Symmetry and the farthest point mapping on convex surfaces

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Abstract. Consider the mapping \( F \) associating to each point \( x \) of a convex surface the set of all points at maximal intrinsic distance from \( x \). We provide a large class of surfaces on which \( F \) is single-valued and involutive. Moreover, we show that there are point-symmetric surfaces of revolution with \( F \) single-valued but not involutive.

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Introduction

Let \( \mathcal{S} \) be the space of all closed convex surfaces (i.e. boundaries of open bounded convex sets) in the 3-dimensional Euclidean space, endowed with the usual Pompeiu–Hausdorff metric.

Denote by \( F_x \) the set of farthest points from \( x \) (absolute maxima of the intrinsic distance from \( x \)) and by \( F \) the farthest point mapping, i.e., the multivalued mapping associating to \( x \in S \) the set \( F_x \).

Our results are related to a conjecture of Steinhaus saying that if on a convex surface \( S \) the mapping \( F \) is single-valued and \( F \circ F = \text{id}_S \) (\( F \) is an involution) then the surface is a sphere ([3], p. 44). This conjecture was disproved by the first author [10]. He constructed a large family \( \mathcal{R} \) of convex surfaces with both axial and central symmetry, on which \( F \) is single-valued and involutive (with \( F_x = \{-x\} \)). Then the following question naturally arose [10]. Is it true, for convex surfaces on which \( F \) is single-valued, that \( F \) is involutive? Or is this at least true for point-symmetric convex surfaces of revolution?

On the (point-symmetric) boundary \( K \) of a 1-times-1-times-2 box (recall the Knuth–Kotani puzzle), the mapping \( F \), even restricted to the vertex set, is not single-valued. In this paper we see that suitable bounds on the curvature of \( S \), or on curvature and radius, guarantee the farthest point mapping to be a homeomorphism. Nevertheless, the answer to the preceding questions will be shown to be negative. We also extend the family \( \mathcal{R} \) given in [10] to the family \( \mathcal{I} \) of all convex surfaces of revolution.

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constant intrinsic radius, and we characterize the centrally symmetric surfaces which verify Steinhaus’ conditions.

In the last years, several questions about farthest points proposed by H. Steinhaus (see the chapter A35 of the book [3] by H. T. Croft, K. J. Falconer and R. K. Guy) have been answered by the second author (see [12], [14], [15]).

It is now known, for example, that on any convex surface \( S \) for almost all points \( x \in S \) (in the sense of measure), \( F_x \) contains a single point [15]. J. Rouyer [7] showed that similar results hold true (in the framework of Riemannian geometry) for surfaces which are homeomorphic to the 2-sphere, but not necessarily convex.

1 Definitions and notation

For \( S \in \mathcal{S} \) and \( x, y \in S \), \( p(x, y) \) will be the geodesic (intrinsic) distance between \( x \) and \( y \), and \( \rho_x \) the distance function from \( x \), \( \rho_x(y) = p(x, y) \).

Also, let \( C_x \) be the set of all points in \( S \) joined to \( x \) by at least two segments, i.e. shortest paths, \( M_x \) be the set of all relative maxima for \( \rho_x \), and \( C(x) \) be the cut locus of \( x \), i.e., the set of all points \( y \in S \) such that some segment from \( x \) to \( y \) is not extendable as a segment beyond \( y \). Of course, \( C_x \subset C(x) \) and \( M_x \subset C(x) \).

The 1-dimensional Hausdorff measure (length) of the set \( A \subset S \) is denoted by \( \lambda A \), while card \( A \) denotes its cardinality.

For \( x \in S \), let \( T_x \) denote the space of tangent directions at \( x \); it can be regarded as a closed curve, the intersection of the tangent cone at \( x \) with the 2-dimensional unit sphere. Thus, \( \lambda T_x \leq 2\pi \).

It is well-known that the mapping \( F \) is upper semicontinuous. We call \( F \) injective if \( F_x \cap F_y = \emptyset \) for any pair of distinct points \( x, y \in S \). Also, we call \( F \) surjective if for every point \( y \in S \) there is some point \( x \in S \) with \( y \in F_x \). When we say that \( F \) is bijective or a homeomorphism, we implicitly state that \( F \) is single-valued.

