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The strange aspect of most compacta

To Solomon Marcus, the best of my teachers, for his Real Functions with wonderful features.

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Abstract. In this paper we describe a compact set K in $\boldsymbol{R}^d$ which is typical from the point of view of Baire categories, as it appears when seen from some point x of $\boldsymbol{R}^d$. It matters whether x belongs to K or not! If $x \notin K$, then K looks porous (this is easily seen). If $x \in K$, then K looks only σ-porous, but dense at least in a half-sphere (of directions from x). If x is a typical point of K, then K looks even dense (in the whole sphere).

1. Introduction.

Let $\mathscr{K}$ be the complete space of all compact sets in the Euclidean space $\boldsymbol{R}^d (d \geq 2)$, with the usual Pompeiu-Hausdorff metric. Let $\mathscr{C} \subset \mathscr{K}$ be its closed subspace of all continua (i.e. compact connected sets), and $\mathscr{S} \subset \mathscr{C}$ the closed subset of all starshaped compact sets.

We say that *most* (or *typical*) elements of a Baire space enjoy property $\boldsymbol{P}$ if those elements not enjoying $\boldsymbol{P}$ form a first category set.

Very curious topological properties of most continua have been discovered as early as in 1930 by Mazurkiewicz [**7**], and later by Bing [**1**]. The striking fact is that most continua are pseudoarcs, objects with quite counterintuitive properties. Several properties of most compact sets have been established by the author [**11**], Gruber [**3**], Wieacker [**10**] and Myjak and Rudnicky [**8**]. Most starshaped compact sets have also been investigated, see [**13**], [**5**], [**14**].

The surveys [**15**] and [**4**] provide fairly complete descriptions of the typical convex bodies.

Suppose a point lives in a typical compact world. It can be miop (like me or even worse), but still it may look from time to time to the surrounding world. How does the world appear to it? The answer shows an astonishing regularity. We formulate it before giving definitions; so it cannot be precise but suggestive: seen from most of its points, the world looks dense, from all its other points the world looks at least half-dense, and from densely many points exactly half-dense. So, if points are regarded as stars of this compact universe then, seen from most stars, the other stars lie densely in the sky; and, seen from any star, at least a hemisphere of the sky is densely covered by stars. However, again strangely enough, seen from any place outside, our typical universe looks nowhere dense (see [**13**])!

A typical convex body has by a result of Klee from 1959 ([**6**], see also [**2**] and [**12**])

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a C^1 boundary and provides a crude example of a compact set enjoying the first two properties. But most compacta, most continua and most compact starshaped sets are much thinner than most convex bodies, in particular they are nowhere dense and have Hausdorff dimension zero.

For those not satisfied with the above description or reluctant to simply believe the author, we have to become more formal.

2. Definitions and notation.

We denote by $\mathrm{int}A$, $\mathrm{bd}A$, $\overline{A}$, $\mathrm{conv}A$ the interior, boundary, closure and convex hull of the set A, respectively.

For any point $x \in \boldsymbol{R}^d$ and set $A \subset \boldsymbol{R}^d$, let

$$D_x(A) = \left\{ \frac{y-x}{\|y-x\|} : y \in A \setminus \{x\} \right\}.$$

We shall make use of open convex sets of the following kind. Let $\varDelta_\alpha(z)$ be the open disc on S^{d-1} of centre $z \in S^{d-1}$ and radius $\alpha > 0$, i.e.

$$\varDelta_\alpha(z) = \{u \in S^{d-1} : \|u - z\| < \alpha\},$$

and let

$$C_\alpha(z) = \{\lambda x : x \in \varDelta_\alpha(z), 0 < \lambda < \alpha\}.$$

Let us say that a compact set $K \in \mathscr{K}$ *looks dense from* $x \in K$ if for any neighbourhood N of x, the set of directions $D_x(N \cap K)$ is dense in S^{d-1}.

We also say that $K \in \mathscr{K}$ *looks at least half-dense from* $x \in K$ if there is a closed half-space H^+ with the origin $\mathbf{0}$ on its boundary, such that for any neighbourhood N of x, the set $D_x(N \cap K)$ is dense in $S^{d-1} \cap H^+$. We remark that, $\overline{D_x(N \cap K)}$ being an increasing function of N with respect to inclusion, this definition is equivalent to the following. The set $K \in \mathscr{K}$ looks at least half-dense from $x \in K$ if, for any neighbourhood N of x, there is a closed half-space H^+ with $\mathbf{0} \in \mathrm{bd}H^+$ such that $\overline{D_x(N \cap K)} \supset S^{d-1} \cap H^+$.

