POINTS JOINED BY THREE SHORTEST PATHS ON CONVEX SURFACES

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ABSTRACT. Let \( S \) be a convex surface and \( x \in S \). It is shown here that the set of all points of \( S \) joined with \( x \) by at least three shortest paths can be dense in \( S \). It is proven that, in fact, in the sense of Baire categories most convex surfaces have this property, for any \( x \). Moreover, on most convex surfaces, for most of their points, there is just one farthest point (in the intrinsic metric), and precisely three shortest paths lead to that point.

1. INTRODUCTION

Let \( S \subseteq \mathbb{R}^3 \) be a (closed) convex surface and \( x \in S \). The set \( T_x \) of all points joined with \( x \) by at least three segments, i.e., shortest paths in \( S \), is known—and easily seen—to be at most countable. The set \( C_x \) of all points joined with \( x \) by at least two segments, is not very large either. It is proven in [6] that \( C_x \) is \( \sigma \)-porous and therefore of first Baire category and of (2-dimensional) Hausdorff measure 0. However, \( C_x \) must be uncountable if it contains more than one point, because it is arcwise connected [9].

The set \( C_x \) can be dense in \( S \); in fact this is the generic behaviour, as shown in [6].

About \( T_x \) even the following question seems to be an open problem: "Does every convex surface \( S \) possess a point \( x \) with \( T_x \neq \emptyset \)?" This problem and a rapid investigation of common convex surfaces already show how thin \( T_x \) usually is. The first result of this paper is therefore very surprising, at least for the author. It states that \( T_x \) can be dense in \( S \), too, and that this is the generic behaviour!

H. Steinhaus raised the more restrictive, exciting question whether there always exists a point \( x \in S \) admitting a farthest point in \( T_x \) ([3], p. 44 (iii)). A related question of Steinhaus, less concrete, asks for a description of \( F_x \), the set of all farthest points from \( x \in S \) ([3], p. 44 (iv)). A few steps towards an answer were made in [9], where we proved, for example, that \( F_x \subseteq C_x \) and, if \( S \) is of class \( C^1 \), \( F_x \subset C_x \). Of course, in general \( F_x \notin T_x \). However, it may happen that \( F_x \subset T_x \). We will show that this happens indeed, generically, for most points \( x \in S \).

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3513
We shall make use of the following lemma.

**Lemma 1.** Let \( S \in \mathcal{P} \) and \( x \in S \), suppose \( y, z \) are distinct points in \( C_x \) and consider two segments from \( x \) to \( y \) and another two from \( x \) to \( z \). Then there is a Jordan arc \( A \) joining \( y \) and \( z \), lying (except for the endpoints \( y \) and \( z \)) in the domain (i.e., connected open set) of \( S \) which has all four preceding segments on its boundary, and decomposing it into two subdomains \( \Delta, \Delta' \), such that each point of \( A \) is joined with \( x \) by a segment in \( \Delta \) and another segment in \( \Delta' \).

This lemma follows immediately from Theorem 1 in [9] and its proof. We shall also need the following result, first proven in [9].

**Lemma 2.** Let \( S \in \mathcal{P} \), \( x \in S \) and \( y \in F_x \). Then any angle between two tangent directions at \( y \) measuring (on the tangent cone) more than \( \pi \) contains the tangent direction of a segment from \( y \) to \( x \). Thus, if the full angle of \( S \) at \( y \) is larger than \( \pi \), then \( y \in C_x \).

A basic result of Aleksandrov [1] on the convergence of angles will also be needed.

**Lemma 3.** Let \( S, S_n \in \mathcal{P} \), \( x \in S \), \( x_n \in S_n \), and assume that the full angle of \( S \) at \( x \) is \( 2\pi \), \( x \) is an endpoint of the segments \( \Sigma, \Sigma' \subset S \) and \( x_n \) is an endpoint of the segments \( \Sigma_n, \Sigma'_n \subset S_n \). If \( S_n \) converges to \( S \), \( x_n \) converges to \( x \), \( \Sigma_n \) converges to \( \Sigma \) and \( \Sigma'_n \) converges to \( \Sigma' \), then the angle between \( \Sigma_n \) and \( \Sigma'_n \) at \( x_n \) converges to the angle between \( \Sigma \) and \( \Sigma' \) at \( x \).

For any Jordan arc \( J \) with definite directions at its endpoints the notions of a right and a left swerve can be introduced (see, for example, [2], pp. 108–110).

