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Tiling the pentagon

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Dedicated to in honor of Ludwig Danzer on his seventieth birthday

Abstract

Finite edge-to-edge tilings of a convex pentagon with convex pentagonal tiles are discussed. Such tilings that are also cubic are shown to be impossible in several cases. A finite tiling of a polygon P is *equiangular* if there is a 1-1 correspondence between the angles of P and the angles of each tile (both taken in clockwise cyclic order) so that corresponding angles are equal. It is shown that there is no cubic equiangular tiling of a convex pentagon and hence it is impossible to dissect a convex pentagon into pentagons directly similar to it. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Finite tiling; Dissection; Pentagonal tiling; Cubic tiling; Equiangular tiling

0. Introduction

In 1940, Langford asked for a classification of plane figures that can be dissected into four congruent pieces, each similar to the original one [3]. In 1964, Golomb studied the general case with n pieces instead of four [1]. Ten years later, Valette and Zamfirescu completely solved Langford's problem and extended the question to dissections into not necessarily congruent pieces each similar to the original, and to dissections into pieces that have just their corresponding angles the same as those of the original [4].

In the present paper we treat this last extension and discuss the problem of dissecting a convex pentagon P into finitely many pieces, each of which is also a convex pentagon. Throughout the paper, the term *pentagon* always refers to a convex pentagon.

When a pentagon P is dissected into a finite number of pentagons, we obtain a *finite tiling of P* with pentagonal tiles, in the sense of Grünbaum and Shephard [2] (see Fig. 1). In what follows, we use the usual terminology from tiling theory [2].

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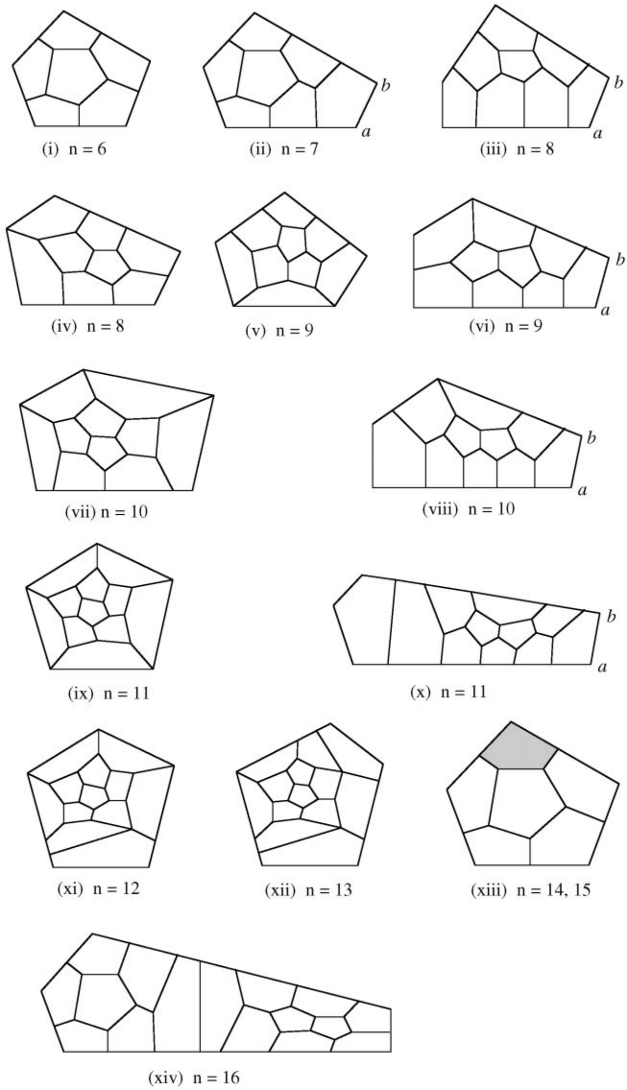

Fig. 1.

So, for example, every tiling has edges and vertices. The pentagon P has vertices and edges, too, but in order to avoid confusion, we shall instead refer to the *corners* and *sides* of P. If every edge of each pentagon (including P) is an edge of the tiling, then the tiling is usually called edge-to-edge. The tiling in Fig. 1(ix) is an edge-to-edge tiling in this strict sense. However, we shall focus our attention only on the interior of the pentagon P when we speak about an *edge-to-edge tiling of P*. In this sense, every tiling in Fig. 1 is edge-to-edge. In what follows, the term *tiling* always refers to an edge-to-edge *tiling* of P, and *tile* refers to a convex pentagonal tile.

