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Farthest points on convex surfaces

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Introduction

In the very enjoyable book ([3], p. 44) of Craft, Falconer and Guy we read: "We take... a... convex surface C in $\mathbb{R}^3$.... Steinhaus... asked... what can be said qualitatively about the set of all "farthest points" from a point x." Our aim here is to investigate this question.

Let $\mathscr{S}$ be the space of all *closed convex* surfaces (i.e. boundaries of open bounded convex sets) in $\mathbb{R}^3$ and denote, for any $S \in \mathscr{S}$ and $x \in S$, by F_x the set of all farthest points from x and by C_x the set of all points joined with x by at least two *segments* (i.e. shortest paths).

We shall see that to any point x on a closed convex surface we may associate in a natural way a point or a Jordan arc $J_x \supset F_x$ lying in the cut locus of x. This will provide the topological characterization of F_x.

Further we shall prove an easily stated, remarkable geometric property of F_x: any three of its points (if it contains at least three) form an obtuse or right geodesic triangle.

Moreover, the preceding result is shown to hold for a large class of geodesic triangles with vertices in the cut locus. Interestingly, the extreme case of a right triangle does not imply the degeneracy of the surface.

For the reader, to be familiar with Aleksandrov's book [1] (see also [2], [4]) would be of considerable help.

The usual intrinsic metric of $S \in \mathscr{S}$, induced by the Euclidean distance in $\mathbb{R}^3$ will be denoted by ρ.

Let $x \in S$, and denote by E_x the *cut locus* of x, i.e. the set of all endpoints different from x of maximal (by inclusion) segments starting at x. Also, let E be the set of all *endpoints* of S, i.e. points not interior to any segment of S. The set

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E may be quite large. We proved in [6] that, in the sense of Baire categories, on most convex surfaces most points are endpoints. Our cut locus E_x includes, of course, not only F_x and C_x which are always small (see the Proposition in [8] and Theorem 1 in [7]), but also $E \setminus \{x\}$.

A *domain*, i.e. a connected open set, in $S \in \mathscr{S}$ with a Jordan closed curve as boundary is called a *Jordan domain*.

A Jordan domain in $S \in \mathscr{S}$ the boundary of which is the union of three segments is called a *geodesic triangle*.

Auxiliary results

Lemma 1. *Let $y, z \in E_x$ be distinct, and suppose Σ_y, Σ'_y are possibly coinciding segments from x to y and Σ_z, Σ'_z are possibly coinciding segments from x to z. By Lemma 1 in [8], there exists a domain Δ with boundary $\Sigma_y \cup \Sigma'_y \cup \Sigma_z \cup \Sigma'_z$. Then there is a Jordan arc J_{yz} in $C_x \cup \{y, z\}$ joining y with z. This is the unique Jordan arc J joining y with z such that $J \setminus C_x$ be finite. Moreover, every point in $J_{yz} \setminus \{y, z\}$ belongs to Δ and can be joined with x by two segments the union of which separates y from z.*

Proof. Let $T_x \subset S^2$ be the Jordan closed curve of all tangent directions at x, and $\tau_y, \tau'_y, \tau_z, \tau'_z \in T_x$ the tangent directions at x of $\Sigma_y, \Sigma'_y, \Sigma_z, \Sigma'_z$. Assume w.l.o.g. that $\tau_y, \tau'_y, \tau'_z, \tau_z$ lie in this order on T_x. Let I, I' be two arcs in T_x with disjoint interiors, I having endpoints τ_y, τ_z and I' the endpoints τ'_y, τ'_z. Every point in Δ can be joined with x by a segment Σ with tangent direction τ_Σ at x. Let $\Delta_0 \subset \Delta$ be the set of all points in Δ admitting a segment Σ with $\tau_\Sigma \in I$. Similarly, let Δ'_0 be the set of all points in Δ for which $\tau_\Sigma \in I'$. We shall show that

$$J_{yz} = (\Delta_0 \cap \Delta'_0) \cup \{y, z\}$$

is a Jordan arc.