The union of two segments from \( x \) to some point \( y \in S \), which make an angle equal to \( \pi \) at \( y \), will be called a loop at \( x \).

If \( \sigma_1, \sigma_2 \) are two segments with precisely one common endpoint \( a \), then \( \angle \sigma_1 \sigma_2 \) denotes the angle between the tangent directions of \( \sigma_1 \) and \( \sigma_2 \) at \( a \). For \( a \neq b \), \( ab \) means the segment from \( a \) to \( b \) when that segment is unique or clearly identifiable from the context. \( \angle xyz \) means \( \angle(xyz) \).

A geodesic triangle in a Riemannian manifold or a convex surface is a collection of three segments \( \gamma_1, \gamma_2, \gamma_3 \) such that \( \gamma_1, \gamma_2, \gamma_3 \) have a common endpoint \( a_{i+2} \) (the indices are taken modulo 3). We shall denote the triangle by \( \langle \gamma_1, \gamma_2, \gamma_3 \rangle \) or \( a_1a_2a_3 \).

Let \( K \) denote the sectional curvature of a given Riemannian manifold, and \( M_H \) the simply connected 2-dimensional space of constant curvature \( H \).

Put \( \rho_x(A) = \inf_{y \in A} \rho_x(y) \) for \( A \subset S \). For a point \( x \in S \), the intrinsic radius at \( x \) is \( \rho_x(F_x) \). The radius of \( S \) is defined by \( \text{rad} S = \inf_{x \in S} \rho_x(F_x) \), its diameter is defined by \( \text{diam} S = \sup_{x \in S} \rho_x(F_x) \) and its injectivity radius by \( \text{inj} S = \inf_{x \in S} \rho_x(C_x) \).

We denote by \( D(x, \varepsilon) \) the open disc around \( x \) of geodesic radius \( \varepsilon \).

Two segments \( \sigma_1, \sigma_2 \) with a common endpoint bifurcate if \( \sigma_1 \cap \sigma_2 \) includes a non-degenerate arc but none of the two segments.

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\text{WDG (170 x 240mm) MMaths J-1445 Adv. in Geom. 6-2 PMU:ICvNIL10/1/2006 pp. 345–353 1445_6-2_11 (p. 346)} \]
2 Auxiliary results

The set $S_2$ of all convex surfaces possessing some point $x \in S$ with disconnected $M_x$ is obviously of second Baire category in $\mathcal{S}$. It was introduced by the second author in [15], where he showed that, in the sense of Baire category, on most $S \in S_2$ there exist a point $x$ and a Jordan arc in $C(x)$ containing infinitely many points of $M_x$.

Other properties of $F$ were established in [11], where the following two lemmas are proved.

**Lemma 1.** The mapping $F$ is injective on any surface $S \in \mathcal{S}$ without conical points, in particular on any surface of class $C^1$.

**Lemma 2.** Let $S \in \mathcal{S}$. If $F$ is continuous then it is surjective. If $F$ is surjective then $F_x$ is connected for each $x \in S$.

The following two results were established in [12].

**Lemma 3.** For any surface $S \in \mathcal{S}$ and any point $x \in S$, each component of $F_x$ is a point or an arc.

**Lemma 4.** If $S \in \mathcal{S}$, $x \in S$ and $y \in M_x$, then each arc in $T_y$ of length $\pi$ contains the tangent direction of a segment from $y$ to $x$. Thus, if $\lambda T_y > \pi$, then there are at least two segments from $x$ to $y$, and if $S$ is differentiable at $y$ and there are only two segments from $x$ to $y$ then these have opposite tangent directions at $y$.

We shall make use of the next lemma, a proof of which can be found in [15] (see also [13]).

**Lemma 5.** Let $S \in \mathcal{S}$, $x \in S$ and $y, z \in C_x$. Let $J$ be the arc joining $y$ to $z$ in $C_x$. If $u \in J \setminus \{y, z\}$ is a relative minimum of $\rho_x|_J$, then $u$ is the midpoint of a loop $\Lambda$ at $x$ and, except for the two subarcs of $\Lambda$ from $x$ to $u$, no segment connects $x$ to $u$.

Moreover, we shall need the following classical relation of Clairaut (see, for example, [4] p. 257).

