claiming that every surface on which all geodesics from an arbitrary point meet again in another point is a sphere.

We shall make extensive use of the methods of ALEKSANDROV [1]. In most cases we will precisely refer to the used results from [1]. But generally speaking knowledge of large parts of [1] would be of much help for the reader.

Thanks are due to the referee for his or her comments.
On multijoined points

Let $S \subset \mathbb{R}^3$ be an arbitrary (closed) convex surface, i.e., the boundary of an open bounded convex set, and denote by $\varrho$ its intrinsic metric. Let $x \in S$. A point $y \in S$ is called here multijoined to $x$ if there are at least two segments from $x$ to $y$.

It is easily seen that the set of all points $y$ admitting at least 3 segments from $x$ to $y$ is at most countable. This has been independently observed by P. Gruber.

The set $C_x$ of all points of $S$ multijoined conjugate to $x$ is small from both the measure theoretic and Baire category points of view. Indeed, we proved in [14] that $C_x$ is $\sigma$-porous and therefore of (2-dimensional) measure 0 and of first category. We are going to show here that $C_x$ is always connected. This was known to differential geometers under stronger smoothness assumptions and conjectured for arbitrary convex surfaces in [15].

**Theorem 1.** For any point $x$ on a convex surface $S \subset \mathbb{R}^3$ the set $C_x$ is arcwise connected.

The proof of the theorem makes use of the following simple lemma.

**Lemma 1.** Assume that $a, b, y, z$ belong to a convex surface $S \subset \mathbb{R}^3$ and are all distinct (except possibly for $a, b$). If the distinct segments $\Sigma_{ay}$ from $a$ to $y$ and $\Sigma_{by}$ from $b$ to $y$ are equally long and the distinct segments $\Sigma_{az}$ from $a$ to $z$ and $\Sigma_{bz}$ from $b$ to $z$ are equally long too, then

$$\Sigma_{ay} \cap \Sigma_{bz} = \Sigma_{az} \cap \Sigma_{by} = \{a\} \cap \{b\}.$$

Proof. It suffices to prove the second equality. Suppose, on the contrary, $c \in \Sigma_{az} \cap \Sigma_{by} \setminus \{a, b\}$. Since

$$\varrho(a, c) + \varrho(c, y) \geq \varrho(a, y) = \varrho(b, y),$$

it follows that $\varrho(a, c) \geq \varrho(b, c)$. Analogously, $\varrho(b, c) \geq \varrho(a, c)$, whence $\varrho(a, c) = \varrho(b, c)$.

Let $\Sigma_1$ be the subsegment of $\Sigma_{az}$ from $a$ to $c$ and $\Sigma_2$ the subsegment of $\Sigma_{by}$ from $c$ to $y$. Since

$$\varrho(a, c) + \varrho(c, y) = \varrho(b, c) + \varrho(c, y) = \varrho(b, y) = \varrho(a, y),$$

$\Sigma_1 \cup \Sigma_2$ is a segment, different from $\Sigma_{by}$ but having with it the common arc $\Sigma_2$, in contradiction with basic properties of segments on convex surfaces (see [1], p. 84–85). Since $a \in \Sigma_{by}$ only if $a = b$, because $\varrho(a, y) = \varrho(b, y)$, and, analogously, $b \in \Sigma_{az}$ only if $a = b$, the second equality is verified.

**Proof of Theorem 1.** If $C_x$ is a single point there is nothing to prove. If $y, z \in C_x$ we choose two segments $\Sigma_y^1, \Sigma_y^2$ from $x$ to $y$ and another two $\Sigma_z^1, \Sigma_z^2$ from $x$ to $z$. The mentioned basic properties of segments on convex surfaces ([1], p. 84–85) imply that

$$\Sigma_y^1 \cap \Sigma_z^2 = \{x, y\}, \quad \Sigma_z^1 \cap \Sigma_y^2 = \{x, z\},$$