The proof of Theorem 1 in [8], which we shall not repeat here, shows that J_{yz} is an arc. The fact that we now possibly have only one segment from x to y or z is irrelevant. The only important fact is that the maximality of any segment from x to y or z prevents a segment Σ joining a point of $\Delta_0 \cup \Delta'_0$ with x from including it. Thus, for any such Σ, τ_Σ lies in the interior of I or I'. Therefore $\Delta_0 \cap \Delta'_0 \subset C_x$. To see that J_{yz} is a Jordan arc, it suffices to take an arbitrary point $u \in \Delta_0 \cap \Delta'_0$ and prove that it is a cutpoint of J_{yz} (see [5], p. 54). Indeed, u and x are joined by (at least) two segments Σ_u, Σ'_u with $\tau_{\Sigma_u} \in I$ and $\tau_{\Sigma'_u} \in I'$ and therefore $\Sigma_u \cup \Sigma'_u$ separates y from z; moreover $(\Sigma_u \cup \Sigma'_u) \setminus \{u\}$ is disjoint from C_x and does not contain y or z, whence

$$J_{yz} \cap (\Sigma_u \cup \Sigma'_u) \setminus \{u\} = \emptyset\ .$$

Now, suppose J is another Jordan arc joining y to z with $J \setminus C_x$ finite. Let $j \in J \setminus J_{yz}$. If $j \notin \Delta$ then J must cross $\Sigma_y \cup \Sigma'_y \cup \Sigma_z \cup \Sigma'_z$ in a point different from y and z. So we may suppose w.l.o.g. that $j \in \Delta_0$. Consider a maximal subarc J_0 of J whose interior lies in $\Delta_0 \setminus J_{yz}$. Let $v \in J_0$. If $v \in C_x$, let Σ_v, Σ'_v be two segments from v to x and let $\Delta_v \subset \Delta_0$ be the Jordan domain with boundary $\Sigma_v \cup \Sigma'_v$. If $J_0 \cap \Delta_v \neq \emptyset$ then J_0 has at least two points in $\Sigma_v \cup \Sigma'_v$, and we associate to v one of these points, different from v. If $J_0 \cap \Delta_v = \emptyset$, we associate to v the domain Δ_v. Thus, $v \notin C_x$ or $J_0 \cap \Delta_v \neq \emptyset$ holds for finitely many v, while $J_0 \cap \Delta_v \neq \emptyset$ holds for at most countably many v, which is absurd.

We shall mostly tacitly, but repeatedly make use of the following result of Aleksandrov.

Lemma 2. *The angles of any geodesic triangle on a convex surface are not smaller than the corresponding angles of the Euclidean triangle with the same side-lengths.*

For a proof, see [1], p. 132.

Lemma 3. *If one angle of a geodesic triangle T on a convex surface is equal to the corresponding one of the Euclidean triangle T_0 with the same side-lengths, then either T and T_0 are isometric or there is a segment passing through T and joining the other two vertices of T. In the second case both other angles of T are larger than the corresponding angles of T_0.*

Proof. Let a, x, y be the vertices of the geodesic triangle T on $S \in \mathscr{S}$ and a_0, x_0, y_0 those of T_0; assume that the angle of T at a is equal to that of T_0 at a_0.

First suppose there is no segment from x to y passing through T. If the angle of T at x is larger than that of T_0 at x_0 then, by Hilfssatz 4 in [1], p. 279, there is a point $x' \in ax$, for which the angle at a'_0 of the Euclidean triangle $a'_0x'_0y'_0$ with the same side-lengths as $ax'y$ is larger than the angle at a_0 of T_0. By Lemma 2, this implies that the angle at a of T is larger than that at a_0 of T_0, contradicting the hypothesis. Hence the angles at x and x_0 are equal. Analogously, the angles at y and y_0 are equal too and the triangles T and T_0 are isometric.

Suppose now, there is a segment from x to y meeting T_x. Then, of course, it makes with ax an angle smaller than the angle at x of T, but larger or equal to the angle at x_0 of T_0. Hence the angle of T at x is larger than that of T_0 at x_0 and, analogously, the angle at y is larger than that at y_0.

Lemma 4. *Let $S \in \mathscr{S}$, $x \in S$, and $y \in F_x$. If the full angle of S at y is larger than π then $y \in C_x$.*

This is part of Theorem 2 in [8].

