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On the length of the cut locus for finitely many points

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There is a priori no nonvanishing lower bound for the length of the cut locus on a surface with respect to a point. Since the sphere is responsible for this, we considered in [5] the case of surfaces not homeomorphic to the sphere. With the same motivation in mind, we investigate here the length of cut loci with respect to finite sets containing more than one point, on various surfaces.

Throughout this paper, each appearing surface is a compact 2-dimensional Alexandrov space with curvature bounded below, as defined in [2] or [6]. Thus, it is equipped with an intrinsic metric and is a topological 2-manifold [2].

For any compact surface S and closed subset $M \subset S$, ρ denotes the intrinsic metric on S and $d(M)$ the intrinsic diameter of M. A *segment ab* is a shortest path joining a to b (and has length $\rho(a, b)$). Moreover, $C(M)$ denotes the *cut locus* of M in S, i.e., the set of all points $x \in S$ such that some shortest path from x to M cannot be extended as a shortest path to M beyond x, and $C^{cp}(M)$ its *cyclic part*, i.e., the set of points in $C(M)$ admitting two segments to M which do not form a null-homotopic closed curve.

Let Σ_x be the (1-dimensional) space of directions at $x \in S$ (for a definition, see [2]). If the diameter of Σ_x is less than π, x is called *singular*. Let us call *strongly singular* those points $x \in S$ for which Σ_x has diameter at most $\pi/2$.

We denote by λ the length (1-dimensional Hausdorff measure).

We are going to give in Theorems 1–7 various lower bounds for the length of the cut locus with respect to a set M, which will always be assumed finite and containing more than one point.

Theorem 1. *On any surface S homeomorphic to the sphere $\mathbb{S}^2$ there is a pair of points p_1, p_2 such that $\lambda C^{cp}(p_1, p_2) \geqslant 2d(S)$.*

Proof. Let a, b be at maximal intrinsic distance on the surface S homeomorphic to

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$\mathbb{S}^2$. It is easily proved that the cyclic part of the cut locus of any pair of points is a closed Jordan curve.

Let σ be a segment from a to b, and $\{s\} = C^{cp}(a,b) \cap \sigma$. Let $t \in C^{cp}(a,b) \setminus \{s\}$. If $\rho(t,a) = \rho(s,a)$, put $p_1 = s$, $p_2 = t$. If $\rho(t,a) > \rho(s,a)$, then the two subarcs of $C^{cp}(a,b)$ joining s with t must contain points p_1, p_2 such that $\rho(p_1,a) = \rho(p_2,a)$.

With this choice of p_1, p_2, obviously $a, b \in C^{cp}(p_1,p_2)$. Hence $\lambda C^{cp}(p_1,p_2) \geqslant 2\rho(a,b)$. □

Theorem 2. *For any surface S homeomorphic to* $\mathbb{S}^2$ *without strongly singular points, for example diffeomorphic to* $\mathbb{S}^2$, *and for any integer* $n \leqslant 4$, *there are points* $p_1, \ldots, p_n \in S$ *such that* $\lambda C^{cp}(\{p_1, \ldots, p_n\}) \geqslant nd(S)$.

Proof. Let again $\rho(a,b) = d(S)$. This time, since Σ_a has diameter larger than $\pi/2$, there are at least two segments σ_1, σ_2 from a to b (see the proof of Theorem 2 in [9]; for S differentiable, see Steinhaus [7]). They cut $C^{cp}(a,b)$ at s_1, s_2, say.

If $\rho(t,a) = \rho(s_1,a)$ for infinitely many points $t \in C^{cp}(a,b)$, we choose n of them. For these, the cyclic part of the cut locus equals the union of the n arcs (of equidistant points) from a to b determined by the n pairs of neighbouring points. Thus, its length is at least $nd(S)$.

