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Hamiltonian Cycles in T-Graphs

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Abstract. There is only one finite, 2-connected, linearly convex graph in the Archimedean triangular tiling that does not have a Hamiltonian cycle.

The vertices and polygonal edges of the planar Archimedean tilings 4^4 and 3^6 of the plane, partially shown in Figs. 1 and 2, respectively, are called the *square tiling graph* (*STG*) and the *triangular tiling graph* (*TTG*). (See [1].)

A subgraph G of *TTG* is *linearly convex* if, for every line L which contains an edge of *TTG*, the set $L \cap G$ is a (possibly degenerate or empty) line segment. Such a line L is called a *grid line*. Linearly convex subgraphs of *STG* are defined similarly. A *T-graph* (respectively, *S-graph*) is any nontrivial, finite, linearly convex, 2-connected subgraph of *TTG* (respectively, *STG*). For example, the graph G shown in Fig. 3 is linearly convex even though it has three components including an isolated vertex v, and G has vertices x and y whose midpoint z is a vertex of *TTG* but not of G. (Each component of G is 2-connected, and the two nontrivial components are each T-graphs.) If a nontrivial graph G is *Hamiltonian* (i.e., has a Hamiltonian cycle), then it is clearly 2-connected. Zamfirescu and Zamfirescu [5] investigated which S-graphs have a Hamiltonian cycle. The situation is much easier for T-graphs, as we will show. With only one exception, any T-graph is Hamiltonian.

Let D denote the T-graph shown in Fig. 4—the linearly-convex hull of the Star of David. Even though D is 2-connected and linearly convex, it is clearly not Hamiltonian.

Theorem. *Every T-graph, other than D, is Hamiltonian.*

Proof. Let G be any T-graph. The *boundary of G*, denoted ∂G, is the boundary of the unbounded component of the complement of G in the plane. It is clear that ∂G is a cycle