and Lemma 1 (with $x = a = b$) yields

$$(\Sigma_y^1 \cup \Sigma_y^2) \cap (\Sigma_z^1 \cup \Sigma_z^2) = \{x\}.$$

Thus the four segments divide the surface into three domains (i.e., open connected sets), one of which has all four on its boundary. Let $D$ be this domain. We may assume that a
small circle $\Gamma$ (in the intrinsic metric of $S$) of centre $x$ and radius $\epsilon$ meets the segments in the order $\Sigma_1^1, \Sigma_1^2, \Sigma_2^1, \Sigma_2^2$. Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be the arcs in which the preceding segments divide $\Gamma$ ($\Gamma_1$ between $\Sigma_1^1$ and $\Sigma_1^2$, $\Gamma_2$ between $\Sigma_2^1$ and $\Sigma_2^2$, etc.). Let $\mathcal{Q}$ be a (full, topologically closed) square of vertices $x, y, x', y'$ in the Euclidean plane (see Figure 1). If $\mathcal{Q}'$ denotes $Q$ with identified vertices $x, x'$, then there obviously exists a homeomorphism $\varphi$ between the closure $\bar{D}$ of $D$ and $\mathcal{Q}'$ such that $x' = x_1 = \varphi(x), y' = \varphi(y)$ and $z' = \varphi(z)$. This obviously induces a multifunction, which will also be denoted by $\varphi$, from $\bar{D}$ to $\mathcal{Q}$ (with $\varphi(x) = \{x_1, x_2\}$). Equip $\mathcal{Q}\setminus\{x_1, x_2\}$ with the distance given by the length of the shortest path between the corresponding points in $\bar{D}$, which does not contain $x$. Define the distance between a point in $\mathcal{Q}\setminus\{x_1, x_2\}$ and $x$ to be the length of the shortest path from the corresponding point of $\bar{D}$ to $x$ crossing $I$ for $\epsilon$ arbitrarily small. Also, let the distance between $x_1$ and $x_2$ be the length of the shortest closed curve through $x$, not contractible in $\bar{D}$. Denote by $\delta$ this metric of $\mathcal{Q}$ and put

$$Q_1 = \{u \in \mathcal{Q} : \delta(u, x_1) < \delta(u, x_2)\},$$

$$Q_2 = \{u \in \mathcal{Q} : \delta(u, x_1) > \delta(u, x_2)\},$$

$$Q_3 = \{u \in \mathcal{Q} : \delta(u, x_1) = \delta(u, x_2)\}.$$

Let $\tau_1, \tau_2$ be the tangent directions of $\Sigma_1^1, \Sigma_1^2$ in $x$. They belong to the closed Jordan curve $T_x \subset S^2$ of all tangent directions at $x$. Let $I$ be the arc on $T_x$ from $\tau_1$ to $\tau_2$ not containing the tangent directions of $\Sigma_1^1, \Sigma_1^2$ in $x$. Now, a direction $\tau \in T_x$ is called singular if no segment starts at $x$ in direction $\tau$ (\cite{1}, p. 213). For any nonsingular $\tau \in I$, let $e(\tau)$ be the endpoint of the maximal (by inclusion) segment $\Sigma(\tau)$ in $S$ starting at $x$ in direction $\tau$. If $\tau$ is singular or $\varphi(e(\tau)) \in Q_1$, let $e'(\tau), e''(\tau)$ be the endpoints of the maximal (by inclusion) open arc in $T_x$ containing $\tau$ such that $\varphi(e(\sigma)) \in Q_1$ for any nonsingular $\sigma$ in the arc.

Suppose $e(e'(\tau)) \neq e(e''(\tau))$ for some $\tau \in I$. Of course $\varphi(e(e'(\tau)))$ and $\varphi(e(e''(\tau)))$ are distinct and belong to $Q_3$. Consider the following four segments: $\varphi(\Sigma(e'(\tau)))$, a segment from $\varphi(e(e'(\tau)))$ to $x_2'$, another segment from $x_2'$ to $\varphi(e(e''(\tau)))$, and $\varphi(\Sigma(e''(\tau)))$. Their union is, by Lemma 1, a closed Jordan curve $J$. Let $C$ be a Jordan arc from $x_1'$ to $x_2'$ whose interior
points lie in the Jordan domain bounded by \( J \) and included in \( Q \). Since \( x_1 \in Q \) and \( x_2 \in Q_2 \), there must be a point \( c \) in \( C \cap Q_3 \). Then, for any segment \( \Sigma \subset Q \) from \( x_1 \) to \( c \), the segment \( \varphi^{-1}(\Sigma) \subset S \) must have a tangent direction \( \sigma \) between \( e'(\tau) \) and \( e''(\tau) \), while \( \varphi(e(\sigma)) \notin Q_1 \), a contradiction. Hence \( e(e'(\tau)) = e(e''(\tau)) \). Then the map \( f: I \to S \) defined by

\[
f(\tau) = \begin{cases} e(\tau) & \text{if } \varphi(e(\tau)) \in Q_3, \\ e(e'(\tau)) & \text{otherwise}, \end{cases}
\]

is continuous (use [1], p. 76) and \( f \) provides an arc whose points are all multijoined to \( x \).

