Geometriae Dedicata **62:** 99–105, 1996.

Conjugate Points and Closed Geodesic Arcs on Convex Surfaces

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(Received: 25 January 1995; in final form: 16 August 1995)

Abstract. This paper discusses conjugate points on the geodesics of convex surfaces. It establishes their relationship with the cut locus. It shows the possibility of having many geodesics with conjugate points at very large distances from each other. It also shows that on many surfaces there are arbitrarily many closed geodesic arcs originating and ending at a common point. To achieve these goals, Baire category methods are employed.

Mathematics Subject Classifications (1991): 53C45, 53C22.

Key words: conjugate points, cut locus, closed geodesic arcs, Baire categories.

1. Introduction

In this paper, we propose a new definition for the notion of a conjugate point to a point x of an arbitrary closed convex surface, along a geodesic starting at x. Rinow in [7] and Kunze in [5] present two further definitions of conjugacy. The exact mutual relationship between the three notions will not be investigated here. They all correspond to the differential-geometric notion of 'first conjugate point'. Note that the one in [7] extends to second, third, etc., conjugate points too. Ours seems to be the simplest.

We examine the relationship between the set of conjugate points to x and the cut locus E_x of x, i.e. the set of all endpoints of maximal (by inclusion) segments starting at x.

Then we show that, in the sense of Baire categories, on most convex surfaces there are densely many points x and geodesics G starting at x, with conjugate points arbitrarily far (on G) from x. For some of these points x, G can even be chosen to have no self-intersections! This strengthens Theorems 1 and 2 from [14].

Finally it is shown that, on most convex surfaces, there are points x with arbitrarily many closed geodesic arcs at x. This result complements Theorem 3 in [14].

The behaviour is in both cases very 'irregular' from the differential-geometric point of view, even impossible in the analytic case. But it fits well to the multitude of exotic properties already discovered for most convex surfaces (see, for example, [8]–[15] or the surveys [2], [16]).

Let S be a convex surface in $\mathbb{R}^3$. A *geodesic* is the image of an interval $I \subset \mathbb{R}$ through a continuous mapping $c : I \to S$, such that every point in I has a neighbourhood N in I such that c is an isometry on N and $c(N)$ is a segment, i.e. a shortest path in S between two points of S. If $I = \mathbb{R}$ and c is periodic, then $c(I)$ is called a *closed geodesic*. If I is a compact interval $[a, b]$, then $c(I)$ is called a *geodesic arc*. If, moreover, $c(a) = c(b)$, then $c(I)$ is said to be a *closed geodesic arc at* $c(a)$.

We denote by λC the length of the arc $C \subset S$.

The space of all convex surfaces (always closed here) in $\mathbb{R}^3$, endowed with the usual Pompeiu–Hausdorff metric, is a Baire space, and 'most' means 'all, except those in a set of first Baire category'.

2. On Conjugate Points

Let S be a convex surface and $x \in S$. We consider an arbitrary (but nonempty) maximal (by inclusion) geodesic G containing x. Let y be a point of G such that, for some neighbourhood $\mathcal{N}$ of a subgeodesic G_y of G from x to y, the minimal length of arcs from x to y in $\mathcal{N}$ is realized uniquely by G_y. (Here, the space of arcs is endowed with the usual compact-open topology.) The existence of such a point y is guaranteed by the definition of a geodesic in conjunction with the basic property that a proper subarc of a segment realizes uniquely the minimum distance between its endpoints. We call such an arc G_y a *length-minimizing arc* of G.

Let + be the direction from x to y on G (corresponding to increasing length on G_y), and – the opposite one. Define, according to +, an order on G and let

$$z_+ = \sup\{y \in G : G_y \text{ is length-minimizing}\},$$

allowing for z_+ the value $+\infty$ too (which will never occur if G is a geodesic arc). We call z_+ the *conjugate* (point or symbol) of x on G in direction +.

Now, define c_+ to be the mapping $x \mapsto z_+$ and, analogously, the mapping c_-. Also, let

$$z_- = \lim_{x \to z_+ -} c_-(x).$$

The limit exists because c_-, like c_+, is a monotone increasing function.