If, in addition, $\cap_{N \ni x} D_x(N \cap K) \subset H^+$, we say that K *looks half-dense from* x.

And if, moreover, there exists a neighbourhood N_0 of x such that $D_x(N_0 \cap K) \subset H^+$, then we say that K *looks precisely half-dense from* x.

For x in $\boldsymbol{R}^d$, or S^{d-1}, and $\delta > 0$, $B_\delta(x)$ denotes the closed ball, or disc, of centre x and radius δ.

Let $0 < \alpha < 1$. A set $A \subset S^{d-1}$ is called (α)-*porous* if in any ball $B_\delta(x) \subset S^{d-1}$ there is another ball $B_{\alpha\|x-y\|}(y)$ (of radius α times the distance from its centre to x) disjoint from A. A set is called *porous* if it is (α)-porous for some α. Any countable union of porous sets is called σ-*porous*.

$\mathscr{X}$ will be one of the spaces $\mathscr{K}, \mathscr{C}, \mathscr{S}$.

3. Viewing from arbitrary points.

As already mentioned, most sets $K \in \mathscr{X}$ are nowhere dense and of Lebesgue measure zero. They are even porous [**11**]. This property is inherited in a weaker form by the images $D_x(K)$. The precise result is formulated in the next theorem and uses the notion of σ-porosity.

Theorem 0. *Suppose the dimension d is at least 3 if $\mathscr{X}$ is $\mathscr{C}$ or $\mathscr{S}$. Then, for most sets $K \in \mathscr{X}$ and any point $x \in K$, $D_x(K)$ is σ-porous.*

Proof. First we remark that we shall work with the space $\mathscr{S}'$ of all starshaped sets K having a single-point kernel $\{k_K\}$ instead of $\mathscr{S}$, knowing that $\mathscr{S}'$ is residual in $\mathscr{S}$ [**13**].

We are going to prove that, for most $K \in \mathscr{X}$ and any $x \in K$, $D_x(K)$ is a countable union of $(\frac{1}{2})$-porous sets.

Let $\mathscr{X}_{n,m}$ be the family of all $K \in \mathscr{X}$ for which there are a point $x \in K$, a ball $B_{1/n}(x) \subset \boldsymbol{R}^d$ and a ball $B_\delta(z)$ of centre $z \in S^{d-1}$ and radius $\delta \geq 1/m$ such that $S^{d-1} \cap B_\delta(z) \setminus D_x(K \setminus (B_{1/n}(x) \cup B^*_{1/n}(k_K)))$ contains no closed disc of radius half the distance from its centre to z $(m, n \in \boldsymbol{N})$, where $B^*_{1/n}(k_K)$ means $B_{1/n}(k_K)$ if $\mathscr{X} = \mathscr{S}'$ and is void otherwise. That is

$$\mathscr{X}_{n,m} = \{K \in \mathscr{X} : \exists x \in K, \exists z \in S^{d-1} \text{s.t.} \forall z' \in S^{d-1},$$
$$B_{1/m}(z) \setminus D_x(K \setminus (B_{1/n}(x) \cup B^*_{1/n}(k_K))) \not\supset B_{\|z'-z\|/2}(z')\}.$$

We show that $\mathscr{X}_{n,m}$ is nowhere dense in $\mathscr{X}$. To this end, approximate (in $\mathscr{K}$) an arbitrary element K of $\mathscr{X}$ by a finite subset F of $\eta\boldsymbol{Z}^d$. Join neighbouring points (i.e., at distance η) in F by line-segments as necessary to get an approximating set in $\mathscr{C}$ if $\mathscr{X} = \mathscr{C}$, and join all points of F with k_K if $\mathscr{X} = \mathscr{S}'$. We obtain a set K'.

We claim that, for $\mathscr{X} = \mathscr{C}$, the orthogonal projection K'' of K' on an arbitrary hyperplane H is $(\frac{1}{2})$-porous in H.

Indeed, there are only d distinct directions of edges in the graph K'. So, any point of K'', like any point of K', has degree at most $2d$.