### On farthest points

We shall prove here that the set \( F_x \) of all farthest points on \( S \) from \( x \in S \) is strongly related to the set \( C_x \). In any case \( F_x \subset C_x \), often \( F_x \subset C_x \).

**Theorem 2.** Let \( S \subset \mathbb{R}^3 \) be a convex surface and \( x \in S \). Then \( F_x \subset C_x \). Any angle between two tangent directions at a point \( y \in F_x \) measuring (on the tangent cone) more than \( \pi \) contains the tangent direction of a segment from \( y \) to \( x \). So if the full angle of \( S \) at \( y \) is larger than \( \pi \), then \( y \in C_x \), and if \( S \) is differentiable at \( y \) and there are only two segments from \( x \) to \( y \), then these have opposite tangent directions at \( y \).

Again we establish a lemma before proving the theorem.

**Lemma 2.** Let \( \Gamma \) be a Jordan closed curve on a convex surface \( S \subset \mathbb{R}^3 \) and let \( a, b, x \in S \setminus \Gamma \). Suppose that between every point of \( \Gamma \) and \( x \) there is a unique segment, and \( a, b \) do not belong to any such segment. Then \( \Gamma \) does not separate \( a \) from \( b \).

**Proof.** Suppose, on the contrary, that \( \Gamma \) separates \( a \) from \( b \). Denote by \( \Theta \) the union of all segments joining points in \( \Gamma \) with \( x \).

On one hand, since \( a, b \notin \Theta \), but \( \Gamma \subset \Theta \), the set \( \Theta \) is not contractible.

On the other hand, if \( p(v, r) \) denotes the point of the segment (supposed unique) from \( x \) to \( v \), at distance \( r \) from \( v \), the function \( p \) is continuous in both variables (use, for example, (10.5), (10.5'), (11.3) in [3]). Now, there is indeed a unique segment from \( x \) to \( v \) for any \( v \in \Theta \). Then the homotopy \( H: \Theta \times [0, 1] \to \Theta \) defined by

\[
H(v, t) = p(v, t_0(x, v))
\]

shows (see [5], p. 362) that \( \Theta \) is contractible. A contradiction is found.

**Proof of Theorem 2.** Let \( \Gamma_\varepsilon \) be a small circle of radius \( \varepsilon \) around \( y \), homeomorphic to \( S^1 \) (whose existence is guaranteed for \( \varepsilon \) small enough, see [1], p. 383). Suppose that \( \Gamma_\varepsilon \cap C_x = \emptyset \). Then, by Lemma 2, every point on \( S \) separated from \( y \) by \( \Gamma_\varepsilon \) lies on the segment from \( x \) to some point of \( \Gamma_\varepsilon \). Let \( \Sigma_y \) denote the segment from \( x \) to an arbitrary point \( u \in \Gamma_\varepsilon \). Also, let \( \Sigma \) be a segment from \( x \) to \( y \) and consider \( s \notin \Sigma \). If \( \Gamma_\varepsilon \cap C_x = \emptyset \) for arbitrarily small \( \varepsilon > 0 \), let \( \varepsilon \) be so that \( \Gamma_\varepsilon \) separates \( s \) from \( y \) and consider the point \( u \in \Gamma_\varepsilon \) with \( s \in \Sigma_u \). By taking a sequence of numbers \( \varepsilon \) converging to \( 0 \), we get a sequence of segments with \( x \) as an endpoint, all containing \( s \). This sequence converges to a segment from \( x \) to \( y \) containing \( s \). Thus \( y \in C_x \). Otherwise, if \( \Gamma_\varepsilon \cap C_x \neq \emptyset \) for a sequence of numbers \( \varepsilon \) converging to \( 0 \), obviously \( y \in C_x \).
Suppose that there is an angle $T$ at $y$ between $\tau_1$ and $\tau_2$ measuring on the tangent cone at $y$ more than $\pi$, but not containing the tangent direction of any segment from $y$ to $x$. Let $\tau_0$ be the middle point of $T$ (viewed as an arc of the rectifiable curve $T_\tau$). Clearly, the angles from $\tau_1$ and $\tau_2$ to $\tau_0$ measure more than $\pi/2$. Let $\Sigma'$ be a segment with an endpoint in $y$ and with a tangent direction $\tau'$ at $y$ so close to $\tau_0$ that the angles from $\tau_1$ and $\tau_2$ to $\tau'$ are still larger than $\pi/2$. The existence of $\Sigma'$ is guaranteed by the fact that the singular directions form a set of measure 0 (see [1], p. 213). Let $t \in \Sigma' \setminus \{y\}$ (see Figure 2). Of course, $q(x, t) \leq q(x, y)$. For $t \to y$ choose the segment $\Sigma_t$ from $t$ to $x$ so that $\Sigma_t$ converges to an arc $\Sigma$. Then $\Sigma$ is a segment from $y$ to $x$ and therefore its tangent direction $\sigma$ at $y$ is not in $T$. Hence the angle from $\sigma$ to $\tau'$ is larger than $\pi/2$. Consider the Euclidean triangle with side lengths $q(x, y), q(y, t), q(t, x)$. Its angle $\alpha$ opposite to the side of length $q(t, x)$ is not larger than its angle opposite to the side of length $q(x, y)$ and therefore smaller than $\pi/2$. The angle between $\sigma$ and $\tau'$ is smaller than $\alpha + \omega$; here $\omega$ is the curvature of the triangle with sides $\Sigma, \Sigma', \Sigma_t$, where $\Sigma_t$ is the subarc of $\Sigma'$ from $y$ to $t$ (see [1], p. 215). Since $\omega \to 0$ as $t \to y$,