Let \( M_1, \ldots, M_n \) be two-dimensional manifolds, each with its own intrinsic metric. For every \( i \) consider an open set \( D_i \) with \( \overline{D_i} \subset M_i \) and whose boundary is the union of pairwise disjoint rectifiable Jordan curves \( C_i, \ldots, C_{n_i} \). We say that the manifold \( M \) is obtained by gluing together \( \overline{D_1}, \ldots, \overline{D_n} \) if all \( C_i^j \) are decomposed into Jordan arcs which are pairwise identified in such a way that any two identified subarcs of these identified Jordan arcs have the same length, while \( \overline{D_1} \cup \ldots \cup \overline{D_n} = M \). We shall also make use of the following fundamental result of Aleksandrov [1].

**Aleksandrov's gluing theorem.** Let \( M_1, \ldots, M_n \) have nonnegative curvature, and let the swerve have bounded variation on any subarc of any \( C_i^j \). The manifold \( M \) obtained by gluing together \( \overline{D_1}, \ldots, \overline{D_n} \) has nonnegative curvature if and only if for any identified subarcs \( A_i \subset C_i^j \) and \( A' \subset C_k^l \), the sum of the swerve of \( A_i \) in \( M_i \) towards \( D_i \) and the swerve of \( A' \) in \( M_j \) towards \( D_j \) is nonnegative.
and for any point $p$ belonging to more than two sets $D_i$ the sum of the angles of these $D_i$ at $p$ is at most $2\pi$.

3. Generic density of $T_x$

The purpose of this section is to prove the following.

**Theorem 1.** On most convex surfaces $S \in \mathcal{S}$, for each point $x \in S$, the set $T_x$ is dense in $S$.

**Proof.** Consider a surface $S \in \mathcal{S}$ satisfying $\overline{C_x} = S$ for some point $x \in S$. Let $O \subset S$ be open. We choose a point $y \in O \cap C_x$ and another point $y' \in C_x$. Let $\Sigma_y$, $\Sigma_y'$ be two segments from $x$ to $y$ and $\Sigma_{y'}$, $\Sigma_{y'}'$ another two from $x$ to $y'$. By Lemma 1, there is a Jordan arc $A \subset C_x$ joining $y$ to $y'$ with the following properties:

(i) $A$ lies, except for its endpoints, in the domain of $S$ bounded by $\Sigma_y \cup \Sigma_{y'} \cup \Sigma_{y'}'$ and decomposes it into two subdomains $\Delta_0$ and $\Delta'_0$.

(ii) Each point of $A$ is joined with $x$ by (at least) two segments, which lie, except for their endpoints, one in $\Delta_0$, the other in $\Delta'_0$.

Suppose w.l.o.g. that $\Sigma_y \cup \Sigma_{y'} \cup A$ is the boundary of $\Delta_0$ and $\Sigma_y' \cup \Sigma_{y'}' \cup A$ the boundary of $\Delta'_0$. Since $A$ is locally connected, there is a point $z \in A \cap O$ such that the whole subarc $A'$ of $A$ from $y$ to $z$ lies in $O$. Let

$$\Sigma_z \subset \Delta_0, \quad \Sigma'_z \subset \Delta'_0$$

be two segments joining $x$ with $z$. Denote by $\Delta$ the domain of $S$ which is bounded by $\Sigma_y \cup \Sigma_z \cup A'$ and does not meet $\Sigma_y'$. Similarly, let $\Delta'$ be the domain bounded by $\Sigma_{y'} \cup \Sigma'_z \cup A'$ and not meeting $\Sigma_y$.

Choose a point $u \in \Delta \cap C_x$. Let $\tau_u$, $\tau'_u$ be the tangent directions of two segments from $x$ to $u$, at $x$. Also, let $\tau_y$ and $\tau_z$ be the tangent directions at $x$ of $\Sigma_y$ and $\Sigma_z$, respectively. We may suppose w.l.o.g. that $\tau_y$, $\tau_u$, $\tau'_u$, $\tau_z$ lie in this order on the closed Jordan curve of all tangent directions at $x$.

Let now the point $v$ move on $A'$ from $y$ to $z$, join it in $\Delta$ by a segment $\Sigma_v$ with $x$ and consider the tangent direction $\tau_v$ of $\Sigma_v$ at $x$. Since $\tau_v = \tau_y$ for $v = y$, $\tau_u = \tau_z$ for $v = z$ and $\tau_v$ never lies between $\tau_u$ and $\tau'_u$, it follows that there is a a point $w \in A'$ which is a limit point of points $v$ with $\tau_v$ between $\tau_y$ and $\tau_u$ and, also, of points $v$ with $\tau_v$ between $\tau'_u$ and $\tau_z$. Therefore $w$ is joined by two distinct segments with $x$ in $\Delta$.

Since there is, in addition, a segment from $w$ to $x$ in $\Delta'$, we have $w \in T_x$. This shows that $O \cap T_x \neq \emptyset$. Hence $\overline{T_x} = S$.

By Corollary 2 in [6], most surfaces $S \in \mathcal{S}$ satisfy $\overline{C_x} = S$ for any $x \in S$. This ends the proof.