**Lemma 6.** Let $S$ be a surface of revolution. For a variable point $x$ on a geodesic $\gamma$ of $S$, denote by $r_x$ the distance from $x$ to the axis of revolution, and by $\theta_x$ the angle made at $x$ by $\gamma$ with the meridian through $x$. Then $r_x \sin \theta_x$ is constant as $x$ varies on $\gamma$.

We shall also use the following results.

**Lemma 7** ([10]). For an arbitrary point $x$ on a closed convex surface $S$ centrally symmetric about the origin, $F_x = \{-x\}$ if and only if $F_y = \{x\}$ for all $y \in F_x$.

**Lemma 8** ([1]). On any convex surface, segments do not bifurcate.
Lemma 9 ([1]). If \( ab, bc \) are segments on \( S \in \mathcal{S} \), and \( x \in ab \) converges to \( b \) while \( xc \) converges to \( bc \), then \( \angle axc \) converges to \( \angle abc \).

The following comparison theorem can be found, for example, in [1].

Lemma 10. Let \( S \) be a convex surface, \((\gamma_1, \gamma_2, \gamma_3) \subset S \) a geodesic triangle and \((\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3) \) a planar triangle with \( \lambda_{\gamma_i} = \lambda_{\overline{\gamma}_i} \). Then

\[
\angle \gamma_i \gamma_{i+1} \leq \angle \overline{\gamma}_i \overline{\gamma}_{i+1} \quad (i = 1, 2, 3 \text{ mod } 3).
\]

Toponogov’s well-known comparison theorem, reproduced here as Lemma 11, can be found, for example, in [2].

Lemma 11. Let \( M \) be a complete manifold with \( K \geq H \), and \((\gamma_1, \gamma_2, \gamma_3) \) a geodesic triangle in \( M \). If \( H > 0 \), suppose \( \lambda_{\gamma_1} \leq \pi/\sqrt{H} \) for all \( i \). Then there exists in \( M_H \) a geodesic triangle \((\gamma_1, \gamma_2, \gamma_3) \) such that \( \lambda_{\gamma_i} = \lambda_{\gamma_1} \) and the corresponding angles satisfy \( \angle \gamma_i \gamma_{i+1} \leq \angle \overline{\gamma}_i \overline{\gamma}_{i+1} \) \( (i = 1, 2, 3) \).

If the inequality \( K \leq H \) is assumed, then the inequalities \( \angle \gamma_i \gamma_{i+1} \geq \angle \overline{\gamma}_i \overline{\gamma}_{i+1} \) hold \( (i = 1, 2, 3) \).

We shall also use Pogorelov’s rigidity theorem ([6] p. 167):

Lemma 12. Any two isometric convex surfaces are congruent.

The following is folklore.

Lemma 13. The orthogonal projection of any arc outside a compact convex surface onto the surface is not longer than the arc itself.

3 Surfaces with \( F \) a homeomorphism, but not an involution

The classical Sphere Theorem (see [2], [9], or [5]) gives sufficient conditions for a complete, simply connected manifold \( M \) with Gauß curvature \( K \) to be homeomorphic to the unit sphere: \( 1/4 < K < 1 \). In fact, this 1/4-pinching of the curvature proves sufficient for \( F \) to be a homeomorphism.

Theorem 1. Let \( S \) be a convex surface of class \( C^2 \) with Gauß curvature \( K \geq 1 \). If \( \text{rad} \, S > \pi/2 \) then \( F \) is a homeomorphism.

Proof. The injectivity of \( F \) follows from Lemma 1. The single-valuedness of \( F \) and its surjectivity can in fact be found inside the proof of Theorem 3 in [5]. For the readers convenience, we give here a short direct proof.

Suppose there exists \( x \in S \) with \( \text{card} \, F_x \geq 2 \). Then choose two points \( y, z \in F_x \) an arc \( J \) joining \( y \) to \( z \) in \( C \), and a relative minimum \( u \in J \setminus \{ y, z \} \) of \( p_x \mid J \). By Lemma 5, \( u \) is the midpoint of a loop \( \Lambda \) at \( x \), so both subarcs of \( \Lambda \) from \( x \) to \( u \) are segments and make the angle \( \pi \) at \( u \).
Let $uy$ be a segment from $u$ to $y$. Then one of the two angles determined by $\Lambda$ and $uy$ is at most $\pi/2$. Consider the triangle $xuy \subset S$ containing that angle, and a triangle $\bar{x}\bar{y}\bar{u} \subset M_1$ isometric to $xuy$.

