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How to Hold a Convex Body?

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Abstract. Convex bodies are often used for mathematical tests. They occasionally try to escape. Can the testing mathematician hold them still by using a circle? Rarely not.

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The Problem and the Answer

There have been various ways in which people tried to carry a convex body. For a trip by train a stiff bag is probably appropriate. This means a polytope and, to get an economical one, its facets must touch the convex body. Such approximating polytopes have been often considered in the literature (see, for example, P. Gruber's survey [5]). If, instead, the total length of the edges should be minimal then, in the case of the unit ball, the polytope must be a cube, as Besicovitch and Eggleston showed [1].

For a walk in fresh air a cage might be the right holding instrument. Coxeter [4] asked about its minimal total length in the case of the unit ball and the question was investigated by Besicovitch [3] and Valette [6].

For a visit on a market the use of a net might be appropriate. The minimal length in the case of the unit ball was determined by Besicovitch [2].

Can a very simple instrument such as a (rigid) circle be used to hold a convex body? Certainly not in the case of a ball. More generally, it cannot be used to hold any ellipsoids, (bounded) circular cylinders, or other usual convex bodies. F. Caragiu asked: 'Do there exist any convex bodies which can be held using a circle?'

We shall see here not only that these convex bodies exist, but also that they appear to form a large majority!

Let $\mathcal{B}$ be the space of all convex bodies and $\mathcal{C}$ the space of all circles in $\mathbb{R}^3$, both endowed with the usual Pompeiu–Hausdorff metric δ.

Mathematically, for a convex body B to be *held by the circle* C means that $C \cap \operatorname{int} B = \emptyset$ and, for some number $m \in \mathbb{N}$, there is no continuous mapping $f : [0, 1] \to \mathcal{C}$ such that $f(0) = C$, $\delta(f(0), f(1)) > m$ and, for all $t \in [0, 1]$, $f(t)$ is congruent with C and $f(t) \cap \operatorname{int} B = \emptyset$.

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THEOREM. *The convex bodies which cannot be held by a circle form a nowhere dense subset of $\mathcal{B}$.*

Prerequisites

Before giving a proof to our theorem we formulate and prove a lemma.

Let C_0 be a circle and $a_1, b_1, c_1 \in C_0$ be the vertices of an acute triangle. Now let $a_i, b_i, c_i \in C_0$ $(i = 2, 3)$ be other six points such that the triangles $a_ib_ic_i$ be similar to $a_1b_1c_1$ and

$$||a_i - a_{i+1}|| = ||b_i - b_{i+1}|| = ||c_i - c_{i+1}|| = \varepsilon > 0 \quad (i = 1, 2).$$

The number $\varepsilon < 1$ is chosen so small that the triangle $a_ib_jc_k$ be acute for any indices i, j, k. A small rotation leaving C_0 invariant plus a small translation in a direction perpendicular to the plane of C_0 can be chosen such that, for the new positions a_i', b_i', c_i' of a_i, b_i, c_i, the triangles $a_i'a_ia_{i+1}$, $b_i'b_ib_{i+1}$, $c_i'c_ic_{i+1}$ be equilateral $(i = 1, 2)$.

Consider the polytope

$$Q = \text{conv}\{a_1, a_1', a_2, a_2', a_3, b_1, b_1', b_2, b_2', b_3, c_1, c_1', c_2, c_2', c_3\}$$

and the circle Γ circumscribed to the intersection of Q with the plane P situated at mid-distance between the plane of C_0 and that through a_i', b_i', c_i' $(i = 1, 2)$.

LEMMA. *The polytope Q can be held by a circle coplanar and concentric with Γ, disjoint from Q.*

Proof. First, we claim that each circle $\Gamma^\dagger$ congruent with Γ and satisfying

$$0 < \delta(\Gamma, \Gamma^\dagger) \leq \frac{\varepsilon}{4}$$

meets $\text{int}\, Q$. To prove this, let k be a congruence mapping Γ into $\Gamma^\dagger$ and remark that each edge in the set

$$\mathcal{E} = \bigcup_{i=1}^{2} \{a_ia_i', a_i'a_{i+1}, b_ib_i', b_i'b_{i+1}, c_ic_i', c_i'c_{i+1}\}$$

meets the plane $k(P)$ inside of $\text{conv}\, \Gamma^\dagger$ if $\Gamma^\dagger \cap \text{int}\, Q = \emptyset$.

The orthogonal projection Γ' of $\Gamma^\dagger$ onto P is an ellipse whose long axis equals the diameter of Γ. The orthogonal projection of $\cup \mathcal{E}$ on P consists of three (congruent) broken lines circumscribed to Γ. Any halfcircle of Γ meets at most two of these broken lines (remember the initial choice of the points $a_i, b_i, c_i, a_i', b_i', c_i'$).

Now, either

(i) $\Gamma = \Gamma'$

or

(ii) $\Gamma \cap \text{conv}\, \Gamma'$ is contained in a half of Γ.

In case (i), $k(P)$ is parallel to, but different from, P. But $k(P) \cap Q$ has a circumscribed circle larger than, and concentric with, $\Gamma^\dagger$. This implies $\Gamma^\dagger \cap \operatorname{int} Q \neq \emptyset$.