If a side of P contains no vertex in its relative interior, the side is said to be *simple*. So in Fig. 1(ix), each side of P is simple, while in Fig. 1(v), P has two non-simple sides and in Fig. 1(vii), P has one non-simple side. When we dissect a tile according to a special pattern, we may say the pattern is *embedded* into the tile. For example, if we embed the pattern of Fig. 1(v) into a boundary tile of the tiling in Fig. 1(i), with non-simple sides on the boundary sides of P, we obtain a new edge-to-edge tiling (Fig. 1(xiii)). The *valence of a vertex* is defined to be the number of edges incident to the vertex.

1. Infinitely many tilings

It is easily seen that a pentagon can always be dissected to produce a tiling by n pentagons for $n \geqslant 6$. Once tilings have been obtained for $6 \leqslant n \leqslant 10$ tiles (see Fig. 1), the tiling for $n = 6$ can be embedded in one pentagon tile in each of these figures, giving a tiling with $n + 5$ pentagons. This process can be repeated so as to give tilings for all $n \geqslant 6$. This procedure will not produce edge-to-edge tilings for $n \geqslant 11$. However, we can prove that edge-to-edge tilings are possible.

Theorem 1. *A pentagon P can always be dissected into n pentagons which form an edge-to-edge tiling of P, for any* $n \geqslant 6$.

Proof. Fig. 1 shows edge-to-edge tilings for $6 \leqslant n \leqslant 13$. New edge-to-edge tilings can be constructed from these in at least three ways: (1) embed tiling (v) for $n = 9$ into a tile with two sides on the boundary of P such that the two non-simple sides go to the boundary sides; (2) embed tiling (vii) for $n = 10$ into any boundary tile with the only non-simple side on the boundary; (3) embed the tiling (ix) for $n = 11$ into any tile of the tiling. In particular, edge-to-edge tilings for $n = 14$ and 15 are obtained by procedures (1) and (2) applied to the shaded tile in a tiling for $n = 6$ (see Fig. 1). The procedure described in (3) produces from an m-tiling a new $(m + 10)$-tiling, so that repeated application of the procedure to the tilings for $6 \leqslant n \leqslant 15$ produces edge-to-edge tilings for all $n \geqslant 6$. □

For any edge-to-edge tiling $\mathscr{T}$ of a pentagon P, we introduce the following notation. f is the number of tiles in $\mathscr{T}$, v is the number of vertices of $\mathscr{T}$, and e is the number of edges of $\mathscr{T}$. Recall that Euler's formula for planar graphs states $v + f = e + 1$. We also will use the following notation: v_c is the number of 2-valent vertices of $\mathscr{T}$ (these are necessarily at corners of P), v_b is the number of remaining vertices of $\mathscr{T}$ on the boundary of P, and v_i is the number of vertices of $\mathscr{T}$ in the interior of P. Then $v = v_b + v_c + v_i$.

For edge-to-edge tilings, Euler's formula gives a simple lower bound on the number of tiles.

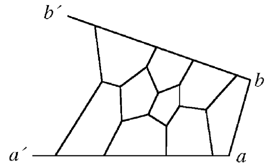

Fig. 2.

Proposition 1. *Let $\mathcal{T}$ be any edge-to-edge tiling of a pentagon P. Then* $f \geqslant v_b - v_c + 6$ *and in particular* $f \geqslant v_b + 1$.

Proof. Each tile in the tiling counts 5 vertices; each interior vertex is counted by at least three tiles, each vertex on the boundary is counted by at least two tiles, except that each 2-valent vertex is counted by only one tile. Thus,

$$3v \leqslant 5f + v_b + 2v_c. \tag{1}$$

Let e_b be the number of edges on the boundary of P; note $e_b = v_b + v_c$. Each tile in the tiling counts 5 edges; each interior edge is counted by two tiles while each edge on the boundary of P is counted by only one tile. Thus,

$$2e = 5f + e_b = 5f + v_b + v_c. \tag{2}$$

Using (1) and (2), we solve for v and e and substitute in Euler's formula, then multiply by 6 to obtain the inequality

$$10f + 2v_b + 4v_c + 6f \geqslant 15f + 3v_b + 3v_c + 6$$

and the result follows. □

2. Cubic tilings

Now, we restrict our attention to edge-to-edge tilings of a pentagon P in which each vertex of the tiling has valence three, with the exception of the corners of P, which may have valence two or three. We call such a tiling a *cubic* tiling. Although we do not know if there are cubic edge-to-edge tilings of P for every $n \geqslant 6$, we can prove that there are infinitely many of such tilings.