4. Generic existence of points in $F_x \cap T_x$

We treat generically here Steinhaus' first question. This is less spectacular, and less easy too. Concretely, we obtain the following result.

**Theorem 2.** On most surfaces $S \in \mathcal{S}$, for most points $x \in S$, the set $F_x$ consists of a single point, joined with $x$ by precisely three segments.
Proof. Let $S_x$ denote the set of all segments from $x$ to points in $F_x$. Also, for any $S \in \mathcal{P}$, let

$$A_0(S) = \{x \in S: \text{card} S_x \leq 2\},$$
$$A_n(S) = \{x \in S: \text{there are four segments in } S_x \text{ at mutual distances at least } n^{-1}\},$$
$$B_n(S) = \{x \in S: \text{diam } F_x \geq n^{-1}\}.$$

For any number $n \in \mathbb{N}$, both sets $A_n$ and $B_n$ are closed in $S$. Let

$$\mathcal{P}' = \{S \in \mathcal{P}: \text{card} F_x \neq 1 \text{ or card} S_x \neq 3 \text{ is of 2nd category}\},$$
$$\mathcal{A}_n = \{S \in \mathcal{P}: A_n(S) \text{ is not nowhere dense} \ (n \in \{0\} \cup \mathbb{N})\},$$
$$\mathcal{B}_n = \{S \in \mathcal{P}: B_n(S) \text{ is not nowhere dense} \ (n \in \mathbb{N})\}.$$

We show that $\mathcal{P}'$ is of first category. We observe, indeed, that

$$\mathcal{P}' \subset \bigcup_{n=0}^{\infty} \mathcal{A}_n \cup \bigcup_{n=1}^{\infty} \mathcal{B}_n.$$

We shall say that a point in $\mathbb{R}^3$ is rational if its coordinates are rational. Also, for $S \in \mathcal{P}$ and $z \in \mathbb{R}^3$, let $\delta(z, S) = \min_{x \in S} \|x - z\|$. If a closed subset of $S$ is not nowhere dense, it must include a disc $D$ (in the intrinsic metric of $S$) on $S$. For every such disc $D$ we may find a rational point $z \in \mathbb{R}^3$, at distance at most $q^{-1}$ from $S$ ($q \in \mathbb{N}$), such that $B(z, 2q^{-1}) \cap S \subset D$. So, for each $m \in \mathbb{N}$,

$$\mathcal{A}_0 \cup \mathcal{A}_m \cup \mathcal{B}_m \subset \bigcup_{z, q} \mathcal{A}_{m, z, q},$$

where, for any $m, q \in \mathbb{N}$ and rational $z \in \mathbb{R}^3$,

$$\mathcal{A}_{m, z, q} = \{S \in \mathcal{A}_0 \cup \mathcal{A}_m: \delta(z, S) \leq q^{-1} \text{ and } B(z, 2q^{-1}) \cap S \text{ is included in } A_0 \text{ or } A_m \text{ or } B_m\}.$$

To show that $\mathcal{A}_{m, z, q}$ is nowhere dense, let $\mathcal{O} \subset \mathcal{P}'$ be open. If $\mathcal{A}_{m, z, q} \cap \mathcal{O} = \emptyset$, there is nothing to show. If $S_0 \in \mathcal{A}_{m, z, q} \cap \mathcal{O}$, we choose a polytopal surface in $\mathcal{O}$ approximating $S_0$, with a vertex $x$ in $S_0 \cap B(z, 2q^{-1})$ and with small spherical images at all vertices. Then, by Lemma 2, every arc of length $\pi$ on the Jordan curve $J$ (of length close to $2\pi$) of all tangent directions at some point $y \in F_x$ contains the tangent direction of a segment in $S_x$. It follows that there are two segments from $x$ to $y$ such that the lengths of the arcs determined by their tangent directions at $y$ on $J$ are at most $\pi$, or there are three segments from $x$ to $y$ such that the lengths of the three arcs in which their tangent directions at $y$ divide $J$ are less than $\pi$.

In the case of two segments, we consider a small number $\alpha > 0$ and a small triangle $\Theta$ of angles $\alpha, \alpha, \pi - 2\alpha$ and sides $a, a, b$, say. We also consider two congruent isosceles triangles with sides $r, r, a$, where $r$ equals the distance from $x$ to $y$ on $S$, and a third isosceles triangle with sides $r, r, b$ and a small angle $3\alpha$. Now cut $P$ along the two segments from $x$ to $y$. By Aleksandrov's gluing theorem, for $\alpha$ sufficiently small, the four triangles and the two pieces of $P$ can be glued in an obvious way together, the two congruent long thin isosceles triangles becoming adjacent. The nonnegative
swerve of a segment on both sides and our choice of the angles and sides of the involved triangles imply the assumptions in Aleksandrov’s gluing theorem. This construction works even if \( y \) lies on a facet of \( P \). This cannot happen in the present case of two segments if \( \text{card} \, S_x \leq 3 \) but may well happen if \( \text{card} \, S_x \geq 4 \). For instance if there are precisely four segments from \( x \) to \( y \) and the tangent directions at \( y \) are pairwise opposite.