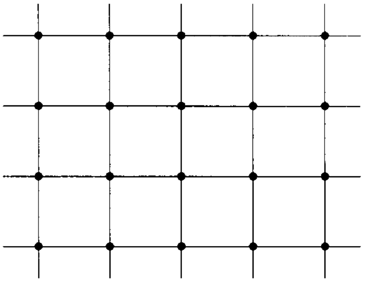

Fig. 1

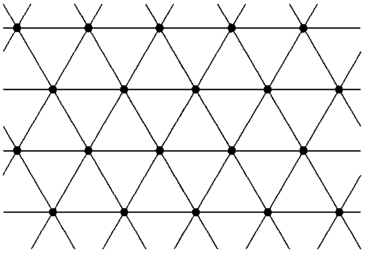

Fig. 2

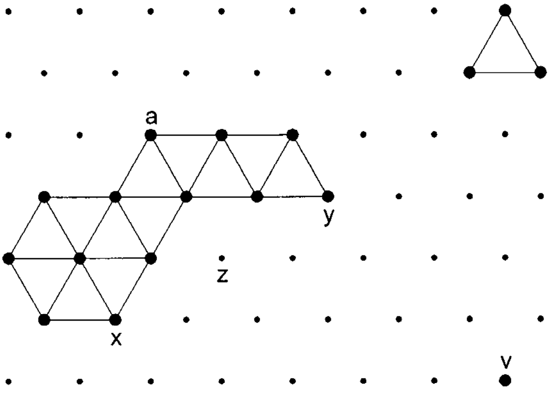

Fig. 3

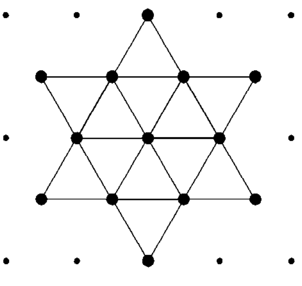

Fig. 4

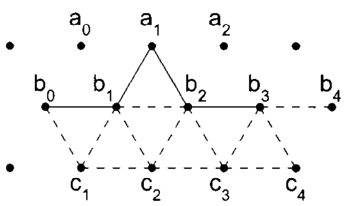

Fig. 5

in G which surrounds G, and any vertex of G not on ∂G is called an *interior vertex* of G. A vertex a on ∂G is said to be a *boundary vertex of type* 1 (respectively, of *type* 2) if the boundary of G forms an interior angle of size $\pi/3$ (respectively, of size $2\pi/3$) at a. If a is any boundary point of G of type 1 or 2, let $G' = G'(a)$ denote the subgraph of G formed by removing vertex a and edges of G adjacent to a. If a is of type 1, then either $G'(a)$ is a line segment and G is a triangle with a Hamiltonian cycle, or else $G'(a)$ is also linearly convex and 2-connected and therefore a T-graph. However, if vertex $a \in \partial G$ is of type 2, then $G'(a)$ need not be a T-graph unless a is also adjacent to an interior vertex of G. (For example, see vertex a in Fig. 3.) The key step in an inductive proof will be to remove a boundary vertex of type 1 or 2, inductively assume that $G'(a)$ is a T-graph with a Hamiltonian cycle (H-cycle), and extend that to an H-cycle for G. We may assume G has interior points, for otherwise it is clear that ∂G is an H-cycle for G. Thus the T-graphs without interior points form a basis for the inductive proof.

Case 1. Assume there exists a boundary vertex a_1 of G of type 1, let b_1 and b_2 be the (unique) two vertices of G adjacent to a_1, and label the vertices of G with the coordinate system shown in Fig. 5. The boundary cycle ∂G which includes the path $b_1a_1b_2$ could not continue from b_2 toward a_2 or G would not be linearly convex, nor toward b_1 or else G would fail to have an interior point or fail to be 2-connected. If ∂G continued from b_2 toward c_2, then any H-cycle H' of $G'(a_1)$ would necessarily use edge b_1b_2 (since b_2 is of type 1 in G'). Replacing edge b_1b_2 by path $b_1a_1b_2$ would extend H' to an H-cycle for G.

Next suppose ∂G included the path $b_1a_1b_2c_3$, and edge b_1b_2 does not lie in any H-cycle H' of G'. Then any H-cycle of G' would have to include the path $c_3b_2c_2$. As this cycle left c_2 it might go directly to b_1 via edge c_2b_1. In this case the H-cycle of G' could be modified to an H-cycle of G by replacing $c_3b_2c_2b_1$ by $c_3c_2b_2a_1b_1$. Otherwise (i.e., edge c_2b_1 is not in any H-cycle of G'), to include b_1 in H', another part of H' would have to include the path $b_0b_1c_1$. Then H' may be modified by replacing path $b_0b_1c_1$ by b_0c_1 and replacing $c_3b_2c_2$ by $c_3b_2a_1b_1c_2$, giving an H-cycle for G.

Thus (using symmetry) we may assume that any T-graph with a boundary vertex a_1 of type 1 has an H-cycle except possibly in the case when ∂G contains the path $b_0b_1a_1b_2b_3$. In this case, it follows from the linear convexity of G that the grid line L, through a_1 and parallel to edge b_1b_2, may meet G only at vertex a_1. Hence the path $b_0b_1a_1b_2b_3$ in ∂G will continue along the b-level grid line to a last vertex b_n ($n \geq 3$) and then drop to a c-level vertex of G. If $n = 3$ and the next edge of ∂G is b_3c_3, then b_3 is a boundary vertex of type 1, and the same arguments may be applied with b_3 in the role

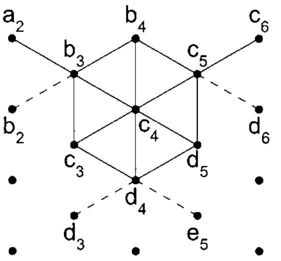

Fig. 6

of a_1. (Six repeated iterations would produce the T-graph D.) Otherwise the next edge of ∂G is b_3c_4 and b_3 is of type 2, or the next edge of ∂G is b_3b_4, so $n \geq 4$ and we may assume b_n is a boundary vertex of type 2, by the reasoning in Case 1. If the segment $b_0b_1a_1b_2b_3 \cdots b_nc_{n+1}$ of the boundary cycle ∂G would continue from c_{n+1} to c_n, then c_{n+1} would be of type 1 and of the form already eliminated above. It follows that as ∂G continues from c_{n+1} it eventually must meet the d-level of vertices, by the 2-connected property. A symmetric argument shows that ∂G continues from b_0 and meets the d-level of vertices. Thus linear convexity forces c_n to be an interior vertex of G. So b_n is of type 2 and adjacent to an interior vertex of G. It follows that $G'(b_n)$ must be 2-connected and linearly convex, and hence a T-graph, and hence we may assume that $G'(b_n)$ has an H-cycle H'.