In case (ii), $\operatorname{conv} \Gamma'$ does not meet one of the three broken lines mentioned above, which implies that $\operatorname{conv} \Gamma^\dagger$ does not meet each edge in $\mathcal{E}$ and, therefore, $\Gamma^\dagger \cap \operatorname{int} Q \neq \emptyset$.

Hence our claim is verified. Now, suppose there exists a sequence $\{\Gamma_n\}_{n=1}^\infty$ of circles converging to Γ, all coplanar and concentric with Γ such that, for no n, the polytope Q can be held by Γ_n. Choose $m = 1$. There is, for each n, a path $f_n : [0, 1] \to \mathcal{C}$ with the required properties. Since $\delta(\Gamma_n, f_n(1)) > 1$, there is some number t_n such that $\delta(\Gamma_n, f_n(t_n)) = \varepsilon/4$. Some subsequence of $\{f_n(t_n)\}_{n=1}^\infty$ must converge and let Γ^* be its limit. Clearly, the circle Γ^* is congruent to Γ, $\Gamma^* \cap \operatorname{int} Q = \emptyset$ and $\delta(\Gamma, \Gamma^*) = \varepsilon/4$. Here we use the assertion claimed above with $\Gamma^\dagger = \Gamma^*$, and a contradiction is obtained. Hence Q can be held by all circles in a whole neighbourhood of Γ in $\{C \in \mathcal{C} : C$ is coplanar and concentric with Γ and $C \cap \operatorname{int} Q = \emptyset\}$. Just select one of these circles different from Γ and the lemma is proved.

The proof

Proof of Theorem. Let $\mathcal{O}$ be an open set in $\mathcal{B}$. Choose a convex body $B \in \mathcal{O}$. Select a section S of B such that the circle C circumscribed to S be maximal. Then the cylinder U with C as orthogonal section encloses $\operatorname{int} B$ and $B \cap U \subset S$. Indeed, together with any point in $B \setminus (S \cup \operatorname{int} \operatorname{conv} U)$ we would find another one or two in $S \cap C$ determining a larger circumcircle.

If $a \in C \cap B$, then either (i) the diametrically opposite point $a^* \in C$ belongs to B or (ii) there is an acute triangle abc with $b, c \in C \cap B$.

Now consider a polytope $L \in \mathcal{O}$ approximating B such that

(*) in case (i) $L \cap C$ consists of a^* plus two vertices a^+ and a^-, symmetrical with respect to the line through a, a^* and close to a,

(**) in case (ii) $L \cap C = \{a, b, c\}$,

(***) all vertices of L except for a^+ and a^- (in case (i)) lie in B.

Now remember the circle C_0 considered prior to the lemma, put $C_0 = C$ and take the points a_1, b_1, c_1 from there to be a^+, a^-, a^* in case (i), or a, b, c in case (ii). The polytope Q constructed there will now be used in the following way. We keep the circle C_0 circumscribed to one facet of Q identified with our circle C and take the convex hull K of $L \cup Q$. If the number ε used to construct Q is small enough then $K \in \mathcal{O}$, because $\lim_{\varepsilon \to 0} Q \subset L$ and, consequently, $\lim_{\varepsilon \to 0} K = L$.

The lemma provides a circle C' disjoint from Q, with which Q can be held. Let γ be the angle between any triangular facet of Q and $\operatorname{aff} C$. Since $U \cap L \subset C$, L lies in a circular cone V over C with the apex on the same side of $\operatorname{aff} C$ as Q. This and $\lim_{\varepsilon \to 0} \gamma = \pi/2$ imply $C' \cap V = \emptyset$ and therefore $C' \cap K = \emptyset$, for ε sufficiently small. Now take a parallel body K_ν of K at distance ν. If ν is small enough, $C' \cap K_\nu = \emptyset$ too. Consider a neighbourhood $\mathcal{N}$ of K_ν such that $C' \cap D = \emptyset$ and $K \subset D$ for every $D \in \mathcal{N}$. Then, obviously, each $D \in \mathcal{N}$ includes Q and therefore can be held by C'.

This ends the proof.

References

1. Besicovitch, A. S. and Eggleston, H. G.: The total length of the edges of a polyhedron, *Quart. J. Math. Oxford Ser. (2)* **8** (1957), 172–190.
2. Besicovitch, A. S.: A net to hold a sphere, *Math. Gaz.* **41** (1957), 106–107.
3. Besicovitch, A. S.: A cage to hold a unit-sphere, *Proc. Sympos. Pure Math.* Vol. VII, Amer. Math. Soc., Providence, RI, 1963, pp. 19–20.
4. Coxeter, H. S. M.: Review 1950, *Math. Reviews* **20** (1959), 322.
5. Gruber, P. M.: Aspects of approximation of convex bodies, in P. Gruber and J. Wills (eds), Handbook of Convex Geometry, Elsevier Science Publishers, 1993.
6. Valette, G.: A propos des cages circonscrites à une sphère, *Bull. Soc. Math. Belg.* **21** (1969), 124–125.