We now prove by induction that at any point p of K'' there is a $(d-1)$-dimensional small open bounded cone $\Gamma^{(d-1)}_\lambda$ with apex p, centre line λ and based on a $(d-2)$-dimensional sphere of angular radius $\pi/6$, disjoint from K'' (this implies the $(\frac{1}{2})$-porosity of K'' in H). In the proof we only use the property of K'' of being a graph with line-segments as edges and of maximal degree $2d$.

For $d = 3$, any 3 concurrent lines in a plane form 6 angles, one of which is at least $\pi/3$. Assuming this true in dimension d, let the dimension be $d+1$. Choose one of the lines through $p \in K''$ and take the ($(d-1)$-dimensional) hyperplane H' of H through p orthogonal to that line. Project all other (at most d) lines through p onto H. By induction there is a $(d-1)$-dimensional cone $\Gamma^{(d-1)}_\lambda$ in H' disjoint from all above projections. Then, the (d)-dimensional cone $\Gamma^{(d)}_\lambda$ is disjoint from K''. This proves the claim.

Now we state that the set K' does not belong to $\mathscr{X}_{n,m}$. This is easily seen for $\mathscr{X} = \mathscr{K}$. For $\mathscr{X} = \mathscr{C}$, $D_x(K' \setminus B_{1/n}(x))$ is a graph in S^{d-1} with arcs of great circles as edges. Even if z happens to be a vertex of this graph, the existence of a suitable ball $B_{\|z'-z\|/2}(z')$ follows from the above claim. For $\mathscr{X} = \mathscr{S}'$, $D_x(K' \setminus (B_{1/n}(x) \cup B^*_{1/n}(k_{K'})))$ is a finite union of pairwise disjoint arcs of great circles.

Now, it is clear that the parallel set $K' + B_\varepsilon(\mathbf{0})$ is not in $\mathscr{X}_{n,m}$ if ε is chosen small enough. Thus, any $K \in \mathscr{X}$ at Pompeiu-Hausdorff distance at most ε from K' is not in $\mathscr{X}_{n,m}$ and $\mathscr{X}_n = \cup_{m=1}^\infty \mathscr{X}_{n,m}$ is of first Baire category. But $\mathscr{X}_n$ is the set of all $K \in \mathscr{X}$ for which there is a ball $B_{1/n}(x)$ with $D_x(K \setminus (B_{1/n}(x) \cup B^*_{1/n}(k_K)))$ not $(\frac{1}{2})$-porous. It follows that the set $\cup_{n=1}^\infty \mathscr{X}_n$ of all $K \in \mathscr{X}$, for which there is a point $x \in K$ such that $D_x(K \setminus \{k_K\})$ is not a countable union of $(\frac{1}{2})$-porous sets, is of first category. Hence $D_x(K)$ is σ-porous for most $K \in \mathscr{X}$ and any $x \in K$. □

The case when $\mathscr{X} = \mathscr{S}$ and x lies in the kernel of K was treated by Gruber and Zamfirescu in [**5**].

Corollary. *For most sets $K \in \mathscr{X}$ and any point $x \in K$, $D_x(K)$ is of first Baire category and of measure zero on S^{d-1}.*

Indeed, a suitable version of Lebesgue's density theorem (see [**9**], p. 129) guarantees that σ-porosity implies measure zero.

This Corollary motivates further investigation of $D_x(K)$ in the generic case.

Theorem 1. *Most elements of $\mathscr{X}$ look at least half-dense from any of their points.*

Proof. For $K \in \mathscr{K}$ and $\alpha > 0$, let N_α denote the set of all points $x \in K$ such that, for suitable points $z_1, \ldots, z_{d+1} \in S^{d-1}$ at mutual distances at least α, $\mathbf{0} \in \mathrm{conv}\{z_1, \ldots, z_{d+1}\}$, and $x + C_\alpha(z_i)$ is disjoint from K for all i. That is,

$$N_\alpha = \{x \in K : \exists z_1, \ldots, z_{d+1} \in S^{d-1} \text{ s.t. } \mathbf{0} \in \mathrm{conv}\{z_1, \ldots, z_{d+1}\}$$
$$\text{and } \forall i, j \leq d+1, \|z_i - z_j\| \geq \alpha \text{ and } (x + C_\alpha(z_i)) \cap K = \varnothing\}.$$

We shall call *special* a point which belongs to N_α for some $\alpha > 0$.