If $\rho(t,a) > \rho(s_1,a)$ for all but finitely many $t \in C^{cp}(a,b)$, choose two such points t_1 and t_2, one in each of the two arcs into which s_1 and s_2 divide $C^{cp}(a,b)$. For any number r satisfying

$$\rho(s_1,a) < r < \rho(t_i,a)$$

for both $i = 1$ and $i = 2$, there are four points at distance r from a, one on each of the four arcs into which s_1, t_1, s_2, t_2 divide $C^{cp}(a,b)$.

Now choose n out of these four points, and proceed as above. □

Let

$$\mathscr{C}_i^n(S) = \inf\{\lambda C^{cp}(M) : M \subset S, \operatorname{card} M = n\}$$

and

$$\mathscr{C}_s^n(S) = \sup\{\lambda C^{cp}(M) : M \subset S, \operatorname{card} M = n\}.$$

Also, let $i(S)$ denote the injectivity radius of S.

Theorem 3. *For any surface S and integer* $n \geqslant 2$ *we have* $\mathscr{C}_s^n(S) \geqslant nd(S)$.

Proof. Let $\rho(a,b) = d(S)$. Choose $\varepsilon > 0$ arbitrarily small. Consider the circle (in the intrinsic metric) of centre a and radius $d(S) - \varepsilon$. The topological aspect of circles was described in Shiohama and Tanaka's paper [6]. Take a small arc in this circle, and consecutive points $p_1, \ldots, p_n$ on this arc. Each pair p_i, p_{i+1} determines a subarc A_i $(i = 1, \ldots, n-1)$. Let $M = \{p_1, \ldots, p_n\}$.

Inside $ap_i \cup ap_{i+1} \cup A_i$, $C^{cp}(M)$ coincides with $C^{cp}(p_i, p_{i+1})$ and reduces to an arc joining a with the midpoint of A_i, obviously of length at least $d(S) - \varepsilon$.

Outside $ap_1 \cup ap_n \cup \bigcup_{i=1}^{n-1} A_i$, $C^{cp}(M)$ includes an arc from a to the midpoint of A_1, and has therefore length at least $d(S) - \varepsilon$.

Hence $\lambda C^{cp}(M) \geqslant n(d(S) - \varepsilon)$, and the inequality in the statement follows. □

Theorem 4. *For any surface S we have $\mathscr{C}_i^2(S) \geqslant 2i(S)$. If S has genus 1, then $\mathscr{C}_i^2(S) \geqslant 4i(S)$.*

Proof. Let $p_1, p_2 \in S$, take a segment p_1p_2 and put $\{s\} = p_1p_2 \cap C^{cp}(p_1, p_2)$, i.e., $\rho(p_1, s) = \rho(p_2, s)$. As an interior point of a segment, s is not singular. We claim that there is some point in $C^{cp}(p_1, p_2)$, different from s and joined with s by at least two segments.

Indeed, assume the claimed assertion is false. Let t be a variable point on $C^{cp}(p_1, p_2)$. As t goes around on $C^{cp}(p_1, p_2)$, the direction of st moves in Σ_s between a position τ orthogonal to the direction of sp_1 and the antipodal position (at distance π from τ). So, on its way, since it depends continuously on t (due to the uniqueness of the segment between s and t), it must take either the direction of sp_1 or that of sp_2, say the first one.

If $t \in sp_1$, then $\rho(p_1, t) = \rho(p_2, t)$ implies

$$\rho(p_2, t) + \rho(t, s) = \rho(p_1, s) = \rho(p_2, s),$$

which means that $st \cup tp_2$ is a segment; this and sp_1 would bifurcate, which is impossible (see [2]).

It follows that $p_1 \in st$. Since $\rho(p_1, s) = \rho(p_2, s)$ and $\rho(p_1, t) = \rho(p_2, t)$, the arc $sp_2 \cup p_2t$ is also a segment from s to t, different from the previous one because $p_1 \neq p_2$, and this contradicts the assumption.