From convex surfaces to acute triangles

The following result, which will be used to establish the topological characterization of F_x is perhaps interesting in itself too. It constitutes an unexpected link between convex surfaces and triangle geometry.

Theorem 0. *Let S be a convex surface or a doubly covered 2-dimensional convex set, and let $x, z \in S$ be distinct. Assume that x and z are joined by n segments* *and in the interior of each of the n resulting digons there is a point at distance at least* $\rho(x,z)$ *from* x $(n \geq 3)$*. Then* S *is a doubly covered acute triangle,* x *is the centre of its circumcircle and* z *is its orthocentre.*

Proof. Join all n points (from the n digons) mentioned in the statement, by segments, with x and z. Thus S is decomposed into $2n$ geodesic triangles, and let uxz be one of them. Let U', X' Z' be its angles at u, x, z respectively. Consider the Euclidean triangle with the same side-lengths and let U_0, X_0, Z_0 denote its angles. Then $U_0 \leq Z_0$, because the same inequality holds with respect to the lengths of their opposite sides. Hence $X_0 + 2Z_0 \geq \pi$. Then, by Lemma 2, $X' + 2Z' \geq \pi$. Summing up for all $2n$ triangles yields

$$X + 2Z \geq 2n\pi \ ,$$

where X and Z are the full angles of S at x and z, respectively. Writing this in terms of the curvatures $\omega(x)$ and $\omega(z)$ of x and z gives

$$2\pi(n-3) + \omega(x) + 2\omega(z) \leq 0 \ .$$

This holds only if $\omega(x) = \omega(z) = 0$, $n = 3$ and each geodesic triangle has vanishing curvature and is isosceles with apex x.

Thus S is a polytopal surface with only three vertices u, v, w, hence the doubly covered triangle uvw, $\rho(x,z) = \rho(x,u) = \rho(x,v) = \rho(x,w)$, both points x, z lie in the relative interiors of the two different sides of S, and x is the center of the circumcircle of uvw.

It follows that the triangle uvw is acute. Now we only have to look for z. For each point $s \in uvw$ there is a shortest path from s to x which meets the boundary of uvw. It is easily seen – and known – that its maximal length is attained at u, v, w, and at the orthocenter h. Just consider the circumcircle C of uvw and the three circles symmetric to C with respect to the sides of uvw. Every three of these four circles are concurrent: at u, v, w, h. Consequently $z = h$.

The antipode J_x and the topological characterization of F_x

Theorem 1. *If* $u, v, w \in F_x$ *then either* $u \in J_{vw}$ *or* $v \in J_{wu}$*, or* $w \in J_{uv}$*.*

Proof. Suppose the conclusion of the theorem is false. So $u, w \notin J_{uv} \cap J_{vw}$. If $J_{uv} \cap J_{vw} = \{v\}$ then, by the uniqueness established in Lemma 1, $v \in J_{uw}$, which is supposed false. Hence $J_{uv} \cap J_{vw}$ contains a point $z \notin \{u, v, w\}$. By Lemma 1, $J_{uv} = J_{uz} \cup J_{zv}$, $J_{vw} = J_{vz} \cup J_{zw}$, $J_{wu} = J_{wz} \cup J_{zu}$. By the same lemma, since $z \in J_{uv}$, there are two different segments from x to z the union of which separates u from v on S. Similarly there are two different segments from x to z the union of which separates v from w, and another two separating w from u. Among these segments there must be three distinct, because for just two distinct segments Σ, Σ' at least two of the points u, v, w belong to the same component of $S \setminus (\Sigma \cup \Sigma')$. These at most six distinct segments decompose S into equally many Jordan domains, only three of which meet $\{u, v, w\}$. We can obviously choose three of the segments, say $\Sigma_z^u, \Sigma_z^v, \Sigma_z^w$, such that $\Sigma_z^u \cup \Sigma_z^v$ separates w from u, v, $\Sigma_z^v \cup \Sigma_z^w$ separates u from v, w, and $\Sigma_z^w \cup \Sigma_z^u$ separates v from w, u.

Clearly, $\rho(x,u) = \rho(x,v) = \rho(x,w) \geq \rho(x,z)$. Hence, by Theorem 0, $S \notin \mathscr{S}$ and a contradiction is obtained.