Comparing the triangles $xuy$ and $x\bar{u}y$, by Lemma 11, we have $\angle x\bar{u}y \leq \pi/2$. Since $\lambda\bar{y} = \rho(x, y) \geq \text{rad } S > \pi/2$, $\bar{y}$ lies in the open half-sphere of $M_1$ opposite to $\bar{x}$. Then, the inequality $\lambda\bar{u}x \leq \lambda\bar{u}y$ implies $\angle x\bar{u}y > \pi/2$, and a contradiction is obtained. Thus, $F$ is single-valued on $S$ and therefore continuous. By Lemma 2, $F$ is also surjective.

Since $S$ is compact and $F$ is bijective and continuous, its inverse is also continuous.

**Remark 1.** After rescaling, Theorem 1 says that, if $S \in \mathscr{S}$, $K \geq k_0 > 0$ and $\text{rad } S > \pi/(2\sqrt{k_0})$, then $F$ is a homeomorphism.

**Corollary.** If $1/4 \leq K < 1$ then $F$ is a homeomorphism.

**Proof.** By Klingenberg’s inequality in [2], p. 98, $K < 1$ implies $\text{inj } S > \pi$, whence $\text{rad } S > \pi$. Thus, by Remark 1 with $k_0 = 1/4$, $K \geq 1/4$ implies that $F$ is a homeomorphism.

**Remark 2.** The corollary says that, if $S \in \mathscr{S}_2$ has everywhere positive Gauß curvature and $K_{\text{min}}, K_{\text{max}}$ denote its minimum and maximum respectively, then $K_{\text{max}} \geq 4K_{\text{min}}$. The converse is, however, false, as we can easily see on some surfaces in the class $\mathfrak{H}$ defined in [10].

We treat now a special case, which will give the answer to the problem mentioned in the Introduction. We consider surfaces $S$ of class $C^2$ which are of revolution about the axis $\Omega = oZ$ and symmetric with respect to the origin $o$ of the space. The equator $Q$ is the intersection of $S$ with the plane through $o$ orthogonal to $\Omega$. A meridian is the intersection of $S$ with a plane including $\Omega$.

Denote by $\mathcal{N}$ the set of all surfaces $S$ satisfying the following conditions.

i) The curvature $K$ is pinched, $1/4 \leq K < 1$, with the minimum $1/4$ attained on $Q$.

ii) The radius of $Q$ is less than $\sqrt{3}$.

iii) The surface $S$ surrounds the sphere $\Sigma$ of equator $Q$ and $S \cap \Sigma = Q$.

We observe that $\mathcal{N}$ contains the ellipsoids of revolution with semiaxes $\sqrt{2} < a = b < c \leq 2$.

**Theorem 2.** On any surface in $\mathcal{N}$, the mapping $F$ is a homeomorphism but not an involution.

**Proof.** Since the Gauß curvature on $S$ verifies $1/4 \leq K < 1$, $F$ is a homeomorphism of $S$, by the previous corollary. It remains to prove that $F \circ F \neq \text{id}_S$.

Clearly, by symmetry, $x$ and $F_x$ are on the same meridian, for any point $x \in S$. Let $y \in Q$, and let $x \in S \setminus \{y\}$ lie on the same meridian. If $x$ is close enough to $y$, then
there is a unique segment $xy$ joining $x$ to $y$. We show that, for $\rho(x, y)$ small enough, $F_x \neq \{-x\}$. This will imply, by Lemma 7, that $F$ is not an involution.

First remark that, by Lemma 13, any meridian is longer than the equator (in case of equality, $S = \Sigma$), and $Q$ is a geodesic of $S$.

Now, assume $F_x = \{-x\}$. Clearly, any segment $\sigma$ from $x$ to $-x$ intersects $Q$. The existence of two such intersection points, would contradict Lemma 8. Thus, $\sigma \cap Q$ is a single point. No segment $\sigma$ (from $x$ to $-x$) can lie on a meridian $M$, for $x$ close enough to $y$, because $M$ is longer than $Q$. By Lemma 4, some segment $\sigma$ from $x$ to $-x$ makes an angle $\alpha \geq \pi/2$ with $xy$. Let $z$ be the equatorial point of $\sigma$.