The couple z_-, z_+ is called a *conjugate pair*. In particular, a conjugate pair of points on G is the couple of endpoints of a maximal (by inclusion) geodesic arc in G such that each proper subarc is length-minimizing. Clearly, $c_-(u) \leq z_-$ and $c_+(u) \geq z_+$ for any point u between z_- and z_+ on G.

To illustrate these notions, let us consider the following example. Let G be an infinitely long geodesic starting at the vertex x of a polytopal surface. In this case, $c_+(y) = +\infty$ for any $y \in G$ and $c_-(y) = x$ for any $y \in G \setminus \{x\}$. So, x and $+\infty$ are conjugate to y, but only $x, +\infty$ is a conjugate pair.

A geodesic arc whose endpoints form a conjugate pair will be called *stretched.*

The set of all points conjugate to x along the various geodesics starting at x is denoted by D_x.

For any point $x \in S$, the *cut locus* $E_x \subset S$ is defined to be the set of all endpoints different from x of maximal (by inclusion) segments starting at x. An important subset of the cut locus is the set C_x of all points multijoined to x, i.e. joined by at least two segments with x. The following theorem establishes a remarkable connection between these sets E_x, C_x and D_x.

THEOREM 1. *All points in $E_x \setminus C_x$ are conjugate to x.*

Proof. If $E_x = C_x$ there is nothing to prove. So, let $y \in E_x \setminus C_x$. There is a single segment Σ from x to y. If there is no geodesic extending Σ beyond y then, trivially, y is conjugate to x. Suppose now that some geodesic $\Gamma \supset \Sigma$ goes beyond y. Clearly, the conjugate $c_+(x)$ of x on Γ is not interior to Σ. Let $z \in \Gamma \setminus \Sigma$ be close to y and consider a segment Σ_z from x to z. This segment is shorter than the subarc of Γ from x to z, because Σ cannot be extended as a segment beyond y. For $z \to y$, we have $\Sigma_z \to \Sigma$, since Σ is the only segment from x to y.

Let now $v \in \Gamma \setminus \Sigma$, denote by Γ_u the subarc of Γ from $u \in \Gamma$ to v, and consider an arbitrary neighbourhood $\mathcal{N}$ of Γ_x. For $z \in \Gamma_x$ close enough to y, $\Sigma_z \cup \Gamma_z$ belongs to $\mathcal{N}$ and is shorter than Γ_x. Therefore $c_+(x) = y$.

In [5] Kunze proves this same result using his definition of conjugate points.

In our everyday life, E_x is a geometrically realized tree with finitely many 'nodes' (i.e. points at which E_x is not locally homeomorphic to a line), and $E_x \setminus C_x$ is exactly the (very scarce) set of all nodes of degree 1. But this does not happen frequently. Instead, typical is the following quite unexpected, almost 'paradoxal' behaviour.

THEOREM 2. *On most convex surfaces S, for any point $x \in S$, most points of S are conjugate to x and lie in its cut locus, more precisely $E_x \setminus C_x$ is residual in S.*

Proof. By Theorem 1, $E_x \setminus C_x$ consists of points conjugate to x. Clearly, the set E of all endpoints of S is a subset of E_x. By Theorem 1 in [10], E is residual in S for most convex surfaces S.

For every convex surface S, the set C_x is of first category, by Theorem 1 in [15]. It follows that, for most S, $E_x \setminus C_x$ is residual in S.

Note that, by Theorem 2, the set $E_x \setminus C_x$ may well be uncountable (compare [5], p. 201).

3. On Conjugate Pairs

In this section, we present a general convergence theorem on conjugate pairs and then apply it in the case of most convex surfaces. The convergence theorem is a strengthening of Lemma 2 in [14].

Let d_S denote the intrinsic metric of the convex surface S.

THEOREM 3. *Let Γ_0 be a length-minimizing arc on a geodesic of the convex surface S. If Γ_0 has endpoints x_0, y_0, S_n are convex surfaces, x_n, $y_n \in S_n$ ($n \in \mathbb{N}$), $S_n \to S$, $x_n \to x_0$ and $y_n \to y_0$ when $n \to \infty$, then there are stretched geodesic arcs $\Gamma_n \in S_n$ with endpoints x_n, y_n such that $\Gamma_n \to \Gamma_0$.*

Proof. Consider a point in the bounded component of $\mathbb{R}^3 \setminus S$ as centre of a central projection p. For n large enough ($n \geq m$, say), p induces a homeomorphism h_n from S_n to S, according to which $d_{S_n} \to d_S$.