The next theorem shows that, for any point x on a convex surface, for which F_x contains more than one point, there is a Jordan arc J_x including F_x whose relative interior lies in C_x.

Theorem 2. *If, for some point x on $S \in \mathscr{S}$, the set F_x contains more than one point, then there are two points $y_1, y_2 \in F_x$ such that $F_x \subset J_{y_1 y_2}$.*

Proof. Let $y, z \in F_x$ be distinct. By Lemma 1 and using the notation of its proof, there are three associated Jordan arcs J_{yz}, I, and I', such that each point in $J_{yz} \setminus \{y, z\}$ can be joined with x by two segments Σ, Σ' with $\tau_\Sigma \in I$ and $\tau_{\Sigma'} \in I'$. If there are more than two segments from x to y or z, take I and I' to be minimal (by inclusion) and rename them I_{yz} and I'_{yz}. Thus, if there exists a pair of points $u, v \in F_x$ with $\{u, v\} \not\subset J_{yz}$ then, by Theorem 1, $J_{yz} \subset J_{uv}$, $I_{yz} \subset I_{uv}$ $I'_{yz} \subset I'_{uv}$. More generally, for u, u', v, $v' \in F_x$ with $u \neq v$ and $u' \neq v'$, the inclusions $J_{uv} \subset J_{u'v'}$, $I_{uv} \subset I_{u'v'}$ and $I'_{uv} \subset I'_{u'v'}$ are equivalent. Now consider the arcs $\cup_{u,v \in F_x} I_{uv}$ and $\cup_{u,v \in F_x} I'_{uv}$. Let τ_1, τ_2 and τ'_1, τ'_2 be their endpoints in the order $\tau_1, \tau_2, \tau'_2, \tau'_1$ around x (where possibly $\tau_i = \tau'_i$). Then τ_i, τ'_i are limit points of the tangent directions $\tau_{\Sigma_u}, \tau_{\Sigma'_u}$ of two segments Σ_u, Σ'_u (of constant length $\rho(x, y)$) from x to u, which implies that Σ_u and Σ'_u converge to two segments $\Sigma_{y_i}, \Sigma'_{y_i}$ (of same length) from x to some point y_i, where $\Sigma_{y_i} = \Sigma'_{y_i}$ if $\tau_i = \tau'_i$ ($i = 1, 2$). Then $y_1, y_2 \in F_x$ and τ_1, τ_2 are the endpoints of $I_{y_1 y_2}$. Thus, $I_{uv} \subset I_{y_1 y_2}$ and therefore $J_{uv} \subset J_{y_1 y_2}$ for any $u, v \in F_x$. The proof is finished.

We define the *antipode* J_x of x to be F_x if F_x contains a single point, and the Jordan arc given by Theorem 2 otherwise. Thus, J_x is the unique Jordan arc A between farthest points, satisfying $F_x \subset A \subset E_x$.

We know from Theorem 2 in [8] that all points of F_x belong to $\overline{C_x}$. Many of them, for example all smooth ones, belong even to C_x by Lemma 4. In general they may not lie in C_x. However, there are not many such exceptions.

Corollary. *At most two points of F_x lie outside C_x.*

Proof. Suppose F_x has at least three points not in C_x. Then J_x is a Jordan arc. Now, using Theorem 2 and Lemma 1 (and the fact that a Jordan arc has only two endpoints), we get a contradiction.

We proved in [8] that every component of F_x is a point or a Jordan arc, but this was not providing a complete topological characterization of F_x.

Theorem 3. *Let Σ be a segment on the sphere S^2, $S \in \mathscr{S}$, and $x \in S$. Then there exists a homeomorphism $\phi : S \to S^2$ such that $\phi(F_x) \subset \Sigma$. Moreover, for any closed set $\Xi \subset \Sigma$, there exist $S \in \mathscr{S}$, $x \in S$, and a homeomorphism $\phi : S \to S^2$ such that $\phi(F_x) = \Xi$.*

Proof. For the first part, the assertion is obvious if J_x is a single point, and follows from Theorem 2 by taking a homeomorphism $\psi : J_x \to \Sigma$ and extending ψ^{-1} to the whole sphere, if J_x is a Jordan arc.

The second part follows from a construction given in [8], p. 11.