Theorem 2. *There are infinitely many cubic edge-to-edge tilings of P.*

Proof. Fig. 1 shows cubic tilings for $n = 6, 7, 8, 9, 10, 11$, and 16. Where vertices of these graphs are labeled a and b, we can join the corresponding vertices a' and b' of the graph in Fig. 2 to produce a new cubic tiling as in Fig. 3. By repeating this procedure with the new graph, we obtain an infinite family of cubic edge-to-edge tilings. (In fact, this process produces tilings with n tiles for $n = 7 + 10k, 8 + 10k, 9 + 10k, 10 + 10k$, and $11 + 10k$, $k \geqslant 1$.) □

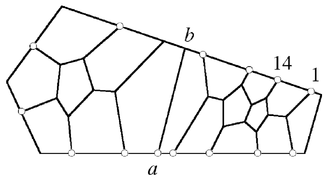

Fig. 3.

Remark 1. If $\mathscr{T}$ is a cubic tiling of a pentagon P, then $\mathscr{T}$ has at least as many 3-valent boundary vertices as boundary tiles. To see this, orient the edges of the tiling $\mathscr{T}$ on the boundary of P in a clockwise manner. Then every boundary tile has at least one 3-valent vertex that is an initial vertex of one of these oriented edges. Fig. 3 illustrates the case where there are more 3-valent boundary vertices than boundary tiles. Here there are 14 boundary tiles, counted by the vertices shown as small circles, beginning with the vertex labeled 1, moving clockwise, and ending with 14. Note there are two additional 3-valent boundary vertices that are not used in the count, since then the two center tiles would be counted twice.

The following theorem shows that the number of tiles in a cubic tiling of a pentagon depends only on the number of vertices on the boundary.

Proposition 2. *Let $\mathscr{T}$ be a cubic tiling of a pentagon P. Then,*
(i) $f = v_b - v_c + 6$,
(ii) $v_i - v_b + 2v_c = 10$,
(iii) $f = v_i + v_c - 4$,
(iv) $2f = v_i + v_b + 2$.

Proof. (i) The argument in Proposition 1 holds with the inequality as an equation since every vertex is 3-valent except the 2-valent corner vertices.

(ii) In a similar counting argument, each interior vertex counts three tiles, each 3-valent boundary vertex counts two tiles, and each 2-valent vertex counts one tile; each tile is counted by five vertices. Thus,

$$5f = 3v_i + 2v_b + v_c. \tag{3}$$

Each interior vertex and each 3-valent boundary vertex counts three edges and each 2-valent vertex counts two edges; each edge is counted by two vertices. Thus,

$$2e = 3v_i + 3v_b + 2v_c. \tag{4}$$

We use $v = v_i + v_b + v_c$ and then (3) and (4) to substitute for v, f, and e in Euler's formula, and multiply by 10 to obtain

$$10v_i + 10v_b + 10v_c + 6v_i + 4v_b + 2v_c = 15v_i + 15v_b + 10v_c + 10$$

from which the result follows.

(iii) results from adding (i) and (ii).
(iv) results from adding (i) and (iii). □

Corollary. *Every cubic tiling $\mathscr{T}$ of a pentagon has an interior tile. Hence, $f \geqslant 6$.*

Proof. Let f_i be the number of interior tiles and f_b be the number of boundary tiles. Then by Proposition 2, $f = f_i + f_b = v_b - v_c + 6$. Since $f_b \leqslant v_b$ (Remark 1), it follows that $v_b - v_c + 6 \leqslant f_i + v_b$, hence $6 - v_c \leqslant f_i$. Since $v_c \leqslant 5$, we have $f_i \geqslant 1$. Choose an interior tile of the tiling. Since all tiles are convex, each of the five edges of the interior tile is shared with a different pentagon tile, so the tiling has at least six tiles. □

Remark 2. If there is a cubic tiling of a pentagon P in which every tile has the same kind of angles as P (i.e., P and each tile in the tiling have the same number of acute, right, and obtuse angles) then the angles of P cannot all be obtuse, because at any 3-valent boundary vertex of the tiling one angle must be acute or right. On the other hand, since the angles of P sum to 3π, P cannot have more than three angles that are acute or right.