There is a neighbourhood $\mathcal{N}$ of Γ_0 in the space of arcs on S from x_0 to y_0, such that Γ_0 is shorter than any other member of $\mathcal{N}$. Let $N = \cup\mathcal{N} \subset S$. Clearly, $x_0, y_0 \in \operatorname{int} N$ (the interior relative to S).

Consider a line-segment $L \subset \mathbb{R}^2$ of endpoints x, y, as long as Γ_0, and an open rectangle $R \supset L$.

Let $f : \overline{R} \to N$ be a mapping such that $f(L) = \Gamma_0$, $f(x) = x_0$, $f(y) = y_0$ and each point of $\overline{R}$ has a neighbourhood V for which $f|_V$ is a homeomorphism. Consider the metric d of $\overline{R}$ defined by

$$d(u, v) = \inf_J \lambda f(J),$$

where J are arcs joining u to v in $\overline{R}$. Now, clearly, f is a local isometry on R.

For each $n \geq m$ and any points $u, v \in \overline{R}$ joined by arcs $J \subset \overline{R}$, let

$$d_n(u, v) = \inf_J \lambda h_n^{-1}(f(J)).$$

It is easily checked that d_n is a metric in $\overline{R}$ and $h_n^{-1} \circ f$ is a local isometry from (R, d_n) to (S_n, d_{S_n}). Since $d_{S_n} \to d_S$, we have $d_n \to d$ too.

Choose in the set $f^{-1}(h_n(x_n))$ a point x'_n closest to x and in $f^{-1}(h_n(y_n))$ a point y'_n closest to y. Since $h_n(x_n) \to x_0$ and f is locally isometric, we must have $x'_n \to x$ and, similarly, $y'_n \to y$ in (R, d).

Let Γ'_n be a shortest arc from x'_n to y'_n in $(\overline{R}, d_n)$ and put $\Gamma^*_n = h_n^{-1}(f(\Gamma'_n))$. From $x'_n \to x$, $y'_n \to y$ and $d_n \to d$, we conclude that $\Gamma'_n \to L$ and $\Gamma^*_n \to \Gamma_0$. Thus, starting with some index $m' \geq m$, every arc Γ'_n lies in R ($n \geq m'$). For these indices, Γ^*_n realizes the minimum length of all arcs from x_n to y_n in a certain neighbourhood (corresponding through $h_n^{-1} \circ f$ to a neighbourhood of Γ'_n consisting of curves lying in R) of Γ^*_n in S_n. Then any proper subarc of Γ^*_n is length-minimizing; this shows that x_n, y_n is a conjugate pair on Γ^*_n, and we choose $\Gamma_n = \Gamma^*_n$ ($n \geq m'$). For the first $m' - 1$ indices, we may choose any segment from x_n to y_n as Γ_n.

THEOREM 4. *For most convex surfaces S the following holds: S is of class C^1 and for any positive number r there are densely many pairs (x, τ) in the sphere bundle T_1S such that some geodesic G through x has the directions $\pm\tau$ at x and possesses a conjugate pair z_-, z_+, both at distance at least r from x.*

Proof. That most convex surfaces S are of class C^1 was first proved by Klee [4]. This enables us to speak about T_1S.

The proof of the theorem uses the proof of Theorem 1 in [14], which will not be repeated here.

We first observe that Theorem 2 can be used in the particular case that S is a polytopal surface, on which an infinitely long (in both directions) geodesic Γ can be considered. Then $z_{\pm} = \pm\infty$ and the arc $\Gamma_0 \subset \Gamma$ can be chosen arbitrarily long.

Thus, using Theorem 3 instead of Lemma 2 from [14], the proof of Theorem 1 in [14] shows precisely that most convex surfaces have a dense set of pairs (x, τ) such that some geodesic arc G of length at least $2r$ has x as middle point, has directions $\pm\tau$ at x and realizes the minimal length among all arcs in a whole neighbourhood of G. Thus G has its endpoints as a conjugate pair and the theorem is proved.

An analogous use of the proof of Theorem 2 in [14] leads to the following.