Case $\alpha = \pi/2$: In this case, by Lemma 6, $\sigma$ is horizontal at $-x$, and the symmetry of the whole geodesic $\Gamma \ni \sigma$ with respect to the line $oz$ implies that $z$ is the midpoint of $\sigma$. Since the curvature of $S$ along $Q$ is $K = 1/4$, in a whole neighbourhood $N$ of $Q$, $K \leq 1/3$. Because $\sigma$ converges to a half-equator if $x$ tends to $y$, $\sigma$ lies in $N$ if $xy$ is short enough.

Therefore, on the sphere $M_{1/3}$ of radius $\sqrt{3}$, the triangle $xyz$ isometric to $xyz$ has its angles at $x$ and $y$ not smaller than $\pi/2$, by Lemma 11. On $M_{1/3}$ this implies, $x$, $y$ being close, that $\rho(y, z)$ is at least a quarter of the circumference $2\pi \sqrt{3}$, i.e., $\rho(y, z) \geq \pi \sqrt{3}/2$, in contradicition with $\rho(y, z) = \pi a/2 < \pi \sqrt{3}/2$, where $a$ is the radius of $\Sigma$.

Case $\alpha > \pi/2$: Now $\sigma$ has some point $x'$ $\neq x$ as a farthest point from $Q$. Let $y'$ be the equatorial point closest to $x'$. By Lemma 6 and by the symmetry of $S$ and of $\Gamma$, there is a unique point $x'' \in \sigma$ between $x$ and $z$ at distance $\rho(x, y)$ from $Q$, and the point $-x$ is either symmetric to $x$ or to $x''$ with respect to the line $oz$. In the first case $\rho(z, y') < \pi a/2$. In the second, $\Gamma$ reaches $-x'$ beyond $-x$, $\rho(-x, -x') = \rho(x'', x') = \rho(x, x')$, and $\rho(z, y') = \pi a/2$.

We obtain the geodesic triangle $x'y'z$, to which we apply the argument from the previous case, and obtain a contradiction.

\[ \square \]

## 4 Surfaces with involutive $F$

Denote by $\mathcal{H}$ the subset of $\mathcal{I}$ of all convex surfaces which satisfy the conditions in Steinhaus’ conjecture mentioned in the Introduction: $F$ is single-valued and an involution.

In [10] the first author constructed a class $\mathcal{R} \subset \mathcal{H}$ of convex surfaces larger than that of all 2-spheres, thus disproving the mentioned conjecture, and asked for a characterization of $\mathcal{H}$.

Here we find an even larger family of examples of surfaces in $\mathcal{H}$, namely the family $\mathcal{I}$ of all surfaces $S \in \mathcal{I}$ of constant intrinsic radius, i.e.,

$$\mathcal{I} = \{S \in \mathcal{I} : \text{rad } S = \text{diam } S\}.$$ 

The relationship between $\mathcal{R}$, $\mathcal{I}$ and $\mathcal{H}$ is not obvious. It is described by the next theorem.

**Theorem 3.** We have $\mathcal{R} \subset \mathcal{I} \subset \mathcal{H}$, where the first inclusion is strict.
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Proof. We show that $\mathcal{F} \subset \mathcal{H}$.

Let $S \in \mathcal{F}$. Notice that $\rho_1(F_x) = \text{diam } S$ for all $x \in S$. Therefore, for any points $x \in S$, $y \in F_x$, we have $x \in F_{y}$, which implies that $F$ is surjective. By Lemma 2, for every $x \in S$ the set $F_x$ is connected, and by Lemma 3 it must be an arc or a point. Clearly, $F$ is involutive if it is single-valued.

We claim that $F_x$ actually reduces to a point for all $x \in S$. Suppose there is a point $x \in S$ with $F_x$ a nondegenerate arc. By Lemma 1, $x$ is a conical point.