This remark and the above results imply that there cannot be particular kinds of cubic tilings.

Proposition 3. *There is no cubic tiling of a pentagon P that satisfies any one of the following conditions*:
(i) *P and each tile in the tiling have three acute angles.*
(ii) *P and each tile in the tiling have two acute angles and one right angle.*
(iii) *P and each tile in the tiling have one acute angle and two right angles.*
(iv) *P and each tile in the tiling have one acute angle and no right angles.*
(v) *P and each tile in the tiling have no acute angles and one or two right angles.*

Proof. (i)–(iii): The number of acute angles and right angles in the tiling is $3f$. At each interior vertex of the tiling there is at most one acute angle or right angle, and at each 3-valent boundary vertex there are at most two angles that are acute or right; there are at most three 2-valent vertices that have acute or right angles. Thus $3f \leqslant v_i + 2v_b + 3$. Adding (iii) and (iv) of Proposition 2 gives $3f = 2v_i + v_b + v_c - 2$, which then implies $v_i - v_b \leqslant 5 - v_c$. Proposition 2(ii) says $v_i - v_b = 10 - 2v_c$, so $5 - v_c \leqslant 0$, and since $v_c \leqslant 5$, we must have $v_c = 5$. So there are no 3-valent boundary vertices at corners of P. Also, $v_i = v_b$ and so $3f = 2v_i + v_b + 3 = 2v_b + v_i + 3$. Let $v_b = v_{ba} + v_{br}$, where v_{ba} is the number of 3-valent boundary vertices at which there are acute angles, and v_{br} is the number of 3-valent boundary vertices at which there are right angles. Then the number of acute and right angles in the tiling is $3f = v_{ba} + 2v_{br} + v_i + 3 = 2v_b + v_i + 3$, hence $v_{ba} + 2v_{br} = 2v_b = 2v_{ba} + 2v_{br}$. This implies $v_{ba} = 0$; there are no acute angles at 3-valent boundary vertices. This shows that case (i) is impossible. For cases (ii) and (iii), recall that the tiling has at least one interior tile. So in case (ii), there is at least one interior vertex with a right angle. Including corner angles, this gives at least $2v_{\mathrm{b}}+2$ right angles and at most $v_{\mathrm{i}}+1$ acute angles in the tiling. But since $v_{\mathrm{i}}=v_{\mathrm{b}}$, this makes it impossible for the tiling to have twice as many acute angles as right angles. In case (iii), there are at least two interior vertices with right angles. Including corner angles, this gives at least $2v_{\mathrm{b}}+4$ right angles and at most $v_{\mathrm{i}}-1$ acute angles in the tiling. But since $v_{\mathrm{i}}=v_{\mathrm{b}}$, it is impossible to have twice as many right angles as acute angles.

(iv) Since no tiles in the tiling have right angles, there is exactly one acute angle at each 3-valent boundary vertex not at a corner of P. At any 3-valent vertices at corners of P there will be two acute angles, and there is one corner 2-valent vertex of P with an acute angle. Thus there are at least $v_{\mathrm{b}}+1$ acute angles in the boundary tiles. But then there are at least $v_{\mathrm{b}}+1$ boundary tiles (since each tile has exactly one acute angle), a contradiction to $f_{\mathrm{b}}\leqslant v_{\mathrm{b}}$ (Remark 1).

(v) If P has k boundary tiles, they contain m right angles, where $k\leqslant m\leqslant 2k$. But there are at least k 3-valent boundary vertices (Remark 1), and at each of these there are $2k$ right angles, and there are an additional one or two right angles at corners of P. However, the inequalities $2k+1\leqslant m$ and $2k+2\leqslant m$ are impossible. □

3. Equiangular cubic tilings

We say a tiling of a pentagon P is *equiangular* if for each tile in the tiling there is a correspondence between the angles of the tile and those of P (each taken in a clockwise cyclic order) so that the corresponding angles of the two tiles are equal.

The long list of equiangular tilings of a quadrilateral by four tiles in [4] contains three cubic tilings (Table I, No. 1 and No. 3k with two tilings). The situation changes when we tile the pentagon. Do equiangular tilings of a pentagon exist? We do not know the answer to this general question, but for cubic tilings, the answer is no. The remainder of this paper is devoted to proving this. An immediate consequence of this fact is that there can be no cubic tiling of a pentagon by tiles all directly similar to the pentagon.