\[
\lim_{t \to y} \sup (\alpha + \omega) \leq \pi/2 ,
\]

whence the angle between $\sigma$ and $\tau'$ is at most $\pi/2$ and a contradiction is obtained.

The remaining assertions of the statement follow immediately.

**The problem of Götz and Rybarski**

Now we pass to the mentioned problem of Götz and Rybarski [6]. The results in the previous sections will be useful here.

**Theorem 3.** Let $S \subset \mathbb{R}^3$ be a convex surface such that whenever points can be joined by two distinct segments then they can be joined by three distinct segments. Then $S$ is a Wiedersehensfläche.

**Proof.** Let $x \in S$ and suppose that $C_x$ contains two points $y, z$. Then, by Theorem 1, $C_x$ includes a whole arc $A$ from $y$ to $z$. Since there are only countably many points in $C_x$ joined with $x$ by at least three segments, for many points in $A$ this cannot happen, in contradiction with the hypothesis. So $C_x$ contains a single point $y$. By Theorem 2, $y$ must
be the unique farthest point of $S$ from $x$. Indeed, if $y' \in F_x \setminus \{y\}$, then $y' \notin C_x$; in this case Theorem 2 tells us that $y' \in C_x \setminus C_x$ which yields the infinity of $C_x$ and a contradiction is obtained. For any nonsingular tangent direction $\tau$ at $x$ let again $e(\tau)$ denote the other endpoint of the maximal segment starting at $x$ in direction $\tau$.

Suppose that $e(\tau) \neq y$ for some nonsingular $r$. Let $\Gamma$ be an intrinsic circle on $S$ of centre $e(\tau)$ and with a radius small enough to guarantee that $\Gamma$ is a Jordan curve (see [1], p. 383) separating $e(\tau)$ from both $x$ and $y$. Since $e(\tau), y$ do not belong to any segment from a point of $\Gamma$ to $x$, Lemma 2 implies that $e(\tau), y$ are not separated by $\Gamma$, and a contradiction is found again. Hence $e(\tau) = y$ for all nonsingular $\tau \in T_x$. Because every tangent direction $\sigma \in T_x$ is the limit of a sequence of nonsingular directions, some subsequence of the corresponding sequence of segments from $x$ to $y$ converges to a segment from $x$ to $y$ having $\sigma$ as tangent direction at $x$ (use [1], p. 158). Hence $\sigma$ is not singular.

Therefore for every $\tau \in T_x$ we have $e(\tau) = y$, which proves that $S$ is a Wiedersehensfläche.

**Corollary.** Each $C^3$ convex surface in $\mathbb{R}^3$ such that whenever points can be joined by two distinct segments then they can be joined by three distinct segments is a sphere.

This follows from Theorem 3 together with Green's result in [7].

**A question of Steinhaus**

We now turn to Steinhaus' question about the set of all farthest points from some given point of a convex surface $S \subset \mathbb{R}^3$. The set $F_x$ of all farthest points from $x \in S$ is the largest possible with centre at $x$. Let $C(x, r)$ denote the circle

$$\{y \in S : g(x, y) = r\}$$

of centre $x$ and radius $r$. Also let $r_x = g(x, z)$, where $z \in F_x$. Clearly $C(x, r) \neq \emptyset$ if and only if $0 \leq r \leq r_x$, and $C(x, r_x) = F_x$. We denote by $\mu_\alpha$ the $\alpha$-dimensional Hausdorff measure. It is well-known that $\mu_1 C(x, r) \leq 2\pi r$. So $\mu_2 C(x, r) = 0$. The following easy proposition confirms this and applies of course in particular to $F_x$. For a definition and applications of porosity and strong porosity, see [12], [13].