Now, once the claim is proved, it is clear that $\lambda C^{cp}(p_1, p_2) \geqslant 2\lambda(st) \geqslant 2i(S)$. In case S is a torus, the same arguments apply to each of the two disjoint cycles forming $C^{cp}(p_1, p_2)$. □

Theorem 5. *Let S be a surface and $M \subset S$. If $d(M) \leqslant d(S)/2$, then $\lambda C(M) \geqslant d(S)/2$.*

Proof. Let again $\rho(a, b) = d(S)$. Assume p_a (respectively p_b) is a closest point of M from a (respectively b). (Possibly, $p_a = p_b$.) Let q_a be a closest point of $C^{cp}(M)$ from p_a. (Thus, q_a is either the midpoint of a segment p_ap or of a segment p_ap_b.) Further, if $p_a \neq a$, let c_a be the cut point of M in the direction of a segment p_aa; if $p_a = a$, set $c_a = q_a$. The points c_b and q_b are defined analogously.

Clearly, $\rho(p_a, q_a) \leqslant d(M)/2$ and $\rho(p_b, q_b) \leqslant d(M)/2$. Therefore

$$\rho(a, p_a) + \rho(b, p_b) \geqslant \rho(p_a, q_a) + \rho(p_b, q_b),$$

otherwise

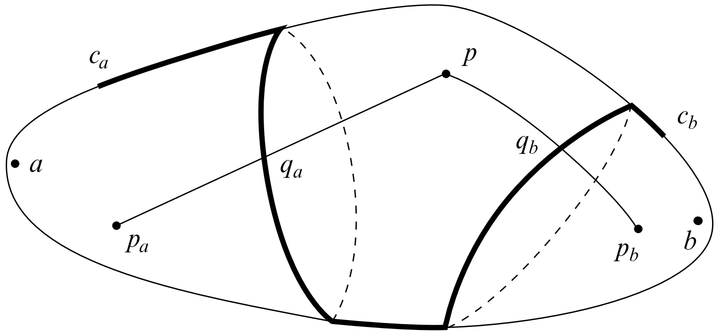

Figure 1

$$\rho(a, p_a) + \rho(p_a, p_b) + \rho(p_b, b) < \frac{d(M)}{2} + d(M) + \frac{d(M)}{2} \leqslant d(S),$$

which is false.

If $\rho(c_a, c_b) \geqslant d(S)/2$, the conclusion of the theorem follows. If $\rho(c_a, c_b) < d(S)/2$, then

$$\rho(a, c_a) + \rho(b, c_b) > \frac{d(S)}{2}.$$

Further, $\rho(q_a, c_a) \geqslant \rho(p_a, c_a) - \rho(p_a, q_a)$ and, similarly, $\rho(q_b, c_b) \geqslant \rho(p_b, c_b) - \rho(p_b, q_b)$. Hence

$$\begin{aligned}\rho(q_a, c_a) + \rho(q_b, c_b) &\geqslant \rho(p_a, c_a) + \rho(p_b, c_b) - (\rho(a, p_a) + \rho(b, p_b)) \\ &= \rho(a, c_a) + \rho(b, c_b) > \frac{d(S)}{2}.\end{aligned}$$

Let S_a, Γ_a denote the set of all points in S, respectively $C^{cp}(M)$, which are not closer to a point in $M \backslash \{p_a\}$ than to p_a. Also, let Γ_a' be the union of all Jordan arcs in $C(M)$ joining points in Γ_a. Then $S_a \backslash \Gamma_a'$ is an open topological disc and Γ_a a union of pairwise disjoint closed Jordan curves, the boundary of S_a.

Clearly, $a, p_a, c_a \in S_a$ and $q_a \in \Gamma_a$. Let $c_a c_a'$ be the shortest arc in $C(M)$ with $c_a' \in \Gamma_a'$. We proceed in the same way with b instead of a, and get $S_b, \Gamma_b, \Gamma_b', c_b'$, with $b, p_b, c_b \in S_b$, $q_b \in \Gamma_b$ and $c_b' \in \Gamma_b'$.