Theorem 3. *There is no cubic equiangular tiling of the pentagon.*

Proof. Suppose we have a cubic tiling of a pentagon P. Throughout the proof, the corners of the pentagon P will be a,b,c,d,e (in clockwise order), having angles $\alpha,\beta,\gamma,\delta,\varepsilon$, respectively. We will also follow the convention of listing the two angles at any 3-valent boundary vertex in clockwise order.

P must have at least one acute or right angle and can have at most three angles that are acute or right (Remark 2). Proposition 3 shows six cases impossible; there are only three remaining cases, which our proof considers separately. The equiangular condition of the tiling is essential in the proofs of these cases. Indeed, for each of these cases,

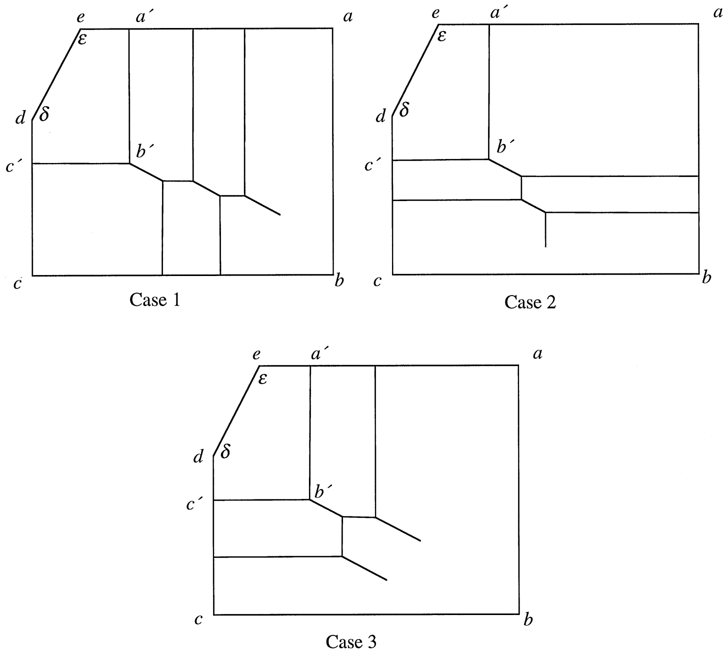

Fig. 4.

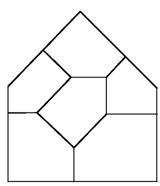

Fig. 5. The pentagon P and its six tiles each have three right angles and two $3\pi/4$ angles.

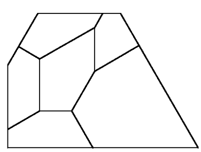

Fig. 6. The pentagon P and its six tiles each have angles of $\pi/3$, $\pi/2$, $2\pi/3$, $2\pi/3$ and $5\pi/6$.

there do exist cubic tilings of a pentagon P that are not equiangular but for which the angles of the pentagon and the angles of the tiles are the same (Figs. 5, 6) or of the same type (Fig. 7).

Case I: *The minimal angle of P is* $\pi/2$. By Proposition 3, P has exactly three right angles. Since P has no acute angles, at each 3-valent boundary vertex of the tiling both angles must be $\pi/2$.

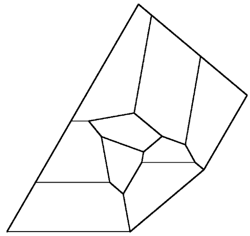

Fig. 7. The pentagon P and its nine tiles each have acute angles $\pi/3$ and $4\pi/9$ and three obtuse angles. The obtuse angles are not the same for all tiles.

Suppose first that the right angles are not consecutive; let them be at corners a,b,d. No tile can have ab as its edge since another vertex of the tile would have to be on ae or bc, so the tile would have 3 consecutive right angles, a contradition. Thus the tile containing corner a has a vertex on the interior of ab and the interior of de. Similarly, the tile containing corner b must have a vertex on the interior of cd. But then the tile containing corner d will have 3 consecutive right angles, a contradition.

Now suppose that the right angles are consecutive; let them be at corners a,b,c. The existence of a 3-valent vertex on de would imply that the tile with vertex d has angle δ between two right angles, a contradiction. Thus there is a tile T with edge de and clearly it has a vertex a' on the interior of ae or cd; we may suppose $a' \in ae$. If T also contains c, then the fifth vertex b' is on bc, and the other tile with the edge $a'b'$ has an obtuse vertex on aa' or bb', impossible. So T has a vertex c' on cd different from c; let b' be its fifth vertex. At this interior vertex b' the angles are $\pi/2$, δ, and ε (indeed, these three angles must occur at every interior vertex of the tiling). The three possibilities for the tiling are shown in Fig. 4. But in the first two cases there is no interior tile, and in the third, the tiling has an interior tile with a right angle between two obtuse angles, hence it cannot have three consecutive right angles, a contrdiction.