**Proposition.** For any point $x \in S$ and any $r \geq 0$, $C(x, r)$ is strongly porous.

**Proof.** Let $y \in C(x, r)$ and consider a segment $\Sigma$ from $x$ to $y$. For any point $z \in \Sigma \setminus \{y\}$ the open ball $B$ of centre $z$ and radius $g(z, y)$ is disjoint from $C(x, r)$. Indeed, for every point $w \in B$,

$$g(x, w) \leq g(x, z) + g(z, w) < g(x, z) + g(z, y) = r$$

and therefore $w \notin C(x, r)$. Thus $C(x, r)$ is strongly porous.

By Lebesgue's density theorem, every porous set has measure 0.

More information on the dimension and measure of $F_x$ is provided by the next theorem. For a definition and further facts on Hausdorff measures, see [9].

In the proof of the next theorem we shall use the following notation. If the segments $\Sigma^*, \Sigma^{**} \subset S$ have an endpoint $h$ in common, let $(\Sigma^* h \Sigma^{**})$ denote the measure (on the tangent cone) of the (smaller) angle between $\Sigma^*$ and $\Sigma^{**}$ at $h$. Also, for $x \in S$, let $v(x) \subset S^2$ denote as usual the spherical image of $x$. 
Theorem 4. For any point $x \in S$, the Hausdorff dimension of $F_x$ is at most 1 and

$$\mu_1 F_x \leq \pi r_x.$$ 

Proof. The set of all points in $F_x$ which are isolated or conical is at most countable. Let $F^*$ be its complement in $F_x$. We only have to prove that $\mu_1 F^* \leq \pi r_x$.

Let $a, a_n \in F^*$ be such that $a_n \neq a$ and $a_n \to a$. Of course $\mu_2 v(a_n) \to 0$ as $n \to \infty$, whence the measure of the full angle of the tangent cone at $a_n$ tends to $2\pi$. So, by Theorem 2, from some index on, for every $n$ there are two segments $\Sigma'_n, \Sigma''_n$ from $a_n$ to $x$; we choose them so that any other segment from $a_n$ to $x$ is separated from $a$ by $\Sigma'_n \cup \Sigma''_n$ (see Figure 3). Then we may assume (take a subsequence if necessary) that $\Sigma'_n \to \Sigma'$ and $\Sigma''_n \to \Sigma''$, say. Let $\Sigma_n$ be some segment from $a$ to $a_n$. Since $q(x, a_n)$ is constant for $n \in \mathbb{N}$, $(\Sigma'a\Sigma_n), (\Sigma''a\Sigma_n), (\Sigma''a\Sigma_n)$, $(\Sigma''a\Sigma_n)$ converge all to $\pi/2$ (see [1], p. 381–382). So $\Sigma'$ and $\Sigma''$ make an angle (the angle toward infinitely many $a_n$'s, not the smaller one, but possibly both of them) equal to $\pi$ at $a$. (This implies that the full angle of $S$ at $a$ measures at least $\pi$ and that if it is precisely $\pi$ then $\Sigma' = \Sigma''$.)

Now take $a \in F^*$. We have $(\Sigma'a\Sigma'') = \pi$. Even though there might be more segments from $a$ to $x$, only $\Sigma'$ and $\Sigma''$ are opposite at $a$. We associate these two segments to any point $a \in F^*$. Fix a certain sense on the rectifiable curve $T_x \subset S^2$ of all tangent directions at $x$.

For any point $a \in F^*$, let $\alpha(a)$ and $\alpha'(a)$ be the tangent directions at $x$ of the segments associated to $a$. That $\alpha(a)$ and $\alpha'(a)$ divide $T_x$ into two arcs of equal lengths may happen for one point $a = a_0$ at most, because segments associated to different points of $F^*$ cannot cross each other. For any point $a \in F^* \setminus \{a_0\}$ choose $\alpha(a)$ and $\alpha'(a)$ on $T_x$ such that the arc of $T_x$ from $\alpha(a)$ to $\alpha'(a)$ in the chosen sense is the smaller one. Let $\Delta$ denote the distance along $T_x$ and set

$$A = \{\alpha(a): a \in F^*\}, \quad A' = \{\alpha'(a): a \in F^*\},$$

$$A_m = \{\alpha(a): \Delta(\alpha(a), \alpha'(a)) \geq m^{-1}\},$$

$$A'_m = \{\alpha'(a): \Delta(\alpha(a), \alpha'(a)) \geq m^{-1}\}.$$
Suppose now \( \alpha(a_n) \to \beta \) on \( T_x \) and \( \alpha(a_n) \in A_m \) for all \( n \). Then \( \alpha'(a_n) \to \beta' \) for some point \( \beta' \) such that the arc from \( \beta \) to \( \beta' \) in the chosen sense is not the larger one and \( \Delta(\beta, \beta') \geq m^{-1} \).