THEOREM 5. *On most convex surfaces there are non-self-intersecting geodesics G with arbitrarily distant (on G) conjugate pairs.*

4. On Closed Geodesic Arcs

On most convex surfaces there are no closed geodesics, as Gruber established in [3], but there are infinitely many closed stretched geodesic arcs; more precisely, there are arbitrarily (finitely) many pairwise disjoint closed stretched geodesic arcs [14]. (In the statement of Theorem 3 of [14], the word 'infinitely' appears inadvertently instead of 'arbitrarily'; that the geodesic arcs are indeed stretched can be seen in its proof.)

We complete here the picture by proving the next result.

THEOREM 6. *On most convex surfaces, for any number $n \in \mathbb{N}$, there are points x admitting n closed stretched geodesic arcs at x, every two of which meet only inside a disc of radius $1/n$ around x.*

Proof. Let $\mathcal{S}_m$ be the family of all convex surfaces S such that, for any point $x \in S$, there are at most $m-1$ closed stretched geodesic arcs at x having (pairwise) no common point at distance at least m^{-1} from x. To prove the theorem it suffices to show that $\mathcal{S}_m$ is nowhere dense.

To this end, let $\mathcal{O}$ be open in the space of all convex surfaces and choose a polytopal surface $S \in \mathcal{O}$ without closed geodesics (see [1], pp. 377–378) and with full angles larger than π. Then S has a simple closed quasigeodesic Q not degenerated to a quasigeodesic arc traversed back and forth (see [1], p. 378 or [6]), and this, in absence of any closed geodesics, must go through a vertex v of S.

We cut S along Q and get two pieces S_1, S_2. Let $v_i \in S_i$ and $Q_i \subset S_i$ correspond to v and Q, respectively ($i = 1, 2$). Let $\mu(v)$ denote the (measure of the) full angle of S at v and choose

$$\varepsilon \leq \frac{2\pi - \mu(v)}{2m}.$$

Let tab be an isosceles triangle with the sides ta, tb of length $\lambda Q/2$ and with the angle ε between them.

By the general gluing theorem of Aleksandrov ([1], p. 362) or its polyhedral variant ([1], p. 317), we can glue together S_1, S_2 and $2m$ copies $t_i a_i b_i$ ($i = 1, 2, ..., 2m$) of tab as follows:

(i) $t_1, t_2, ..., t_{2m}, v_1, v_2$ will coincide,
(ii) a_1, a_{m+1} and the point of Q_1 at distance (on Q_1) $\lambda Q/2$ from v_1 will coincide,
(iii) b_m, b_{2m} and the point of Q_2 at distance (on Q_2) $\lambda Q/2$ from v_2 will coincide,
(iv) $b_i, b_{i+m}, a_{i+1}, a_{i+m+1}$ will coincide ($i = 1, ..., m-1$).

Denote by S_ε the resulting polytopal surface, with a point v' corresponding to v_1 and v_2. Clearly, S_ε admits m closed geodesic arcs Γ_i at v', lying in $t_i a_i b_i \cup t_{i+m} a_{i+m} b_{i+m}$ ($i = 1, ..., m$).

For ε small enough there is a surface $S^* \in \mathcal{O}$ congruent to S_ε. Now, let the convex surfaces S_n converge to S^* for $n \to \infty$. Let $v^* \in S^*$ and $\Gamma_i^* \subset S^*$ correspond to v' and Γ_i ($i = 1, ..., m$), and choose $v_n \in S_n$ such that $v_n \to v^*$. By Theorem 3, there are stretched geodesic arcs $\Gamma_{in} \subset S_n$ with both endpoints at v_n, such that $\Gamma_{in} \to \Gamma_i^*$.

Since $\Gamma_i^* \cap \Gamma_j^* = \{v^*\}$ for $i \neq j$, we have $\operatorname{diam}\Gamma_{in} \cap \Gamma_{jn} \to 0$.

Hence, for every element S' of a whole neighbourhood $\mathcal{N}$ of S^*, there are m closed geodesic arcs at some point $x' \in S'$, with no common points of any two of them at distance at least m^{-1} from x'.

Thus, $\mathcal{S}_m$ is nowhere dense, and the theorem is proved.

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