Case II: *P has precisely one acute angle*. Let $\alpha < \pi/2$. By Proposition 3, P also has exactly one right angle, so we may assume that β and γ are obtuse. Clearly vertex a is 2-valent.

Since a,b,c,d cannot belong to the same tile, the first 3-valent vertex a, on the boundary of P that occurs clockwise after a occurs before d. Label all the consecutive 3-valent vertices $a_1,\dots,a_k$ on the boundary of P, written in clockwise order, between a and the corner with a right angle. Let ξ_i,φ_i be the two angles at a_i (taken in clockwise order). For each i, either both angles at a_i are $\pi/2$ or ξ_i is obtuse and $\varphi_i=\alpha$ (since the next two angles that follow α clockwise in any tile must be obtuse). For each a_i on ab (if there are any), $\xi_i=\beta$ and $\varphi_i=\alpha$, so b is 2-valent. If there are no 3-valent vertices on bc or cd (not even at the corners), then the tile with vertices b,c, and d has a vertex a_i on ab and its fifth vertex v is between d and e. The angles at v must be ε and $\gamma=\pi-\varepsilon$. Since $\varepsilon \geqslant \pi/2, \gamma$ cannot be obtuse, contradiction. So there are some 3-valent vertices on bc or cd. If for all vertices a_i no φ_i equals $\pi/2$, then all φ_i equal α. This leads to a contradiction in the tile T_k containing a_k and the corner with a right angle (if $\delta = \pi/2$, there can be at most one obtuse angle in T_k following α, and if $\varepsilon = \pi/2$, there can be at most two obtuse angles in T_k following α).

Now suppose that some φ_i equals $\pi/2$ and $\delta = \pi/2$. In the tile that contains angle $\xi_i = \pi/2$, that angle must be preceded by two obtuse angles, hence the only vertices of bc are b and c, and a_i belongs to cd and is not a corner. It follows that a_{i+1} also belongs to cd and ξ_{i+1} is obtuse, hence $\varphi_{i+1} = \alpha$. This forces the next vertex a_{i+2}, and all the remaining vertices, up to a_k, to belong to cd. But then the tile with vertices a_k and d has two consecutive non-obtuse vertices, a contradiction.

Finally, suppose that some φ_i equals $\pi/2$ and $\varepsilon = \pi/2$. In the tile that contains angle $\xi_i = \pi/2$ that angle must be preceded by three obtuse angles, hence the only vertices on bc and cd are b, c, and d and a_i belongs to de and is not at a corner. Let T_i be the tile with angle φ_i. Since $\varphi_i = \pi/2$, the next clockwise angle in T_i must be α, hence a_{i+1} is also on de. But then T_i must have two consecutive vertices a_{i-1}, a_{i-2} on ab at which $\xi_{i-1} = \delta$ and $\varphi_{i-2} = \gamma$, a contradiction.

Case III: *P has precisely two acute angles.* Let $\alpha < \pi/2$. By Proposition 3, P has no right angles, so we may assume that β and γ are obtuse.

(i) Suppose that the two acute angles are not equal. Then the same acute angle, say α, must occur at every non-corner 3-valent vertex on the boundary of P, since otherwise four different angles of P would sum to 2π, impossible.

We first assume that a is a 2-valent vertex. As in Case II, there are consecutive 3-valent vertices $a_1, \ldots, a_k$ on the boundary of P, written in clockwise order (some possibly at corners), between a and the corner with the other acute angle (either d or e). Let ξ_i, φ_i be the two angles at a_i (taken in clockwise order). For each a_i on ab (if there are any), $\xi_i = \beta$ and $\varphi_i = \alpha$, so b is 2-valent. If there are no 3-valent vertices on bc or cd (not even at the corners), then the tile with vertices b, c, and d has a vertex a_i on ab and its fifth vertex v is between d and e. The angles at v must be ε and $\pi - \varepsilon$, and one of these angles equals α. By assumption, $\varepsilon \neq \alpha$, so $\alpha = \pi - \varepsilon$. But then the other tile with vertices a_i and v has two angles equal to α, contradiction. Thus there must be some 3-valent vertices a_i on bc or cd. Each such a_i not at a corner has $\varphi_i = \alpha$ since in any tile, α must be followed clockwise by two obtuse angles and no tile can have two angles α. If $\varphi_i = \alpha$ for all i, then in the tile T_k with angle $\varphi_k = \alpha$, angle α will not be followed clockwise by angles β, γ, δ (if $\delta < \pi/2$, there can be at most one obtuse angle in T_k following α, and if $\varepsilon < \pi/2$, there can be at most two obtuse angles in T_k following α), contradiction.

