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On a Theorem of Deutsch and Singer

TUDOR ZAMFIRESCU
Fachbereich Mathematik, Universität Dortmund, 44221 Dortmund, Germany

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Abstract. We introduce the notion of a starshaped set-valued function, and relate it to convex set-valued functions. We prove a theorem which, in the convex case, implies that single-valuedness is exceptional if the function is not everywhere single-valued. A second result shows a remarkable costantness of images for starshaped set-valued functions. Both theorems were inspired from a result of Deutsch and Singer, which they strengthen.

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1. Introduction

The main theorem in Deutsch and Singer's paper [1] says that if a set-valued convex function with nonempty values everywhere on a linear space is somewhere single-valued, then it is so everywhere.

This theorem can be strengthened by dropping the assumption about the nonempty values. By the way, this assumption is rather restrictive. In finite dimensions it yields together with compact-valuedness identical values up to translation, which is not the case without it.

We further refine the result by enlarging the considered class of functions to that of starshaped functions, and then by proving the constantness of dimension.

In the following X and Y are two linear spaces.

Let $A \subset X$. A point $x \in A$ is called *internal* if on every line L through x there is an open line-segment containing x and included in A (see Köthe [3]); the set $\operatorname{int} A$ of all internal points of A will be called the *core* of A (Klee calls this set the *intrinsic core* of A [2]).

Let $\mathcal{P}(Y)$ denote the collection of all subsets of Y. We call a function $F\colon X \to \mathcal{P}(Y)$ *starshaped* if there exists a point $y \in \mathcal{D}(F)$ such that, for any point $x \in \mathcal{D}(F)$ and number $\lambda \in (0, 1)$,

$$F(\lambda x + (1-\lambda)y) \supset \lambda F(x) + (1-\lambda)F(y).$$

Here, $\mathcal{D}(F)$ denotes the *domain* of F, i.e. the set on which F is not empty. Let $\mathcal{S}(F)$ denote the set of all points able to play the role of y.

If $\mathcal{D}(F) = \mathcal{S}(F)$ then F is said to be *convex*.

It is easily seen that for every starshaped function F, the set $\mathcal{S}(F)$ is convex, and $F|_{\mathcal{S}(F)}$ is convex. Also, if F is convex, then $F(x)$ is empty or convex for all $x \in X$.

For $x, y \in X$, the open line-segment from x to y will be denoted by (x, y).

2. Single-Point Values

THEOREM 1. *Let F: $X \to \mathcal{P}(Y)$ be starshaped. If F is single-valued at some internal point of $\mathcal{S}(F)$ then F is single-valued everywhere in $\mathcal{D}(F)$.*

Proof. Suppose the starshaped function F is single-valued at the point $x_0 \in \operatorname{int} \mathcal{S}(F)$, but at some other point $x \in \mathcal{D}(F)$ there are two distinct points $y, z \in F(x)$. Then the line through x_0 and x contains a point $x' \in \mathcal{S}(F)$ such that $x_0 \in (x, x')$. So $F(x') \neq \emptyset$. Take $q \in F(x')$.

Let $\lambda \in (0, 1)$ be such that $x_0 = \lambda x' + (1 - \lambda)x$. Then

$$F(x_0) \supset \lambda F(x') + (1 - \lambda)F(x) \supset \lambda\{q\} + (1 - \lambda)\{y, z\}$$
$$= \{\lambda q + (1 - \lambda)y, \lambda q + (1 - \lambda)z\}.$$

Thus $F(x_0)$ contains more than one point, contrary to the assumption. □

For $\mathcal{S}(F) = \mathcal{D}(F)$ we obtain the following corollary; if moreover $\mathcal{D}(F) = X$, we get Deutsch and Singer's main theorem.

COROLLARY 1. *Let F: $X \to \mathcal{P}(Y)$ be convex. If F is single-valued at some internal point of $\mathcal{D}(F)$ then F is single-valued everywhere in $\mathcal{D}(F)$.*

3. Constantness of Dimension

THEOREM 2. *Let F: $X \to \mathcal{P}(Y)$ be starshaped. Then $x \in \mathcal{S}(F)$ and $y \in \operatorname{int} \mathcal{D}(F)$ imply* $\dim F(x) \leqslant \dim F(y)$. *Consequently* $\dim F(x)$, *as a function of x, is constant on $\mathcal{S}(F) \cap \operatorname{int} \mathcal{D}(F)$. Moreover, this function is constant on* $\operatorname{int} \mathcal{D}(F)$ *if* $\operatorname{int} \mathcal{S}(F) \neq \emptyset$.

Proof. Let $x \in \mathcal{S}(F)$ and let y be an internal point of $\mathcal{D}(F)$ distinct from x. There is a point $z \in \mathcal{D}(F)$ such that $y \in (x, z)$. Hence $y = \lambda x + (1 - \lambda)z$ for some $\lambda \in (0, 1)$. Let $q \in F(z)$.

Suppose $\dim F(x) = n < \infty$. We can find the points $a_0, a_1, \ldots, a_n \in F(x)$ such that $a_1 - a_0, a_2 - a_0, \ldots, a_n - a_0$ are linearly independent. Then

$$F(y) \supset \lambda F(x) + (1 - \lambda)F(z) \supset \lambda\{a_0, \ldots, a_n\} + (1 - \lambda)\{q\} \supset \{b_0, \ldots, b_n\},$$

where

$$b_i = \lambda a_i + (1 - \lambda)q \quad (i = 0, \ldots, n).$$

Since $b_i - b_0 = \lambda(a_i - a_0)$, the points $b_1 - b_0, \ldots, b_n - b_0$ are also linearly independent, and $\dim F(y) \geqslant \dim F(x)$. Of course, the converse inequality is also true, in case both points are internal to $\mathcal{D}(F)$ and belong to $\mathcal{S}(F)$.

If $x \in \operatorname{int} \mathcal{S}(F)$ and, as before, $y \in \operatorname{int} \mathcal{D}(F)$, then there is a point $x' \in \mathcal{S}(F)$ such that $x \in (x', y)$. A similar argument yields now $\dim F(x) \geqslant \dim F(y)$. Thus $\dim F(x) = \dim F(y)$. Since the choice of y in $\operatorname{int} \mathcal{D}$(F) was arbitrary, the last assertion of the theorem is proven. □

For $\mathcal{S}(F) = \mathcal{D}(F)$, we get the following corollary.

COROLLARY 2. *Let* $F: X \to \mathcal{P}(Y)$ *be convex. Then* $\dim F(x)$*, as a function of* x*, is constant on the core of* $\mathcal{D}(F)$ *and not larger elsewhere.*

If, moreover, $\mathcal{D}(F) = X$, we get the following corollary, which also implies Deutsch and Singer's theorem.

COROLLARY 3. *Let* $F\colon X \to \mathcal{P}(Y)$ *be convex and* $\mathcal{D}(F) = X$. *Then* $\dim F(x)$*, as a function of* x*, is constant on* X.

References

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