How Do Convex Bodies Sit ?

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§1. Every homogeneous convex body in $\mathbb{R}^d (d \geq 2)$ put to sit on a horizontal hyperplane finds a position of stable equilibrium. A cube has such 2d such positions and an ellipsoid with pairwise distinct axis–lengths has 2. How many positions of stable equilibrium have most convex bodies?

In term "most" is understood in the Baire category sense. For various other results on most convex bodies, see [3].

§2. Let $\mathcal{B}$ denote the Baire space of all convex bodies in $\mathbb{R}^d$. $\mathcal{B}$ is equipped, as usual, with the Pompeiu–Hausdorff metric.

For $M \subset \mathbb{R}^d$, aff M and conv M denote the affine and the convex hull of M, respectively.

For $\omega \in S^{d-1}$, let $D(\omega, r)$ denote the open ball of all points in S^{d-1}at distance less than r from ω.

Let $\mathcal{B}$ be a convex body, bd B its boundary, $c(B)$ its centroid. We denote by $\delta_B(\omega)$ the distance between $c(B)$and the point of bd B seen in direction $\omega \in S^{d-1}$from $c(B)$.

To be a local minimum for δ_B is, of course, a sufficient condition for a point in $S^{d-1}$to correspond to a position of stable equilibrium for B.

§3. We show here that in most cases there are many sitting positions.

Theorem 1. *Most convex bodies have infinitely many positions of stable equilibrium.*

Proof. Let $\{\sigma_i : i \in \mathbb{N}\}$ be a dense set in $S^{d-1}$and put

$$\mathcal{B}_{i,n} = \{B \in \mathcal{B} : \delta_B \text{ has precisely one local minimum in } D(\sigma_i, n^{-1})\}\,.$$

We show that $\mathcal{B}_{i,n}$ is nowhere dense in $\mathcal{B}$, for any i and n.

Let $0 \subset \mathcal{B}$ be open. We have to find an open set $\mathcal{N} \subset 0 \backslash \mathcal{B}_{i,n}$. This is easily done if $0 \cap \mathcal{B}_{i,n} = \emptyset$, so suppose $B \in 0 \cap \mathcal{B}_{i,n}$. We approximate B by a polytope $P \in 0$ admitting a point $\omega_0 \in D(\sigma_i, n^{-1})$ as a local minimum of δ_P. When choosing P we also arrange that the line L through $c(P)$parallel to ω_0 does not meet the (d - 2)–skeleton $\sum$ of P. Let F_0 be the facet containing the point $x_0 = \text{c (P)} + \delta_P(\omega_0)\omega_0$ and F_1 be the other facet meeting L; put $\{x_1\} = F_1 \cap L$.

Let F_0', F_1' be two congruent regular (d - 1)–simplices disjoint from $\sum$, and $F_i' \subset F_i$ and $c(F_i') = x_i (i = 0, 1)$. We chop off thin slices from P using 2d hyperplanes through the facets of F_0', F_1', such that

- the resulting polytope P' has F_0', F_1' as facets instead of F_0, F_1,
- the polyhedron P^* obtained as intersection of all subspaces containing P where boundaries include facets of P distinct from F_1' be a polytope,
- the points $c(P')$and $c(P)$be close enough to guarantee that the

projection x'_0 of $c(P')$on aff F'_0 lies in int F'_0 and that, x'_1 being the intersection of the line through x_0 and $c(P')$with aff F_1, $d \mid\mid x_1 - x'_1 \mid\mid$ is smaller than the distance from x_1 to any facet of F'_1.

The main idea of the proof is the following construction of a polytope P'' close to P' and having the same centroid, by adding to P' two simplices based on F'_0 and F'_1. Let $x_\alpha$be a point collinear with x'_0 and $c(P')$such that $\mid\mid x_\alpha - x'_0 \mid\mid = \alpha$, $x_\alpha \notin P'$, and $P' \cup S_0$ is convex, where

$$S_0 = \text{ conv } (\{x_\alpha\} \cup F'_0) .$$

The line L' through $c(S_0)$ and $c(P')$intersects aff F'_1 in a point between x_1 and x'_1. Let $y_\alpha \in L' \backslash P'$ be such that $v(y_\alpha) \in P^*$ and $\Delta(y_\alpha)$ is maximal; here $v(y) = x_1 + d(y - x_1)$ and $\Delta(y)$ denotes the distance from y to aff F'_1. We see that $\Delta_0 > 0$, where