So $\varphi_i \neq \alpha$ for some i and $a_i = c$ or $a_i = d$. If $a_i = c$ there must be another vertex a_j on bc clockwise after b with $\xi_j = \gamma$ and $\varphi_j = \alpha$ and hence $\xi_i = \beta$ and $\varphi_i = \delta$ or ε. But then $\alpha + \gamma = \pi$ and either $\beta + \delta = \gamma$ or $\beta + \varepsilon = \gamma$, impossible since then four different angles of P sum to less than 2π. If $a_i \neq c$ but $a_i = d$ then $\xi_i = \beta$ or γ and $\varphi_i = \varepsilon$. If $\xi_i = \beta$, there must be another vertex $a_{i-1} \neq c$ on cd or on bc (if $\beta = \gamma$) with $\varphi_{i-1} = \alpha$ and $\xi_{i-1} = \delta$ or γ. Thus, $\beta + \varepsilon = \delta$ and either $\alpha + \delta = \pi$ or $\alpha + \gamma = \pi$, impossible since then four different angles of P sum to less than 2π. If $\xi_i = \gamma$ then there is a vertex $a_{i-1} \neq c$ on cd (if $\beta = \gamma$) or on bc with $\varphi_{i-1} = \alpha$ and $\xi_{i-1} = \delta$ or γ. Thus, $\gamma + \varepsilon = \delta$ and either $\alpha + \delta = \pi$ or $\alpha + \gamma = \pi$. If $\alpha + \delta = \pi$, then $\alpha + \delta + \gamma + \varepsilon < 2\pi$, impossible.

So $\alpha+\gamma=\pi$ and $\gamma+\varepsilon=\delta$, which implies that $\varepsilon<\alpha$, hence e is 2-valent. Since de must contain a further vertex, a_k is on de and $\xi_k=\alpha$, $\varphi_k=\delta$. But then $\alpha+\gamma=\pi$ and $\alpha+\delta=\pi$, impossible since $\gamma+\varepsilon=\delta$.

Finally, suppose that a is a 3-valent vertex. If the angles at a are ε, ε then there must be a vertex v on ae with $v\neq e$ since otherwise the tile containing vertices a and e would have two angles ε. At the first such v adjacent to a, the angles are α,δ so $\alpha+\delta=\pi$. Let T and T' be the adjacent tiles that share vertex a. Then T and T' have angles α and δ at their other common vertex (on the interior of P), absurd since then the third angle at that vertex must be π.

So the angles at a are δ,δ. Let T be the tile with vertex d and let the other vertices of T be (clockwise) e',a',b',c'. If e' is between d and e, then the angles there are ε,α. The vertex a' is in the interior of P and the angle of T at a' is α. In the adjacent tile with edge $a'e'$, the angle at a' is ε, impossible since then the third angle there is π. So $e'=e$ and e is 2-valent. Since the angle in T at a' is α, vertex a' is between a and e. Either (1) b' is in the interior of P and c' is between c and d, or (2) b' is between b and c and $c'=c$. In (1), the angles at c' are α and $\gamma=\pi-\alpha$. Let T' be the adjacent tile with edge $b'c'$. T' must have angle ε at its vertex v that precedes c' and so either $\alpha+\varepsilon=\pi$ (if v is between c and c'), $\varepsilon=\gamma$ (if $v=c$ is 2-valent), or $\varepsilon+\delta\leqslant\gamma$ (if $v=c$ is 3-valent). In every case, $\alpha+\gamma+\varepsilon+\delta<2\pi$, impossible. In (2), the angles at b' are α,β, hence the angles at a' are also α,β. In the other tile with edge $a'b'$ the next angle clockwise after β must be γ, hence this occurs at a 3-valent vertex on aa', and so $\gamma+\alpha=\pi$. But then $\alpha+\beta+\gamma+\delta<2\pi$, impossible.

