On the critical points of a Riemannian surface

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Abstract. We show that all critical points with respect to a point on a Riemannian surface lie on a subset of the cut locus which is locally a tree and has relatively few endpoints. Moreover, we offer some inequalities involving the length of the set of critical points.

Introduction

Let $S$ be a compact Riemannian (2-dimensional) surface without boundary. For an arbitrary point $x \in S$ we consider the Riemannian distance $\rho_x(y)$ from $x$ to $y \in S$ and the cut locus $C(x)$, defined as the set of all $y \in S$ such that no segment, i.e., shortest path, from $x$ to $y$ can be extended as a segment beyond $y$.

The cut locus was introduced by H. Poincaré, [11]. Further basic properties of the cut locus have been investigated by S. B. Myers [8], [9] and J. H. C. Whitehead [12], and later by many other authors. Among other things, it is well-known that $C(x)$ is connected and locally a tree. For an introduction to the cut locus see, for example [7].

For any set which is locally a tree, a point in the set is called endpoint if its deletion does not disconnect any connected neighbourhood (tree, if small) of the point. Let $E(x)$ denote the set of all endpoints of $C(x)$. A point $y \in S$ is called regular with respect to $x$ (and $\rho_x$) if some open halfplane of $T_yS$ contains the tangent vectors at $y$ of all segments from $y$ to $x$. A point $y \in S$ is called critical with respect to $x$ (and $\rho_x$) if it is not regular, i.e., if for any tangent vector $\tau$ at $y$ there exists a segment from $y$ to $x$ with direction $\sigma$ at $y$ such that $\langle \tau, \sigma \rangle \geq 0$ (see, for instance, [2], p. 2). For example, every relative maximum of $\rho_x$ and every relative minimum of $\rho_x|_{C(x)\setminus E(x)}$ is a critical point. Let $Q(x)$ be the set of all critical points with respect to $x$. All these points lie on $C(x)$.

We may encounter uncountably many critical points on one hand (to see this, take the example in [13], p. 320, and modify it appropriately in order to obtain a Riemannian surface) and, on the other, $C(x)$ may be quite large, for example non-triangulable (see Gluck and Singer [3]).

We point out in this paper that in fact $Q(x)$ cannot be too scattered in $C(x)$; more precisely, it must belong to a single handsome tree in $C(x)$ the number of endpoints of which depends only on the positive curvature of $S$.
The case of a convex surface was treated in [16] without any differentiability assumptions.

Since every farthest point from \( x \) on \( S \) (an absolute maximum of \( \rho_x \)) is also critical, we contribute here to a description H. Steinhaus had asked for (see [1]). For similar work on farthest points in the case of convex surfaces, see [15].

**On the number of terminal points**

The following well-known result can be found, for instance, in [7].

**Lemma.** If the point \( y \) of \( C(x) \) is a relative minimum of \( \rho_x\big|_{C(x)\setminus E(x)} \), then there are two segments from \( x \) to \( y \) forming a closed geodesic arc at \( x \), and there is no other segment from \( x \) to \( y \).

For a generalization to Alexandrov spaces, see [14].

Let \( S^+ \subset S \) be the subset of those points \( z \) in \( S \) where the Gaussian curvature \( K(z) \) is positive and put, for any Borel set \( B \subset S \),

\[
K(B) = \int_B K \, ds, \quad K^+(B) = \int_{B \cap S^+} K \, ds, \quad k = K^+(S).
\]

We shall prove the following result.

**Theorem 1.** All critical points of the surface \( S \) with respect to \( x \in S \) belong to some set which is locally a tree, lies in \( C(x) \) and has less than \( k/\pi \) endpoints.

**Proof.** As usual, \([r] \) denotes the smallest integer larger or equal to \( r \in \mathbb{R} \).

Consider the union \( U \) of all Jordan arcs in \( C(x) \) joining critical points. Obviously, all endpoints of \( U \) are in \( Q(x) \). It will suffice to show that this set \( U \) has less than \([k/\pi]\) endpoints.

Assume, on the contrary, that we can find \([k/\pi]\) points among the endpoints of \( U \). Let \( y \) be one of these \([k/\pi]\) points. By the definition of a critical point, from \( x \) to \( y \) there must either exist

(i) two segments with opposite directions at \( y \), or else

(ii) three segments whose directions at \( y \) enclose the origin in the interior of their convex hull (in the tangent plane).

Only one, say \( D \), of the domains (i.e., open connected sets) into which these two or three segments divide \( S \) meets \( U \). Indeed, the contrary assumption together with the fact that \( U \) is connected but \( y \) is an endpoint of \( U \) implies the existence of a point of \( U \) different from \( x \) and \( y \) on one of these segments, and a contradiction is obtained.