We prove that most $K \in \mathscr{X}$ have no special points. First remark that the set $\mathscr{X}_{(\alpha)}$ of those $K \in \mathscr{X}$ for which $N_\alpha \neq \varnothing$ is closed in $\mathscr{X}$. It suffices to show that $\mathscr{X}_{(\alpha)}$ is nowhere dense, because $\cup_{n=1}^\infty \mathscr{X}_{(1/n)}$ is precisely the set of all $K \in \mathscr{X}$ which have special points. Since $\mathscr{X}_{(\alpha)}$ is closed, it is enough to see that $\mathscr{X} \setminus \mathscr{X}_{(\alpha)}$ is dense.

Indeed, each $K \in \mathscr{X}$ can be approximated by a parallel set, and this is not in $\mathscr{X}_{(\alpha)}$.

Now suppose $K \in \mathscr{X}$ does not look at least half-dense from some point $x \in K$. Then $\overline{D_x(N \cap K)}$ does not contain any hemisphere of S^{d-1} for some neighbourhood N of x. Hence

$$\mathbf{0} \in \mathrm{int\,conv}(S^{d-1} \setminus \overline{D_x(N \cap K)}),$$

otherwise a supporting hyperplane of $\overline{\mathrm{conv}(S^{d-1} \setminus \overline{D_x(N \cap K)})}$ through $\mathbf{0}$ would determine on S^{d-1} such a hemisphere. By Carathéodory's theorem, $\mathbf{0} \in \mathrm{conv} Z$ for some set Z of distinct points

$$z_1, \dots, z_{d+1} \in S^{d-1} \setminus \overline{D_x(N \cap K)}.$$

We have $B_{\alpha'}(x) \subset N$ for some $\alpha' > 0$. Choose $\alpha > 0$ smaller than α', smaller than $\|z_i - z_j\|$ for all distinct i, j, and also smaller than $\|z_i - v\|$ for any index i and point $v \in \overline{D_x(N \cap K)}$. Obviously, $x \in N_\alpha$. Hence most $K \in \mathscr{X}$ look at least half-dense from all points in K. $\square$

Corollary (Wieacker [**10**]). *For most compact sets $K \subset \boldsymbol{R}^d$,* conv$K$ *is smooth.*

Indeed, a nonsmooth point x in the boundary of conv$K$ would be special, i.e. K would not look at least half-dense from x.

4. Viewing from most points.

Here we shall prove the following result, already mentioned in the Introduction.

Theorem 2. *Most elements of $\mathscr{X}$ look dense from most of their points.*

Proof. Consider $K \in \mathscr{X}$ and let $M_\alpha(K)$ be the set of all points $y \in K$ such that

$$(y + C_\alpha(z)) \cap K = \varnothing$$

for some $z \in S^{d-1}$.

A standard compactness argument shows that $M_\alpha(K)$ is closed for each $\alpha > 0$.

All points y such that, for some $z \in S^{d-1}$ and $\alpha > 0$,

$$(y + C_\alpha(z)) \cap K = \varnothing$$

form a set

$$M(K) = \cup_{n=1}^{\infty} M_{n^{-1}}(K).$$

This set $M(K)$ is precisely the set of all $y \in K$ from which K does not look dense. We have to show that $M(K)$ is of second category only for elements K of a first category subset of $\mathscr{X}$.

A set $M_\alpha(K)$ is not nowhere dense if and only if it includes the intersection of K with a closed ball $B_\gamma(x)$ of centre $x \in K$ and radius $\gamma > 0$.

If $M(K)$ is of second category then $M_{n^{-1}}(K)$ is not nowhere dense for some $n \in \boldsymbol{N}$. Thus it must contain a set $B_\gamma(x) \cap K$ for some $x \in K$ and $\gamma > 0$. By taking m larger than both n and γ^{-1}, we get

$$M_{m^{-1}}(K) \supset B_{m^{-1}}(x) \cap K$$

for some $x \in K$.