Put $\alpha = \lambda(\Gamma_a \backslash \Gamma_b)$, $\alpha' = \lambda(\Gamma_a' \backslash \Gamma_a)$, $\beta = \lambda(\Gamma_b \backslash \Gamma_a)$, $\beta' = \lambda(\Gamma_b' \backslash \Gamma_b)$, $\gamma = \lambda(\Gamma_a \cap \Gamma_b)$. We have

$$\lambda(q_a c_a') \leqslant \frac{\alpha + \gamma}{2} + \alpha', \quad \lambda(q_b c_b') \leqslant \frac{\beta + \gamma}{2} + \beta',$$

whence

$$\lambda(q_a c_a') + \lambda(q_b c_b') \leqslant \alpha + \beta + \gamma + \alpha' + \beta' \leqslant \lambda C^{cp}(M) + \alpha' + \beta'$$

and

$$\lambda(q_a c_a) + \lambda(q_b c_b) \leqslant \lambda C(M).$$

Hence, eventually,

$$\lambda C(M) \geqslant \rho(q_a, c_a) + \rho(q_b, c_b) > \frac{d(S)}{2}. \qquad \square$$

A set in the surface S is called *concyclic* if it is included in some intrinsic circle of S. The next theorem refers to a set in S which is not concyclic.

Theorem 6. *Let S be homeomorphic to $\mathbb{S}^2$, $M = \{p, p_a, p_b\} \subset S$ and $\rho(a,b) = d(S)$. Assume that p_a, p_b are closest in M from a, respectively b ($p_a \neq p_b$), and any point of S at the same distance from p_a and p_b has smaller distance from p. If $\rho(p, p_a) + \rho(p, p_b) \leqslant d(S)$, then $\lambda C(M) \geqslant d(S)/2$.*

Proof. We use the notation from the previous proof. As before, we may suppose $\rho(a, c_a) + \rho(b, c_b) > d(S)/2$, otherwise we are done.

From the inequality in the statement we get

$$\rho(p, p_a) + \rho(p, p_b) - \rho(p_a, p_b) \leqslant \rho(a, b) - \rho(p_a, p_b) \leqslant \rho(a, p_a) + \rho(b, p_b).$$

Also,

$$\begin{aligned}\rho(p, q_a) + \rho(p, q_b) - \rho(q_a, q_b) &= \rho(p, p_a) + \rho(p, p_b) \\ &\quad - (\rho(p_a, q_a) + \rho(q_a, q_b) + \rho(q_b, p_b)) \\ &\leqslant \rho(p, p_a) + \rho(p, p_b) - \rho(p_a, p_b) \leqslant \rho(a, p_a) + \rho(b, p_b).\end{aligned}$$

Now, as in the previous proof,

$$\rho(q_a, c_a) + \rho(q_b, c_b) \geqslant \rho(p_a, a) + \rho(a, c_a) + \rho(p_b, b) + \rho(b, c_b) - \rho(p_a, q_a) - \rho(p_b, q_b).$$

Any point in $\Gamma_a \cap \Gamma_b$ would be closer to p_a than to p. Hence Γ_a and Γ_b are disjoint, and there is a unique shortest arc $a'b' \subset C(M)$ with $a' \in \Gamma_a$ and $b' \in \Gamma_b$. Let $A \subset C(M)$ be an arc joining q_a to q_b and missing c_a' if $c_a' \neq a'$ and c_b' if $c_b' \neq b'$. In this manner we also find arcs A_a and A_b joining c_a to q_a and c_b to q_b, and meeting A only in q_a, q_b, and perhaps a', b'.