Suppose that the arcs of \( T_x \) from \( \beta \) to \( \beta' \) are not equally long. Then a suitable subsequence of \( \{a_n\}_{n=1}^{\infty} \) converges to some point \( a \) and either \( \alpha(a_0) = \beta \), \( \alpha'(a_0) = \beta' \) or \( a \) is a conical point. If \( a \) is a conical point, then \( \beta \notin A \cup A' \). If not, \( \beta \in A_m \) and this excludes \( \beta \in A'_m \). Suppose now that the arcs of \( T_x \) between \( \beta \) and \( \beta' \) are equally long. Then a suitable subsequence of \( \{a_n\}_{n=1}^{\infty} \) converges to \( a \), and either \( \alpha(a_0) = \beta \), \( \alpha'(a_0) = \beta' \) or \( \alpha(a_0) = \beta' \), \( \alpha'(a_0) = \beta \).

Thus \( A_m \cap A' \subset \{a'(a_0)\} \), whence

\[
\left( \bigcup_{m=1}^{\infty} \overline{A_m} \right) \cap A' \subset \{a'(a_0)\}.
\]

Since \( T = \bigcup_{m=1}^{\infty} \overline{A_m} \) and \( T' = T_x \setminus T \) are complementary Borel sets in \( T_x \),

\[
\lambda T + \lambda T' = \lambda T_x,
\]

and, because \( A \subset T \) and \( A' \subset T' \cup \{\alpha'(a_0)\} \),

\[
\lambda A + \lambda A' \leq \lambda T_x,
\]

where \( \lambda \) denotes the Lebesgue outer measure on \( T_x \). Since \( \lambda T_x \leq 2\pi \), for one of the sets \( A \), \( A' \), say for \( A \), we have \( \lambda A \leq \pi \).

For any \( \varepsilon > 0 \) and \( \delta > 0 \) there is a covering \( \{A_i\}_{i=1}^{\infty} \) of \( A \) with diam \( A_i < \delta/r_x \) and

\[
\sum_{i=1}^{\infty} \text{diam} A_i < \lambda A + \varepsilon/r_x.
\]

Then \( \{a^{-1}(A_i)\}_{i=1}^{\infty} \) is a covering of \( F^* \) and, for any pair of points \( a, b \) in \( a^{-1}(A_i) \),

\[
g(a, b) \leq r_x \Delta(a(a), a(b)) \leq r_x \text{diam } A_i < \delta
\]

(for the first inequality use Aleksandrov's convexity condition, [1], p. 47) and

\[
\sum_{i=1}^{\infty} \text{diam } a^{-1}(A_i) \leq r_x \sum_{i=1}^{\infty} \text{diam } A_i < r_x \lambda A + \varepsilon.
\]

This yields \( \mu_1 F_x \leq \pi r_x \) and the proof is finished.

**Examples.** The following examples illustrate the various possibilities for \( F_x \) and also the fact that the upper bound in Theorem 4 is best possible.

Consider a (planar) half-disc in \( \mathbb{R}^3 \), take for a small \( \varepsilon > 0 \) its inner parallel convex set \( D_\varepsilon \) at distance \( \pi \varepsilon \), and then the outer parallel convex body (in \( \mathbb{R}^3 \)) of \( D_\varepsilon \) at distance \( 2\varepsilon \).

We obtained a \( C^1 \) convex surface \( S \) containing a point \( x \) (which corresponds to the centre of the initial half-disc) and including a portion \( P \) isometric to a piece of a torus, so that \( F_x \) is almost half the largest circle on the torus. For \( \varepsilon \to 0^+ \), we have \( \mu_1 (A \cup A') \to 2\pi \) and \( \mu_1 F_x - \pi r_x \to 0 \). For fixed \( \varepsilon \) and \( r \to r_x - \varepsilon \), \( \mu_1 C(x, r) \to 2 \mu_1 F_x \).

