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On the Perimeter of a Triangle in a Minkowski Plane

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A *Minkowski plane* is a plane endowed with a norm that is not necessarily Euclidean. The Minkowski length of a curve in a Minkowski plane is defined in a manner analogous to the way it is defined in the Euclidean case. A circle of radius r in a Minkowski plane with norm $\|\cdot\|$ is the locus of points X satisfying $\|X - P\| = r$ for a fixed "center" P. It is a convex curve that is symmetric with respect to its center. Golab's theorem shows that the Minkowski length of a circle of unit radius lies between 6 and 8 (see [**2**]). Let L denote the Minkowski length. We establish the following:

Theorem 1. *Let ABC be a triangle in a Minkowski plane, and suppose that one is the minimum radius of a circle that encloses the triangle ABC. Then* $4 \leq L(ABC) \leq 6$.

Since there is an arbitrarily thin triangle with two sides close to a diameter of the circle, the lower bound 4 cannot be improved in *any* Minkowski plane. For the case of the Euclidean norm it is not difficult to see that the upper bound 6 can be reduced to $3\sqrt{3}$ [**1**]. However, in the max-norm $\|(x, y)\|_\infty = \max\{|x|, |y|\}$ any three corners of the unit circle (which is a Euclidean square) form a triangle with perimeter 6. Hence the upper bound of the theorem cannot be improved in general.

Proof. Let Γ be a circle circumscribed about ABC in the sense that it has smallest radius among all circles that contain A, B, and C in their convex hulls. It is easily seen that either the center of the circle is inside the triangle or on one of its sides, or there exists a congruent circumscribed circle with this property. So we may assume that the center of Γ is the origin O and lies inside ABC or on one of its sides. The circle Γ is supposed to have radius 1. Since $\|A - B\| \leq 2$, $\|B - C\| \leq 2$, and $\|C - A\| \leq 2$, we have $L(ABC) \leq 6$.

We have to show that $L(ABC) \geq 4$. If $\|A - B\| = 2$, then

$$\|A - B\| + \|B - C\| + \|C - A\| \geq \|A - B\| + \|B - A\| = 4.$$

Hence we consider the case that $\|A - B\|$, $\|B - C\|$, and $\|C - A\|$ are all less than 2. Notice that in this case O lies inside ABC, while A, B, and C all lie on Γ.

Let D, E, F, G, H, and I be points on the sides of ABC such that $ADOI$, $BFOE$, and $CHOG$ are parallelograms (see Figure 1). Then

$$\|A - D\| + \|A - I\| = \|A - D\| + \|D\| \geq \|A\| = 1,$$

$$\|B - E\| + \|B - F\| = \|B - E\| + \|E\| \geq \|B\| = 1,$$

$$\|C - G\| + \|C - H\| = \|C - G\| + \|G\| \geq \|C\| = 1.$$

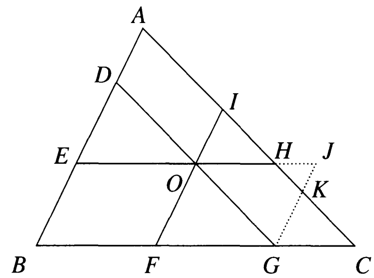

Figure 1.

To prove that $L(ABC) \geq 4$, it is enough to show that

$$\|D - E\| + \|F - G\| + \|H - I\| > 1.$$

Among the similar triangles DEO, OFG, and IOH, we may suppose that DEO is the biggest and IOH the smallest. Then $\|D - E\| \geq \|F\| \geq \|I\|$. Let J be the point such that $OFGJ$ is a parallelogram, and let K be the intersection of GJ and AC. Then the triangle KGC is a translate of IOH, and

$$\begin{aligned}
&\|D - E\| \geq \|F\| = \|J - G\| > \|J - K\|,\\
&\|F - G\| = \|J\|,\\
&\|H - I\| = \|C - K\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|D - E\| + \|F - G\| + \|H - I\| &> \|J - K\| + \|J\| + \|C - K\|\\
&\geq \|C\| = 1.
\end{aligned}$$

■

Remark. The lower bound in the theorem is valid for any closed curve instead of a triangle. More precisely, let Ω be a simple closed curve in a Minkowski plane, and let one be the smallest radius of a circle that encloses Ω. Then $L(\Omega) \geq 4$.

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