(ii) Now suppose that the two acute angles are equal. At any 3-valent boundary vertex not at a corner of P the angles are $\alpha,\pi-\alpha$, whence some obtuse angle of P equals $\pi-\alpha$ and both other obtuse angles are larger (otherwise the two angles not larger than $\pi-\alpha$ plus the two angles equal to α will sum to at most 2π). This implies that at any 3-valent corner (if there are any) there are two angles equal to α. Otherwise, at a 3-valent corner one angle would be α and the other an obtuse angle less than $\pi-\alpha$, which we just saw was impossible.

We first consider the case in which the acute angles are adjacent; we may assume $\alpha=\varepsilon<\pi/2$. Let T be the tile with vertex e and let the remaining vertices of T be (clockwise) a',b',c',d'. Suppose $a'=a$. Then two vertices of T (either b' and d' or b' and c' or c' and d') are 3-valent vertices not at a corner of P. In every case, this implies that T has two obtuse angles equal to $\pi-\alpha$, contradiction. Thus a' is a 3-valent vertex between a and e and the angles at a' are $\alpha,\pi-\alpha$. One of the following must occur: (1) $d'=d$ and c' is between c and d, (2) $d'=d$, $c'=c$, and b' is between b and c, or (3) $d'\neq d$ and d' is between d and e. In (1), the angles at c' are α,γ, and hence the angles at a' are also α,γ. The tile T' that shares vertex a' with T has angle δ at its next clockwise vertex v after a', hence $v\neq a$ and v is a 3-valent vertex on aa'. But then $\delta=\gamma=\pi-\alpha$, contradiction. In (2) the angles at b' are α,β which implies the angles at a' are α,β. Since, the next clockwise vertex after a' on ae must have angle γ, we find (arguing as in (1)) that $\beta=\gamma=\pi-\alpha$, contradiction. In (3), the angles at d' are α,δ, and hence the angles at a' are also α,δ. Any other vertex between a' and

a must also have angles α, δ. This implies that the tile with vertex a must have α as its next angle clockwise after vertex a. Since, every 3-valent vertex between a and b (if there are any) must have angles α, δ, it follows that b is 3-valent and has angles α, α. If there is no other 3-valent vertex on bc then $\gamma = \beta$ and the tile with vertex c has γ as its next angle clockwise after c. This angle either occurs at d or between c and d, and in either case $\gamma = \delta$, contradiction. So there is a 3-valent vertex $v \neq b$ on bc. Either $v = c$ and has angles α, α, or v is between b and c and has angles α, δ (since $\beta \neq \delta$). In the second case, every 3-valent vertex between b and c has angles α, δ which implies that c is 3-valent and has angles α, α. In the same manner, we find that d is 3-valent and has angles α, α. But then $2\alpha = \beta = \gamma = \delta$, contradiction. So this first case is impossible.

Now suppose that the acute angles are not adjacent; we may assume $\alpha = \delta < \pi/2$.

Suppose first that $\varepsilon = \pi - \alpha$. The tile T with vertex d cannot have a 3-valent vertex on cd or bc, otherwise γ or β equals ε, impossible. So T has vertices a', b, c, d, e' with a' on ab and e' on de. The angles both at e' and at a' are ε, α. Thus the adjacent tile with edge $a'e'$ must have angle β at its next clockwise vertex after e'. But this implies $\beta = \varepsilon$, contradiction.

Now suppose that $\gamma = \pi - \alpha$. (By symmetry, this also covers the case $\beta = \pi - \alpha$.) There can be no 3-valent vertices between a and b or between d and e, otherwise β or ε equals γ, impossible. Since vertices b, a, e, d cannot belong to a single tile, the tile T with vertex d must have a 3-valent vertex on ae. Suppose T has no 3-valent vertex on cd. Then T must have a 3-valent vertex b' between b and c with angle β at b', impossible since $\beta \neq \gamma$. Thus T has a 3-valent vertex c' between c and d and c' has angles α, γ. This implies that the vertex e of T must be 2-valent and hence the next clockwise vertex a' of T is between a and e with angles α, γ. The last vertex b' of T has angle β and the other tile with edge $a'b'$ must also have angle β at b'. Since every 3-valent vertex between c and c' has angles α, γ, the other tile with edge $b'c'$ must have angle ε at b'. Hence $2\beta + \varepsilon = 2\pi$. But $\beta + \delta + \varepsilon = 2\pi$ also, which implies $\beta = \delta$, impossible. This completes the proof. □

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