If we take the longest circular arc \( C_\varepsilon \subset D_\varepsilon \) and an arbitrary compact subset \( C'_\varepsilon \) of \( C_\varepsilon \) including the endpoints of \( C_\varepsilon \), and then replace in the above construction \( D_\varepsilon \) by the convex hull of \( C'_\varepsilon \), we get a set \( F_x \) congruent to \( (1 + 2\varepsilon) C'_\varepsilon \).
A similar but nonsmooth example in which $\mu_1(A \cup A') = 2\pi$ can be obtained as follows. Consider the convex hull of a torus. Cut it along a plane of symmetry orthogonal to the plane $\Pi$ of its largest circle. Cut again one of the two resulting pieces along $\Pi$ and obtain two pieces of convex surface, bounded by closed Jordan curves having a common circular arc $A$ on $\Pi$. By Aleksandrov's gluing theorem (see [1], p. 315–320), these pieces can be glued together along the Jordan curve (isometrically) keeping $A$ as a common arc, to form a closed convex surface with the desired properties.

Further facts on farthest points

Also topologically $F_x$ does not behave like the other circles $C(x, r)$. So a component of $F_x$ may never look like the digit 8 or like the letter A, O or P, while $C(x, r)$ with smaller $r$ may well do so. For very small $r$, $C(x, r)$ must be a closed Jordan curve, while $F_x$ can never be that round. The following result presents another aspect of the answer to Steinhaus' question.

**Theorem 5.** For any point $x \in S$, every component of $F_x$ is either a point or a Jordan arc.

**Proof.** Suppose that the component $K$ of $F_x$ has more than a single point. Then it has at least two non-cutpoints $a, b$ (see [11], p. 54). Clearly $a$ and $b$ are not isolated points, so we get as in the preceding proof the (not necessarily distinct) segments $\Sigma'_a, \Sigma''_a$ joining $a$ with $x$ and the (not necessarily distinct) segments $\Sigma'_b, \Sigma''_b$ joining $b$ with $x$ (see Figure 4, where $\Sigma'_a = \Sigma''_b$ and $\Sigma'_b = \Sigma''_a$).

Let $D$ be the domain with boundary

$$\Sigma'_a \cup \Sigma''_a \cup \Sigma'_b \cup \Sigma''_b.$$ 

Since $a$ and $b$ are not cutpoints,

$$K \setminus \{a, b\} \subset D.$$ 

Let now $c \in K \setminus \{a, b\}$. There are points $a', b' \in K$ close enough to $a$, respectively $b$, to ensure that $a', b' \in C_x$ and take them so that, if $\Sigma'_a, \Sigma''_a$ are the segments from $a'$ to $x$ and $\Sigma'_b, \Sigma''_b$ those from $b'$ to $x$ (met in the order $\Sigma'_a, \Sigma''_a, \Sigma'_b, \Sigma''_b$ around $x$), then

$$\Sigma'_a \cup \Sigma''_a \cup \Sigma'_b \cup \Sigma''_b$$

separates $c$ from $a$ and from $b$.

Consider now the arc $A \subset C_x$ joining $a'$ with $b'$, found in the proof of Theorem 1, and let $u \in A \setminus \{a', b'\}$. The union of two of the segments from $u$ to $x$ separates $a'$ from $b'$ (see the mentioned proof). So, if $u \notin F_x$, then $a'$ and $b'$ lie in different components of $F_x$, a contradiction. Hence $A \subset K$. Suppose that for some point $u \in A \setminus \{a', b'\}$ there is a third segment from $u$ to $x$ besides the two mentioned above. Then the two angles (out of three) determined by these three segments at $u$ towards $a'$ and $b'$ measure $\pi$ each, because $u$ is a limit point of sequences of points in $F_x$ lying in the domains bounded by the corresponding pairs of segments; thus nothing remains for the third angle. This contradiction shows that each point $u \in A \setminus \{a', b'\}$ is joined with $x$ by exactly two segments $\Sigma'_u, \Sigma''_u$, and choose the notation so that $\Sigma'_u, \Sigma''_u, \Sigma'_v, \Sigma''_v, \Sigma''_w, \Sigma'_w$ are met in this order around $x$. Then

$$\bigcup_{u \in A} (\Sigma'_u \cup \Sigma''_u) = \overline{D}.$$
Indeed, the existence of a point \( w \) in a component \( D' \) of \( D \setminus A \), not belonging to \( \Sigma_u \cup \Sigma'_u \) for any \( u \in A \) would imply the existence of a point \( v \in A \) joined with \( x \) by two distinct segments through \( D' \) enclosing \( w \) between them; but then there were three segments from \( v \) to \( x \), a contradiction.