Case 2. Assume there exists a boundary vertex b_4 of G of type 2, let b_3 and c_5 be the (unique) two vertices of ∂G adjacent to b_4, and label the vertices of G with the coordinate system shown in Fig. 6. We also assume that any boundary vertices of G of type 1 are of the special form described at the end of Case 1 above, and we may assume the vertex c_4 is in the interior of G. Thus $G' = G'(b_4)$ is a T-graph, and we assume G' has at least one H-cycle H'. If any H-cycle H' of G' contains either edge b_3c_4 or the edge c_4c_5 (say the latter), then replacing c_4c_5 by the path $c_4b_4c_5$ would extend H' to an H-cycle for all of G. So we assume that every H-cycle H' of G' contains neither edge b_3c_4 nor edge c_4c_5. Then each H' must (using symmetry) contain one of the paths $d_4c_4d_5$ or $c_3c_4d_5$.

Case 2.1. Assume some H' contains $d_4c_4d_5$. If H' also contains edge d_5c_5, then replace the path $d_4c_4d_5c_5$ in H' by $d_4d_5c_4b_4c_5$ to extend H' to an H-cycle for G. If edge d_5c_5 is not in H', then H' must also contain the path $c_6c_5d_6$. Replace this path by edge c_6d_6 and path c_4d_5 by $c_4b_4c_5d_5$ to extend H' to an H-cycle for G.

Case 2.2. Assume each such H' contains $c_3c_4d_5$. If H' does not contain edge d_5c_5, then the reasoning of Case 2.1 applies verbatim. So by symmetry we may suppose that every H-cycle H' for G' contains the path $b_3c_3c_4d_5c_5$.

Since c_4 is an interior vertex of G, vertex d_4 is in G and must lie in any H-cycle for $G'(b_4)$. The cycle H' may not extend directly from d_4 to any of its neighbors c_3, c_4, or d_5, by Case 2.2. Thus either edge d_3d_4 or edge d_4e_5 (or both) must be part of H'. By symmetry assume $d_4e_5 \subset H'$. Replace it by $d_4d_5e_5$ and replace $c_4d_5c_5$ by $c_4b_4c_5$. This extends H' to an H-cycle for G as desired.

Case 0. The above two cases were built on the assumption that there exists a vertex of type 1 in ∂G. If ∂G has no vertices of type 1 we need an independent proof of the existence of a vertex of type 2 which has an adjacent interior vertex of G, so that Case 2 may be applied. Vertex a in Fig. 3 shows that not every vertex of type 2 in such a T-graph has this property. It is easy to show if ∂G has no vertices of type 1, and if vertex $b \in \partial G$ is of type 2, and if b is not adjacent to an interior vertex of G, then b cannot be an *extreme* vertex; that is, there is no grid line through b which supports G. It is also easily checked that any extreme vertex of type 2 in ∂G must have an adjacent interior vertex, and by Case 2 the proof is complete. □

Remark 1. An immediate and interesting application may be obtained from the dual formulation of the theorem. A *patch* G in the Archimedean tiling 6^3 of the plane by regular hexagons is any finite subset of at least three of the hexagonal tiles. Each hexagonal tile $H \in G$ is adjacent to six other tiles, whose centers determine six main directions from the center of H. Suppose that for each tile H in a patch G, and for every other tile $H_1 \in G$ that lies in one of the six main directions from H, all tiles between H and H_1 are also in the patch G. In this case we say that G is *visible in each main direction*. Using an analogy from the game of chess, we say that a patch G has a *King's tour* if all hexagons of G may be sequentially ordered, $H_1, H_2, H_3, \ldots, H_k$ so that each H_i has an edge adjacent with H_{i+1}, $i = 1, \ldots, k-1$, and H_k and H_1 have an adjacent edge.

Corollary. *With one exception, if patch G in* 6^3 *is visible in each main direction, and if each pair of hexagonal tiles in G lie in some subpatch that has a King's tour, then G has a King's tour. The only exception is shown in Fig.* 7.

Remark 2. Hamiltonian cycles on Archimedean graphs have been studied in various settings. See [2] and [3] for references on counting the number of Hamiltonian cycles on certain S-graphs, and [4] for results on the areas enclosed by thin Hamiltonian cycles in any of the Archimedean tiling graphs. Earlier references are found in [5].

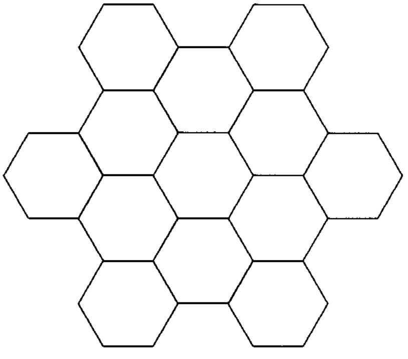

Fig. 7

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