$$\Delta_0 = \lim_{\alpha \to 0} \Delta(y_\alpha) .$$

Because we may consider a sufficiently small number $\alpha > 0$, such that

$$\lim_{\alpha \to 0} \alpha d^{-1} \mid\mid c(S_0) - c(P') \mid\mid . \quad \mid\mid y_\alpha - c(P') \mid\mid^{-1} = 0 ,$$

$$\alpha d^{-1} \mid\mid c(S_0) - c(P') \mid\mid < \Delta_0 \mid\mid y_\alpha - c(P') \mid\mid .$$

Then we can find a point $y \in L'$ between x''_1 and y_α such that

$$(*) \qquad \alpha d^{-1} \mid\mid c(S_0) - c(P') \mid\mid < \Delta(y) \mid\mid y - c(P') \mid\mid .$$

We have

$$v(y) = x_1 + d(y - x_1) \in P^* ,$$

which guarantees that convexity is preserved when $S_1 = \text{ conv } (\{v(y)\} \cup F'_1)$ is added to $P' \cup S_0$: let P'' be the resulting polytope. Clearly, $y = c(S_1)$.

To show $c(P') = c(P'')$ we remark that

(i) the centroid $c(S_0)$, $c(P')$ and $c(S_1)$ are collinear,

(ii) vol $S_0 \mid\mid c(S_0) - c(P') \mid\mid =$ vol $S_1 \mid\mid c(S_1) - c(P') \mid\mid$.

Remark (ii) follows from

$$\text{vol } S_0 = \text{vol conv}(\{x_\alpha\} \cup F'_0) = \alpha d^{-1} \text{vol} F'_0 ,$$

$$\text{vol } S_1 = \text{vol conv}(\{v(y)\} \cup F'_1) = \Delta(y) \text{vol} F'_1 ,$$

$$\text{vol} F'_0 = \text{vol} F'_1 ,$$

and (*). If α is small enough, $P'' \in 0$ and the orthogonal projection c_F of $c(P')$ onto aff F lies in int F and $\mid\mid c_F - c(P') \mid\mid^{-1} (c_F - c(P')) \in D(\sigma_i, n^{-1})$ for any facet F of S_0.

Now, obviously, $\delta_{P''}$ has d local strict minima in $D(\sigma_i, n^{-1}$. Thus, for a whole open neighbourhood $\mathcal{N}$ of P'' in 0 and for each convex body $B \in \mathcal{N}$, δ_B has at least d local minima in $D(\sigma_i, n^{-1}$. Since $d \geq 2$, this implies $\mathcal{N} \subset 0 \backslash \mathcal{B}_{i,n}$.

Obviously $\bigcup_{i,n=1}^{\infty} \mathcal{B}_{i,n}$ contains each convex body B such that δ_B has an isolated local minimum. Hence, for most convex bodies $B \in \mathcal{B}$, δ_B has no isolated local minima, which implies that δ_B has infinitely many local minima and B can sit on a hyperplane in infinitely many ways.

§4. Suppose now we have a travel bag with the shape of a convex surface, which we intend to fill with various clothes, games, bottles, and other useful things. For different distributions of the useful things, the bag may sit in quite different sets of positions, because the corresponding centre of mass distribution varies. This small section is devoted to this case.

We consider mass distributions as different if and only if the corresponding centres are distinct. The following result is known.

For most convex bodies and most points $z \in B$, the distance from z to points in bd B has a function $\delta_{B,z}$ defined on S^{d-1} has infinitely many local minima.

Indeed we proved in [2] that, for most $B \in \mathcal{B}$ and $z \in B$, there are infinitely many normals of bd B passing through z. This was done by showing that $\delta_{B,z}$ has infinitely many local maxima, for most B and z. After only common sense changes applied to the proof in [2], the above result — i. e. with minima instead of maxima — is obtained. Similarly, in [1] it is shown that, for most $B \in \mathcal{B}$ and infinitely many $z \in B$, there are uncountable many normals of bd B passing through z.

These results suggest that there might be infinitely (uncountably?) many positions of stable equilibrium for many distinct mass distributions. We point out that a mass distribution, which can be identified with the corresponding centroid, has to be understood as a (continuous) function defined on the entire space $\mathcal{B}$.

References

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