Let $z_1$, $z_2$ be the endpoints of $F_x$. Denote by $\Delta \subset S$ the maximal open connected set not meeting any segment from $x$ to $z_1$ or $z_2$, but meeting $F_x$, and by $S_1$, $S_2$ the two components of $S \setminus \Delta$ with $z_i \in \text{bd } S_i$ ($i = 1, 2$). Let $\alpha$, $\beta$ be the two angles of $\Delta$ at $x$, and $\gamma_i$ the angle of $S_i$ at $x$ ($i = 1, 2$). Since $\Delta \cup S_i$ meets no segment from $x$ to $z_i$, by Lemma 4, $\gamma_i + \alpha + \beta < \pi$.

For any $v \in F_x \setminus \{z_1, z_2\}$, there is a loop $\Lambda_v$ at $x$ through $v$, by Lemma 5. Since limits of segments are also segments, the same is true for $v \in \{z_1, z_2\}$. So, for any $v \in F_x$, denote by $\sigma_v$, $\sigma_v'$ the two segments from $x$ to $v$ forming $\Lambda_v$, and by $\delta_i(v)$ the angle of $\sigma_v$, $\sigma_v'$ at $x$ toward $S_i$. Since $\delta_1(z_1) = \gamma_1$ and $\delta_1(z_2) = \gamma_1 + \alpha + \beta$, there is some point $z \in F_x$ such that

$$\delta_i(z) = \gamma_i + v,$$

where $v_1 > 0$, $v_1 + v_2 = \alpha + \beta$, and $\gamma_i + v_i \neq \pi/2$ ($i = 1, 2$). Let $\varphi < \pi/2$ satisfy $\varphi > \delta_i(z)$ if $\delta_i(z) < \pi/2$ and

$$\varphi > \max\{\delta_i(z)/2, \pi - \delta_i(z)\}$$

if $\delta_i(z) > \pi/2$.

Let now $\varepsilon > 0$ be such that the planar triangle with one side of length $\varepsilon$, another side of length rad $S/2$ and the angle between them $\varphi$, is obtuse.

By the semicontinuity of $F$, if $y \in \sigma_z$ is close enough to $z$, then $F_y \subset D(x, \varepsilon)$. Let $D_1$, $D_2$ be the two components of $D(x, \varepsilon) \setminus \Lambda_z$, and $E_1$, $E_2$ the two components of $S \setminus \Lambda_z$ satisfying $D_i \subset E_i$ ($i = 1, 2$). Let $y^* \in \sigma_z$ have maximal distance to $y$ in $E_1$. This maximal distance is less than diam $S$. Indeed, let $y^*$ be the point of $\sigma_z'$ at distance $\rho(y, z)$ from $x$. Clearly, $yx \cup y^*x$ is not a segment, $x$ being a conical point, so $\rho(y, y^*) < \text{diam } S$. Also, any other point of $\Lambda_z$ is at distance less than diam $S$ from $y$, whence $F_y \cap \Lambda_z = \emptyset$.

It is easily seen that there is at least one segment $\mu_1$ from $y$ to $y^*$ such that $\angle \mu_1 y^* x \leq \pi/2$, and at least one segment $\mu_2$ from $y$ to $y^*$ such that $\angle \mu_2 y^* x \geq \pi/2$. Indeed, suppose there is no segment of at least one of the two kinds, for instance of the first kind. Then, for $w \in y^*x$ converging to $y^*$, any segment from $y$ to $w$ necessarily converges to $yy^*$. So, by Lemma 9,

$$\angle ywy^* \to \pi - \angle \mu_1 y^* x < \pi/2.$$

Comparing with the planar triangle whose vertex set is isometric to $\{y, w, y^*\}$, we get by Lemma 10 the inequality $\rho(y, w) > \rho(y, y^*)$, which is false.
Now let \( y \to z \). Consequently, \( y' \to x \). Since there is no segment from \( x \) to \( z \) besides \( \sigma_z, \sigma_z' \), we have \( \mu_1 \to \sigma_z \) and \( \mu_2 \to \sigma_z' \). Suppose no segment from \( y \) to \( y' \) is included in \( \Lambda_z \). Then, still, for \( y \) close enough to \( z \), \( \mu_2 \) meets \( F_s \), say at \( z' \). Then

\[
2\rho(x, z') \leqslant \rho(x, y) + \lambda \mu_2 + \rho(y', x) < \lambda \Lambda_z = 2 \text{rad} S
\]

which is false. Hence, \( \mu_2 = yz \cup zy' \).