Geodesic triangles with vertices in the cut locus

Theorem 4 presents a remarkable geometric property of convex surfaces.

Theorem 4. *Let $\Sigma_{xv}, \Sigma'_{xv}$ be two distinct segments between the points x and v on $S \in \mathscr{S}$. If the points $u, w \in S$, separated by $\Sigma_{xv} \cup \Sigma'_{xv}$ on S, are not closer than v to x, then the angle uvw is obtuse or right.*

Proof. Let Σ_{uv}, Σ_{vw} be segments joining u to v and v to w, respectively. Also, let Σ_{xu}, Σ_{xw} be segments from x to u and w. These four segments and the two from the statement decompose the surface S into four geodesic triangles T_u T_w, T'_w and T'_u, of boundaries $\Sigma_{xu} \cup \Sigma_{uv} \cup \Sigma_{xv}$, $\Sigma_{xv} \cup \Sigma_{vw} \cup \Sigma_{xw}$, $\Sigma_{xw} \cup \Sigma_{vw} \cup \Sigma'_{xv}$, and $\Sigma'_{xv} \cup \Sigma_{uv} \cup \Sigma_{xu}$ respectively. Let X_u, X_w, X'_w, X'_u be their angles at x and V_u, V_w, V'_w, V'_u their angles at v, respectively. Let A, α, α' and B, β, β' be the angles of the Euclidean triangles T^*_u and T^*_w of opposite side-lengths $\rho(v, u)$, $\rho(u, x)$, $\rho(x, v)$ and $\rho(v, w)$, $\rho(w, x)$, $\rho(x, v)$ respectively. A comparison between the angles of the four geodesic triangles and their Euclidean counterparts gives

$$A \le X_u \ , \quad A \le X'_u \ , \quad B \le X_w \ , \quad B \le X'_w \ .$$

Since

$$X_u + X'_u + X_w + X'_w \le 2\pi \ ,$$

we have $A + B \le \pi$. Thus, $\alpha \ge \alpha'$, and $\beta \ge \beta'$ imply

$$\alpha + \beta \ge \frac{\pi - A}{2} + \frac{\pi - B}{2} \ge \pi/2 \ .$$

Now, returning to the four geodesic triangles, $V_u \ge \alpha$, $V'_u \ge \alpha$, $V_w \ge \beta$ and $V'_w \ge \beta$, and therefore $V_u + V_w \ge \pi/2$ and $V'_u + V'_w \ge \pi/2$, whence the angle uvw is obtuse or right.

It is interesting to look closer to the case when the angle uvw from the preceding theorem is right.

In order to formulate our result, consider a circle $C \subset \mathbb{R}^2$ of radius r_0, the diameter u_0w_0 of C and two points $v_0, v'_0 \in C$ symmetric with respect to u_0w_0, at distance q_0 from u_0. Let $Q(r_0, q_0) \subset \mathbb{R}^2$ be the bounded domain with the quadrilateral $u_0v_0w_0v'_0$ as boundary.

Theorem 5. *Under the hypotheses of Theorem 4, if the angle uvw is right then a part of S is isometric to $Q(\rho(x, u), \rho(u, v))$.*

Proof. We continue to use the preceding notation.
If, say, $V_u + V_w = \pi/2$ then $\alpha + \beta = \pi/2$, $\alpha = \alpha' = V_u$, $\beta = \beta' = V_w$, $A + B = \pi$, $A = X_u = X'_u$, and $B = X_w = X'_w$. Then $\rho(x, u) = \rho(x, v) = \rho(x, w)$ and

$$X_u + X'_u + X_w + X'_w = 2\pi \ .$$

It follows that Σ_{xu} and Σ_{xw} are the only segments from x to u and w respectively, otherwise the full angle at x would be larger than 2π. By Lemma 3, T_u and T_w must be isometric to T_u^* and T_w^* respectively.