Let \( s_1, s_2 \) be the segments bounding \( D \) and let \( x_x, x_y \) be the angles towards \( S \setminus D \) determined by \( s_1 \) and \( s_2 \) at \( x \), respectively \( y \). Then \( D \) contains all cycles of \( C(x) \) (if any) and by the Gauß–Bonnet formula we have
Clearly, $a_x > 0$ (because $s_1 \neq s_2$) and $a_y \geq \pi$ (indeed, in Case (i) above $a_y = \pi$ and, in Case (ii), $a_y > \pi$).

For each of the $[k/\pi]$ critical points we obtain a domain analogous to $D$; let $D_1, D_2, \ldots, D_{[k/\pi]}$ be these domains. The sets $S \setminus D_i$ ($i = 1, \ldots, [k/\pi]$) have pairwise disjoint interiors because the two segments bounding $D_i$ join $x$ with a critical point which is not in $S \setminus D_j$ ($i \neq j$), and no pair of these segments cross each other. Thus, letting

$$M = \bigcup_{i=1}^{[k/\pi]} (S \setminus D_i),$$

we have

$$K(M) = \sum_{i=1}^{[k/\pi]} K(S \setminus D_i) > \lfloor k/\pi \rfloor \pi \geq k.$$  

However, this contradicts $K(M) \leq K^+(M) \leq k$. Hence $U$ has at most $\lfloor k/\pi \rfloor - 1$ endpoints, q.e.d.

We sketch the construction of examples showing that the bound in Theorem 1 is sharp. To this end we use the non-differentiable example presented in [16], p. 1402.

Take the surface of a regular tetrahedron $abcd$ and approximate it by a Riemannian surface $S$ respecting the symmetries, such that the curvature of each region close to a vertex be slightly larger than $\pi$. (Thus there must also exist points of negative curvature.) The point of $S$ corresponding to the midpoint $m$ of $ab$ certainly has five critical points, four of them close to $a$, $b$, $c$, $d$ and one close to the midpoint of $cd$. In this case $U$ has the first four critical points as endpoints, and $k/\pi$ is slightly larger than 4.

Take now a small ball $B$ with centre in the interior of the facet $abc$ and not collinear with $m$ and $c$, consider the set $abcd \cup \text{conv}(\{m\} \cup B)$, and appropriately approximate (as before) its boundary by a Riemannian surface $S'$. Then the point of $S'$ corresponding to $m$ has an additional critical point, endpoint of $U$, behind $B$, and $k/\pi$ is slightly larger than 5. Further examples are obtained by adding other balls with centres in the interior of $abc$ and in various directions as seen from $m$.

The endpoints of the set $U$ from the preceding proof will be called terminal points of $x$. Of course, every terminal point is critical. By Theorem 1, every point has less than $k/\pi$ terminal points.

**Theorem 2.** All critical points of the orientable surface $S$ with respect to $x \in S$ belong to some tree lying in $C(x)$ and having less than $k/\pi$ endpoints outside the cycles of $C(x)$.
Proof. Consider the set $U$ from the proof of Theorem 1 (which contains all cycles of $C(x)$). Since $S$ is orientable, if $C(x)$ contains the cycle $\Gamma$ then it must contain at least one more cycle having with $\Gamma$ a common branching point of $C(x)$. (The number of cycles in $C(x)$ is finite and depends on the genus of $S$.)

Each cycle must contain some point which is not critical. Indeed, suppose all points of the cycle $\Gamma$ are critical with respect to $x$. Then $\rho_x$ is constant on $\Gamma$. But then each point $y \in \Gamma$ is a relative minimum of $\rho|_{x}$. By the Lemma, there are two segments from $x$ to $y$ forming a closed geodesic arc starting and ending at $x$, and there is no other segment from $x$ to $y$. Since at every point of $C(x)$ the number of branches of the tree $C(x) \cap V$—for a sufficiently small neighbourhood $V$—equals the number of segments from $x$, there are in our case precisely two branches of $C(x)$ at $y$, and thus $y$ is not a branching point, and a contradiction is obtained.

Hence we can choose finitely many points in $U \setminus Q(x)$ so that after their deletion the resulting set $U^*$ remains connected but possesses no cycle. Consider the union of all Jordan arcs in $U^*$ joining critical points. This is obviously a tree included in $C(x)$, which includes $Q(x)$. Its endpoints outside the cycles of $C(x)$ coincide with the endpoints of $U$. By Theorem 1, there are less than $k/\pi$ such endpoints.

In [4], J. Itoh introduced and studied the essential cut locus (compare with our set $U$ employed in Theorems 1 and 2). Also, knowing Theorem 2 of this paper from the author, Itoh provided strengthened variants of it in [6].

For surfaces embedded in $\mathbb{R}^3$ the following concept is a generalization of convexity from genus 0 to arbitrary genus. $S$ has minimal positive curvature if $k = 4\pi$. Of course, a surface in $\mathbb{R}^3$ homeomorphic to $S^2$ has minimal positive curvature if and only if it is convex. Theorems 1 and 2 have the following immediate corollary.