In the case of three segments, we consider the three distances \( \alpha, \beta, \gamma \) determined on \( J \) by their tangent directions. We build a small triangle \( \Theta \) with sides \( a, b, c \) and angles \( \pi - \alpha, \pi - \beta, \pi - \gamma \) and three isosceles triangles with sides \( r, r, a; r, r, b; r, r, c \). We cut \( P \) along the three segments and glue the resulting three pieces together with the four preceding triangles such that the piece containing the angle \( \alpha \) is glued to the isosceles triangles of smallest sides \( b \) and \( c \), etc, \( \Theta \) being glued to all three isosceles triangles.

In both cases, if the initial triangle \( \Theta \) is chosen small enough, then the new polytopal surface \( P' \) can be chosen in \( \mathcal{S}', \) due to the Olovianishnikov-Pogorelov uniqueness theorem [5] (see the comment in [8], p. 114). Of course, the point of \( P' \) corresponding to \( x \) can be kept at \( x \). For \( P' \), the set \( F_x \) consists of a single point \( y' \) inside the set corresponding (through the isometry) to \( \Theta \), at a distance from \( x \) larger than \( r \) (the choice of an angle of \( 3\alpha \) for an isosceles triangle in the first case was made to this end). Clearly, in \( P' \), \( \text{card} \, S_x = 3 \), the full angle at \( y' \) is \( 2\pi \) and the angles between the directions at \( y' \) of the three segments are all less than \( \pi \).

Consider a sequence of convex surfaces \( \{S_n\}_{n=1}^{\infty} \) converging to \( P' \), \( x_n \in S_n \), \( y_n \in F_{x_n} \) and \( x_n \to x \). Then, clearly, \( y_n \to y' \) and any segment from \( x_n \) to \( y_n \) converges to a segment from \( x \) to \( y' \). By Lemma 3, if \( \Sigma_n, \Sigma' \) are two such segments converging to \( \Sigma, \Sigma' \) respectively, then the angle between \( \Sigma_n \) and \( \Sigma'_n \) at \( y_n \) converges to the angle between \( \Sigma \) and \( \Sigma' \) at \( y' \).

Suppose now \( S_n \in \mathcal{A}_{m,z,q} \). Then \( B(z, 2q^{-1}) \cap S_n \) is contained in at least one of the sets \( A_0, A_m, B_m \). This means that (choosing \( x_n \in A_0 \)) in the first case there are precisely two segments from \( x_n \) to \( y_n \) (whose angle at \( y_n \) must converge by Lemma 2 to \( \pi \), because the full angle at \( y_n \) converges to \( 2\pi \)), or there are four segments in \( S_{x_n} \) at distance at least \( m^{-1} \) from each other, or the diameter of \( F_{x_n} \) is at least \( m^{-1} \). But this implies that the angle between two of the three segments in \( S_x \) is \( \pi \) (in the first case), or there are four distinct segments in \( S_x \) (in the second case), or \( F_x \) has more than one point (in the third case), and contradictions are obtained.

Hence there is a ball around \( P' \) in \( \mathcal{S} \) disjoint from \( \mathcal{A}_{m,z,q} \). Thus \( \mathcal{A}_{m,z,q} \) is nowhere dense, and \( \mathcal{A}_0 \cup \mathcal{A}_m \cup \mathcal{B}_m \) of first category in \( \mathcal{S} \). Hence \( \mathcal{S}' \) is of first category, and the proof finds its end.

5. **Open problems**

We conclude the paper with open questions arising from the generic investigation of \( \mathcal{S} \) and related to the results of this paper.

**Problem 1.** Is it true, for most convex surfaces \( S \in \mathcal{S} \), that for any \( x \in S \) and \( y \in F_x \), \( x \) and \( y \) are joined by at most 3 segments?

**Problem 2.** Is it true, for most convex surfaces \( S \in \mathcal{S} \), that any two points of \( S \) are joined by at most 3 segments?
Problem 3. Is it true, for most convex surfaces $S \in \mathcal{P}$, that (the) two farthest points of $S$ are joined by precisely 3 segments?

Examples of polytopal surfaces show that two farthest points may be joined by 5 segments. It follows from recent unpublished work of P. Horja about manifolds of nonpositive curvature that this must be the case if the two points are not vertices.

Problem 4. Is the family of all convex surfaces with two farthest points joined by 5 segments dense in $\mathcal{P}$? Is 5 the right number in Problem 3?

References


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