Hence

$$\begin{aligned}\lambda C(M) &\geqslant \lambda A + \lambda A_a + \lambda A_b \geqslant \rho(q_a, q_b) + \rho(q_a, c_a) + \rho(q_b, c_b)\\ &\geqslant \rho(q_a, q_b) + \rho(p_a, a) + \rho(a, c_a) + \rho(p_b, b) + \rho(b, c_b) - \rho(p_a, q_a) - \rho(p_b, q_b)\\ &\geqslant \rho(p, q_a) + \rho(p, q_b) + \rho(a, c_a) + \rho(b, c_b) - \rho(p_a, q_a) - \rho(p_b, q_b)\\ &= \rho(a, c_a) + \rho(b, c_b) > \frac{d(S)}{2}.\end{aligned}$$ □

Theorem 7. *Let the surface S be convex, and $M = \{p, p_a, p_b\} \subset S$. If M is concyclic, then $\lambda C(M) \geqslant d(S)/2$.*

Proof. If $d(M) \leqslant d(S)/2$, the conclusion follows from Theorem 5. Suppose $d(M) > d(S)/2$. Assume, for example, $\rho(p_a, p_b) = d(M)$. Consider the segments pp_a, pp_b and their midpoints q_a, q_b. From Alexandrov's convexity condition (see [1], p. 47), we have

$$\rho(q_a, q_b) \geqslant \rho(p_a, p_b)/2 > d(S)/4.$$

Let c be the centre of a circle through p, p_a, p_b. There are two possibilities for the topological aspect of $C^{cp}(M)$, pictured in Figure 2.

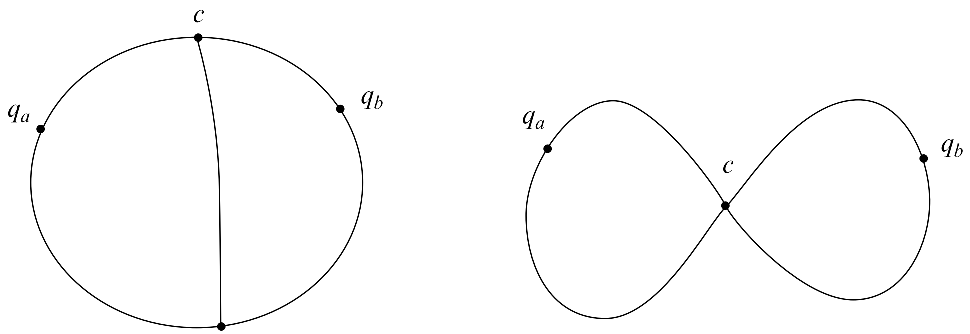

Figure 2

In both cases there are two arcs $A_1, A_2 \subset C^{cp}(M)$, both from q_a to q_b, such that $A_1 \cap A_2 \subset \{q_a, c, q_b\}$. Consequently,

$$\lambda C(M) \geqslant \lambda A_1 + \lambda A_2 \geqslant 2\rho(q_a, q_b) > d(S)/2.$$ □

It is easily seen that $\lambda C(M)/d(S)$ is not bounded below away from 0, for any cardinality of M. If M is a single point then $C(M)$ can be a single point too. If card $M \geqslant 2$, consider a surface as described in Figure 3, with the points of M at the ends of the "legs".

In the convex case, however, we dare to conjecture that this is not true.

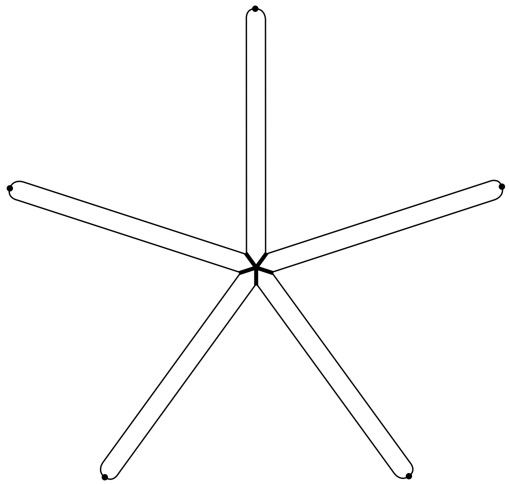

Figure 3

Conjecture 1. *For any smooth convex surface S, if $M \subset S$ contains more than two points, then $\lambda C(M) \geqslant d(S)/2$.*