So, since the set $\mathscr{X}'$ of those $K \in \mathscr{X}$ for which $M(K)$ is of second category is included in $\cup_{m=1}^{\infty} \mathscr{X}_m$, where $\mathscr{X}_m$ denotes the set of all $K \in \mathscr{X}$ for which

$$M_{m^{-1}}(K) \supset B_{m^{-1}}(x) \cap K$$

for some $x \in K$, it only remains to show that $\mathscr{X}_m$ is nowhere dense, for each m.

Again standard compactness arguments show that $\mathscr{X}_m$ is closed. To prove that $\mathscr{X} \setminus \mathscr{X}_m$ is dense in $\mathscr{X}$, take $K \in \mathscr{X}$ and $\varepsilon > 0$ arbitrarily. We claim that the parallel set K' of K at distance $\alpha > 0$ smaller than both ε and m^{-1} does not belong to $\mathscr{X}_m$.

Suppose, on the contrary,

$$M_{m^{-1}}(K) \supset B_{m^{-1}}(x) \cap K'$$

for some $x \in K'$. The set $B_{m^{-1}}(x) \cap K'$ must contain a point $y \in K$. But y cannot belong to $M_{m^{-1}}(K)$ because K' includes $B_\alpha(y)$.

Hence $K' \notin \mathscr{X}_m$, $\mathscr{X}_m$ is nowhere dense, and $\mathscr{X}'$ is of first category in $\mathscr{X}$. □

5. Viewing from other points.

THEOREM 3. *Most elements $K \in \mathscr{X}$ look half-dense from points forming a dense subset of K.*

PROOF. First recall that, by Theorem 1, most $K \in \mathscr{K}$ look at least half-dense from each point of K. Moreover, it is known that most $K \in \mathscr{X}$ are nowhere dense; see, for example, [**11**] for $\mathscr{X} = \mathscr{K}$, [**3**] for $\mathscr{X} = \mathscr{C}$, and [**13**] for $\mathscr{X} = \mathscr{S}$.

So, if $y \in K$ and $\gamma > 0$, there is a ball $B_\delta(z) \subset B_\gamma(y)$ disjoint from K. Let

$$\varepsilon = \sup\{\delta : B_\delta(z) \cap K = \varnothing\}.$$

Clearly, $B_\varepsilon(z) \cap K$ contains some point x. Let

$$H^+ = \{u \in \boldsymbol{R}^d : \langle u, x - z\rangle \geq 0\}.$$

Obviously,

$$\cap_{N \ni x} D_x(N \cap K) \subset \cap_{N \ni x} D_x(N \setminus \operatorname{int} B_\varepsilon(z)) \subset H^+.$$

It follows that K looks half-dense from x.

Hence K looks half-dense from points forming a dense subset of K. □

Theorem 3 can be strengthened in case $\mathscr{X} = \mathscr{K}$.

THEOREM 4. *Most elements $K \in \mathscr{K}$ look precisely half-dense from points forming a dense subset of K.*

PROOF. Recall again that, by Theorem 1, most $K \in \mathscr{K}$ look at least half-dense from each point in K. Consider some sphere in $\boldsymbol{R}^d$. It is easily seen that those $K \in \mathscr{K}$ which meet the sphere form a nowhere dense family. Hence most $K \in \mathscr{K}$ do not meet any sphere of rational centre (i.e. with rational coordinates) and of rational radius. So, for most $K \in \mathscr{K}$, if $y' \in K$ and N is a neighbourhood of y' in $\boldsymbol{R}^d$, there is a ball $B_\gamma(y) \subset N$ with rational γ and y, and with y' in its interior, such that

$$K \cap B_\gamma(y) \subset \mathrm{int} B_\gamma(y).$$

Let x be an extreme point of $K^* = \mathrm{conv}(K \cap B_\gamma(y))$, H a supporting hyperplane of K^* at x, and H^+ the closed half-space determined by H which contains K^*. If, for some point $x' \in H^+ \setminus \{x\}$ and some $\alpha > 0$,

$$x + C_\alpha\left(\frac{x' - x}{\|x' - x\|}\right) \cap K = \varnothing,$$

then K does not look at least half-dense from x (i.e. x is a special point), and a contradiction is obtained.

On the other hand, there is a neighbourhood N_0 of x included in $\mathrm{int} B_\gamma(y)$. For this neighbourhood, $D_x(N_0 \cap K) \subset H^+$. Hence K looks precisely half-dense from x. The proof is finished. □

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