Since no interior point of a segment \( \Sigma'_u \) or \( \Sigma''_u \) belongs to \( F_x \), the chosen point \( c \) must lie on \( A \). Moreover,

\[
(\Sigma'_e \cup \Sigma''_e) \cap F_x = \{c\}
\]

and \( \Sigma'_e \cup \Sigma''_e \) separates \( a \) from \( b \). Therefore \( c \) is a cutpoint of \( K \). Hence \( K \) is a Jordan arc (see [11], p. 54).

**Weak Wiedersehensflächen**

Let \( S \subseteq \mathbb{R}^3 \) be a convex surface and \( x \in S \). Let \( g : [0, l] \to S \) describe a geodesic \( G \) starting at \( g(0) = x \), with the arc-length as parameter, i.e., with \( s \) equal to the distance on \( G \) from \( x \) to \( g(s) \). Let \( z \in S \) and

\[
d_{G,z} : G \to \mathbb{R}_+ \]

be defined by \( d_{G,z}(y) = \varrho(y, z) \). Suppose that, for some \( z \in S \), \( d_{G,z} \circ g \) is non-increasing in a connected neighbourhood of \( 0 \) and let \([0, a]\) be a maximal such neighbourhood (by inclusion). Then we call \( g(a) \) a first proximum of \( G \) from \( z \).

We call a weak Wiedersehensfläche any convex surface \( S \subseteq \mathbb{R}^3 \) such that for any point \( x \in S \) there is some point \( z \in S \) for which every geodesic starting in \( x \) has a first proximum from \( z \) precisely in \( z \). In other words, going from any point \( x \in S \) along a geodesic we eventually reach another point of \( S \) depending on \( x \) but not on the chosen geodesic, such that the distance to that point never increases.
Zamfirescu, On some Questions about Convex Surfaces

We also recall that the specific curvature of a domain $D \subset S$ is $\omega(D)/\mu_2 D$, where $\omega(D)$ is the curvature of $D$ (see [1], p. 418). The surface $S$ is said to have bounded specific curvature if the specific curvatures of all domains in $S$ have a finite upper bound.

We shall prove a statement lying in between HILBERT and COHN-VOSSEN’s conjecture and the (established) Wiedersehensflächen conjecture of BLASCHKE, namely that every $C^3$ weak Wiedersehensflächen is a sphere.

**Theorem 6.** Every weak Wiedersehensfläche with bounded specific curvature is a Wiedersehensfläche.

**Proof.** Let $S$ be a weak Wiedersehensfläche, $x \in S$, and $z$ be the point from the above definition, common to all geodesics starting at $x$. Since $S$ has bounded specific curvature, there is some $r_0 > 0$ so that for every tangent direction $\tau$ at $z$ there is a segment of length $r_0$ starting at $z$ in direction $\tau$ (see [1], p. 420).

Let $G$ be a geodesic starting in $x$ (see Figure 5). It contains $z$ and let $l$ be its length from $x$ to $z$. Then the associated function $d_{G,z} \circ g$ is non-increasing on $[0, l]$.

Suppose $l > q(x, z)$. Take a segment $Z \subset G$ starting in $z$, of length less than $r_0$. Consider the segment $\Sigma_u$ from $x$ to $u \in Z$. If $\Sigma_u \subset G$ for every $u \in Z$, then $G$ is a segment and $l = q(x, z)$, a contradiction. Hence $\Sigma_v \cap G = \{x, v\}$ for some $v \in Z$. The segment $\Sigma_v$ can be extended beyond $v$ to a geodesic $H$ with parametrization $h$, crossing $G$ and ending at $z$ so that $d_{H,z} \circ h$ is non-increasing. Clearly $G$ and $H$ have distinct tangent directions at $z$. Let $Z'$ be the segment of length $r_0$ starting at $z$ in the same direction as $H$. Then $Z' \subset H$. We have

$$d_{H,z}(v) = d_{G,z}(v) < r_0 = d_{H,z}(z'),$$

where $z'$ is the endpoint of $Z'$ different from $z$. Hence $v \neq z'$. But, $Z$ and $Z'$ being different segments starting at $z$, necessarily $v \notin Z'$, so that the monotonicity hypothesis on $d_{H,z} \circ h$ is violated at $v, z'$.

Hence $l = q(x, z)$, and $S$ is a Wiedersehensfläche.

**Corollary.** Each $C^3$ weak Wiedersehensfläche is a sphere.

This follows from Theorem 6 combined with GREEN’s result in [7].
References


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