Consider an ellipsoid in $\mathbb{R}^3$ with a long axis *ab* and both other axes very short. Let M have two points at a, b and the others on a segment σ joining a to b, all close to the midpoint of σ. In this case the length of $C(M)$ is very close to half the diameter of the ellipsoid. Thus, if true, the lower estimate for the length of the cut locus conjectured above is best possible.

Conjecture 2. *There exist surfaces S such that $\mathscr{C}_i^2(S) > 2d(S)$.*

It was shown in [4] and [6] (see also [8]) that $\lambda C(M)$ may be infinite, even for sets M containing a single point. The cut locus $C(M)$ may even fail to be of locally finite length. More precisely, there are convex surfaces S on which, for any point x, every open set in S contains a compact subset of $C(\{x\})$ with infinite 1-dimensional Hausdorff measure [10]. In the Riemannian case this cannot happen, see Hebda [3] and Itoh [4]. It is evident, however, that $\lambda C(M)/d(S)$ has no upper bound depending only on the cardinality of M.

Even concerning the cyclic part of the cut locus, we showed in [5] that there exists a sequence of metrics on the torus, for each of which the diameter of the torus is 1 while the length of the cyclic part of some cut locus tends to ∞.

Concerning the vanishing of $\mathscr{C}_i^n(S)$, we will show the following result.

Theorem 8. *Let S be a convex surface and $n \geqslant 2$. If $\mathscr{C}_i^n(S) = 0$ then S has a strongly singular point.*

Proof. Let $\{M_j\}_{j=1}^{\infty}$ be a sequence of sets, each of cardinality *n*, such that

$$\lim_{j \to \infty} \lambda C^{cp}(M_j) = 0.$$

Then $\lim_{j\to\infty} C^{cp}(M_j)$ is a singleton $\{q\}$, because the existence of two points in this limit at distance ε implies the existence of an arc in $C^{cp}(M_j)$ of length larger than $\varepsilon/2$ for all j large enough (because $C^{cp}(M_j)$ is connected), which is impossible.

Moreover, $\lim_{j\to\infty} M_j$ must also equal $\{q\}$. Indeed, assume this is not the case. Then there exists a subsequence of $\{M_j\}_{j=1}^{\infty}$ converging to an m-point set M, where $m \leqslant n$. Since $\lambda C^{cp}(M) > 0$ and C^{cp} is upper semicontinuous, the corresponding subsequence of $\{\lambda C^{cp}(M_j)\}_{j=1}^{\infty}$ would not tend to 0, and a contradiction is obtained.

We assume that the total length L of Σ_q is greater than π and derive a contradiction.

Let L_δ be the length of the intrinsic circle c_δ of centre q and radius $\delta > 0$. By [1], p. 383, we can choose δ so small that c_δ is a closed Jordan curve. Take $\varepsilon > 0$ such that $L > \pi + 3\varepsilon$. Let abc be a planar triangle with $\lambda(ab) = \delta/2$, $\angle abc = (\pi/2) - \varepsilon$ and $\angle acb > \pi/2$. Set $v = \lambda(bc)$. Clearly, $v < \delta/2$.