About the other two geodesic triangles, T'_u and T'_w we know that $X'_u = A$ and $X'_w = B$. Let Σ'_{uv} be the segment from u to v which separates in T'_u the point x from all other segments from u to v lying in $\overline{T'_u}$ (if any). Then, by Lemma 3, the geodesic triangle T''_u included in T'_u, of sides Σ_{xu}, Σ'_{uv}, Σ'_{xv}, is isometric to the planar triangle T_u^*. The same argument leads to an analogous geodesic triangle $T''_w \subset T'_w$ isometric to T_w^*. Since the curvature at x vanishes too,

$$(T_u \cup T_w \cup T''_u \cup T''_w \cup \Sigma_{xu} \cup \Sigma_{xw} \cup \Sigma_{xv} \cup \Sigma'_{xv}) \setminus \{u, v, w\}$$

is isometric to the domain $Q(r_0, q_0)$ with $r_0 = \rho(x, u)$ and $q_0 = \rho(u, v)$.

An example. It is interesting to remark that, if the angle uvw is right, then $v \in C_u \cup C_w$ must hold, but possibly $v \notin C_u \cap C_w$.

Indeed, if $v \notin C_u \cup C_w$ then S must be isometric to $Q(r_0, q_0)$ with pairwise glued sides u_0v_0, $u_0v'_0$ and $w_0v_0, w_0v'_0$. Thus the curvature is concentrated in $u_0, v_0 = v'_0$ and w_0, which implies that S degenerates to a doubly covered right triangle, hence $S \notin \mathscr{S}$.

We present now an example with $v \notin C_u \cap C_w$, for which, in addition, $u, v, w \in F_x$.

Let $T \subset \mathbb{R}^3$ be the planar triangle of vertices $u = (u_0, 0, 0)$, $w' = (0, w_0, 0)$, $v = (0, 0, 0)$ ($u_0, w_0 > 0$). Choose a point s' of negative first coordinate inside the circumcircle C of T. We produce a small isometric deformation of the quadrilateral $uvs'w'$ by keeping uv fixed, the triangles uvs' and $us'w'$ respectively congruent to themselves, and the third coordinate of w' zero. Thus w' takes a position w with a negative first coordinate and vanishing third coordinate, s' takes a position s with unchanged first coordinate and non-vanishing, say positive, third coordinate, and the intersection t' of us' with vw' takes a position t with still vanishing first coordinate, but positive third coordinate.

Let $s^*, t^* \in \mathbb{R}^3$ be symmetric to s, t (respectively) with respect to the plane uvw, and consider the boundary S of $\mathrm{conv}\{u, v, s, s^*, w\}$. Then, if the deformation is small enough, no point of S is farther from the mid-point x of uw than u, and there are precisely the two segments vtw and vt^*w from v to w on S. Obviously, uv is the only segment from u to v, and $F_x = \{u, v, w\}$.

To produce an analogous example in which $F_x \neq \{u, v, w\}$. it suffices to take $s' \in C$, otherwise proceed as above.

Theorem 4 implies the following property of the cut locus of any convex surface.

Theorem 6. *Let $S \in \mathscr{S}$, $x \in S$, and $u, w \in E_x$ be distinct. If $v \in J_{uw} \setminus \{u, w\}$, $\rho(x, v) \le \rho(x, u)$ and $\rho(x, v) \le \rho(x, w)$, then the angle uvw is obtuse or right.*

Proof. By Lemma 1, there are two segments $\Sigma_{xv}, \Sigma'_{xv}$ from x to v whose union separates u from w.

Now the conclusion follows from Theorem 4.

Theorem 7. *If $S \in \mathscr{S}$, $x \in S$, $u, w \in F_x$ and $v \in J_{uw} \setminus \{u, w\}$, then the angle uvw is obtuse or right. If it is right then the full angle of S at each of the points u and v is not larger than π.*

Proof. By Theorem 6, the angle uvw is obtuse or right.
Suppose now the angle uvw is right. Then, again with the preceding notation, Σ_{xu} and Σ_{xw} must be the only segments from x to u and w respectively, as established in the proof of Theorem 5. Therefore, by Lemma 4, the full angle of S at u and w cannot be larger than π.

Theorem 8. *Any geodesic triangle on $S \in \mathscr{S}$ with vertices in F_x is obtuse or right. If it is right at one vertex then at both other vertices the full angle of S is not larger than π.*

Proof. Let u, v, w be distinct points in F_x. In view of Theorem 1, we may suppose w.l.o.g. that $v \in J_{uw} \setminus \{u, w\}$.

Now, the conclusions follow from Theorem 7.

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