There are two possibilities for \( F_y \) to meet \( D_1 \). Let \( u \in F_y \cap D_1 \).

Case 1. \( u \) lies in the digon of sides \( \mu_1, \mu_2 \).

Case 2. \( u \) lies in the triangle \( (\mu_1, y'x, xy) \).

In the following discussion we assume for both cases that \( \delta_1(z) > \pi/2 \). (The contrary situation can be treated analogously and is simpler.)

In Case 1, some triangle \( yy'u \) has its angle at \( y' \) not larger than \( \angle zy' \mu_1 / 2 \) and therefore less than \( \phi \) for \( y \) close enough to \( z \), because \( \angle zy' \mu_1 \to \delta_1(z) \) by Lemma 9. Looking at the Euclidean triangle with vertex set isometric to \( \{y, y', u\} \), we get \( \rho(y, u) \leqslant \rho(y, y') \) by Lemma 10; this contradicts \( u \in F_y \).

In Case 2, the triangle \( yy'x \) either has its angle at \( y' \) less than \( \phi \), or its angle at \( x \) less than \( \phi \), for \( y \) close enough to \( z \). In the first situation, the triangle \( yy'u \) has its angle at \( y' \) less than \( \phi \); hence, as before, \( \rho(y, u) < \rho(y, y') \), and a contradiction is obtained. In the second situation, the triangle \( yux \) has its angle at \( x \) less than \( \phi \), whence \( \rho(y, u) < \rho(y, x) \), again a contradiction.

Thus \( F_y \) does not meet \( D_1 \). Analogously, it does not meet \( D_2 \). Since we showed that \( F_y \) does not meet \( \Lambda \); either, we found that \( F_y \cap D(x, \varepsilon) = \emptyset \) and a final contradiction is obtained. The proof is almost finished.

From Theorem 9 and Remark 10 of [10], it follows that \( \mathcal{R} \subset \mathcal{I} \). J. Rouyer [8] showed that this inclusion is strict. He proved that the boundary of a half-ball belongs to \( \mathcal{I} \setminus \mathcal{R} \) \( \square \)

For a given convex surface \( S \), endow the space \( \mathcal{P}(S) \) of all compact subsets of \( S \) with the induced Pompeiu–Hausdorff metric \( \mathcal{H}_p \).

**Theorem 4.** The surface \( S \in \mathcal{I} \) is a centrally symmetric surface in \( \mathcal{H} \) if and only if the associated mapping \( F \) is an isometry.

**Proof.** Let \( S \in \mathcal{H} \) be centrally symmetric about the origin. By Lemma 7, \( F_x = \{-x\} \) for all \( x \in S \), hence \( F \) is the restriction to \( S \) of the symmetry with respect to the origin, and therefore an isometry of \( S \).

Conversely, if there is a point \( x \in S \) with \( \text{card } F_x > 1 \), then we can find a sequence \( x_n \) tending to \( x \) such that card \( F_{x_n} = 1 \), and \( F_{x_n} \) converges to a point \( z \in F_x \) (see Theorem 5 in [14]). In this case, we have \( \rho(x_n, x) \to 0 \) and \( \mathcal{H}_p(F_{x_n}, F_x) \to \mathcal{H}_p(\{z\}, F_x) > 0 \). Thus, \( F \) is an isometry between the metric spaces \( (S, \rho) \) and \( (\mathcal{P}(S), \mathcal{H}_p) \) if and only if it is single-valued and an isometry of \( (S, \rho) \). So, let \( F \) be such a mapping.

From \( \rho(x, F_x) = \rho(F_x, F_{F_x}) \), it follows that \( F_{F_x} = x \), hence \( S \in \mathcal{H} \). By Pogorelov’s rigidity theorem (Lemma 12), the isometric convex surfaces \( S \) and \( F(S) \) are congru-
ent via an extension $f$ of the isometry $F$ to the whole space. Since $f$ leaves $S$ invariant and has no fixed points on $S$, it must be the symmetry with respect to the midpoint of some line-segment joining a point $x$ to its (unique) farthest point.

 References


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