Let $K(B)$ be the curvature of the Borel set $B \subset S$. For the open disc $D_\delta \subset S$ of centre q and radius δ,

$$K(D_\delta) = K(D_\delta \backslash \{q\}) + (2\pi - L).$$

Here, $K(D_\delta \backslash \{q\}) \to 0$ as $\delta \to 0$. We choose δ such that $K(D_\delta \backslash \{q\}) < \varepsilon$. Thus,

$$K(D_\delta) < \varepsilon + 2\pi - (\pi + 3\varepsilon) = \pi - 2\varepsilon.$$

For j large enough, $M_j \subset D_{v/2}$. Let q_j, z_j be points in M_j, respectively c_δ, closest to each other. Also, choose $q_j' \in M_j \backslash \{q_j\}$. There exists a point $z_j' \in c_\delta$ such that q_j' lies on a segment from q_j to z_j' or there are two segments γ_1, γ_2 from q_j to z_j' surrounding q_j' inside c_δ.

In the first case z_j' is closer to q_j' than to q_j. In the second, let Δ_j be the open connected set containing q_j' and having $\gamma_1 \cup \gamma_2$ as boundary. Since $\Delta_j \subset D_\delta$,

$$K(\Delta_j) \leqslant K(D_\delta) < \pi - 2\varepsilon.$$

By the Gauß–Bonnet theorem, the angle of Δ_j at q_j' is less than $\pi - 2\varepsilon$. Then at least one of the two angles made by a segment $q_j q_j'$ with γ_1 and γ_2 must be less than $(\pi/2) - \varepsilon$. Let α_j denote that angle.

Take the planar triangle $a'b'c'$ with $\lambda(a'b') = \rho(z_j', q_j)$, $\angle a'b'c' = \alpha_j$ and $\lambda(b'c') = \rho(q_j, q_j')$. A quick comparison of the triangles abc and $a'b'c'$, in which the inequalities

$$\rho(z_j', q_j') \geqslant \rho(z_j', q) - \rho(q, q_j') > \delta - (v/2) > \delta/2,$$

$\alpha_j < (\pi/2) - \varepsilon$ and $\rho(q_j, q_j') < v$ hold, leads to $\angle a'c'b' > \angle acb > \pi/2$; therefore $\lambda(a'c') < \lambda(a'b')$.

By considering the geodesic triangle $z_j' q_j q_j'$ and the planar triangle $a'b'c'$, Toponogov's comparison theorem (hinge version) implies

$$\rho(z_j', q_j') < \lambda(a'c') < \lambda(a'b') = \rho(z_j', q_j).$$

Since, obviously, $\rho(z_j, q_j') \geqslant \rho(z_j, q_j)$, there must exist a point $z_j^* \in c_\delta$ such that $\rho(z_j^*, q_j') = \rho(z_j^*, q_j)$.

We found in this way a point of $C^{cp}(M_j)$ on c_δ. On the other hand the midpoint $m_j \in C^{cp}(M_j)$ of a segment joining the two points of M_j closest to each other is at distance less than v from q. Hence $\rho(m_j, z_j^*) > \delta - v > \delta/2$. So, the arc in $C^{cp}(M_j)$ joining m_j with z_j^* has length larger than $\delta/2$ for all large j, and a contradiction is reached. □

It is easily verified that, if S is a convex polyhedral surface with a vertex x such that the diameter of Σ_x is less than $\pi/2$, then $\mathscr{C}_i^n(S) = 0$. This leads us to the following conjecture.

Conjecture 3. *If the diameter of Σ_x is less than $\pi/2$ for some point $x \in S$, then $\mathscr{C}_i^n(S) = 0$.*

Consider the standard sphere $\mathbb{S}^2$. Clearly, $\mathscr{C}_i^j(\mathbb{S}^2) = \mathscr{C}_s^j(\mathbb{S}^2) = j\pi$ $(j = 2, 3)$. For any $n \geqslant 3$, we have $\mathscr{C}_i^n(\mathbb{S}^2) = 3\pi$. To see this it is enough to consider n points in some small disc on $\mathbb{S}^2$ and arrange that their convex hull be a (spherical) triangle. However, determining $\mathscr{C}_i^n(S)$ for other surfaces S is less easy.

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