**Corollary 1.** All critical points of the surface $S \subset \mathbb{R}^3$ of minimal positive curvature, with respect to $x \in S$, belong to some tree lying in $C(x)$ and having at most 3 endpoints outside the cycles of $C(x)$.

In particular, if $S$ is convex, they belong to some tree lying in $C(x)$ and having at most 3 endpoints.

In the convex case, this also follows from Theorem 4 in [16]; by that theorem, if no differentiability of $S$ is assumed there exists an exceptional case in which $Q(x)$ does not belong to any tree with 3 endpoints (but to one with 4) lying in $C(x)$. The exceptional case is that of a tetrahedron with curvature $\pi$ at every vertex.

**On the measure of the set of critical points**

Otsu and Shioya showed that the cut locus has 2-dimensional Hausdorff measure 0 on any Alexandrov surface [10]. But, on such surfaces, the length of the cut locus can be infinite (see [5], [16]).

In our case of a compact Riemannian surface, the cut locus has dimension at most 1, and has finite length [5]. However this length may be very large. How large can the
length of \( Q(x) \) be? Let \( \lambda A \) denote the length of \( A \). Also, let \( r_x \) denote the radius of \( S \) at \( x \), i.e., \( r_x = \max_{y \in S} \rho_S(y) \), and put \( \kappa = \min_{x \in S} K(x) \).

Let \( L_\kappa(r) \) be the length of the intrinsic circle of radius \( r \) on the simply connected Riemannian surface of constant curvature \( \kappa \).

**Theorem 3.** For any point \( x \in S \),

\[
\lambda Q(x) \leq \frac{L_\kappa(r_x)}{2},
\]

with strict inequality if \( S \) is orientable.

**Proof.** For any arc \( \Lambda \subset Q(x) \) and any point \( y \in \Lambda \) we have a loop at \( x \) with midpoint \( y \). Let \( y_1, y_2 \) be the endpoints of \( \Lambda \). The loops at \( x \) through \( y_1, y_2 \) determine two angles \( \beta'_\Lambda, \beta''_\Lambda \) at \( x \). Let \( \beta_\Lambda = \min\{\beta'_\Lambda, \beta''_\Lambda\} \). By Toponogov’s comparison theorem (hinge version), \( \lambda \Lambda \leq \beta_\Lambda L_\kappa(r_x)/2\pi \).

If \( Q(x) \) includes no cycle of \( C(x) \), summing over all (pairwise disjoint) maximal arcs \( \Lambda \subset Q(x) \) gives

\[
\lambda Q(x) \leq \sum_\Lambda \frac{\beta_\Lambda L_\kappa(r_x)}{2\pi} \leq \sum_\Lambda \frac{(\beta'_\Lambda + \beta''_\Lambda) L_\kappa(r_x)}{4\pi} \leq \frac{L_\kappa(r_x)}{2},
\]

since \( \sum_\Lambda (\beta'_\Lambda + \beta''_\Lambda) \leq 2\pi \).

If \( Q(x) \) does include an entire cycle of \( C(x) \), this cycle equals \( C(x) \) (see the proof of Theorem 2), so \( S \) is a projective plane and, analogously,

\[
\lambda Q(x) \leq \frac{L_\kappa(r_x)}{2}.
\]

Assume now \( S \) is orientable and \( \sum_\Lambda (\beta'_\Lambda + \beta''_\Lambda) = 2\pi \). Then there are only two angles measuring \( \beta'_\Lambda, \beta''_\Lambda \) corresponding to a single arc \( \Lambda \). Since \( S \) is orientable, the sides of one of them cannot separate those of the other. So the loop at \( x \) through an endpoint of \( \Lambda \) makes a non-zero angle at \( x \) of interior disjoint from the above two angles, which yields \( \sum_\Lambda (\beta'_\Lambda + \beta''_\Lambda) < 2\pi \), in contradiction with our assumption.

It is interesting to note the following result concerning the set \( F_x \) of all absolute maxima of \( \rho(x) \) in the convex case. Although \( F_x \) is usually much smaller than \( Q(x) \), no better estimate can be obtained.

**Corollary 2.** For any point \( x \) on the convex surface \( S \) the following inequality holds:

\[
\lambda F_x \leq \pi r_x.
\]

This was essentially already proven in [13]. An example in [13], suitably modified, shows that the above upper bound is best possible.
If, however, $x$ has 3 terminal points we expect a much lower upper bound for $\lambda F_x$. This suggests a fruitful interplay with the previous section.

Suppose we know the number of terminal points of the point $x \in S$. Then we can be more precise concerning the length of $Q(x)$.

**Theorem 4.** Let $S$ be orientable. For any point $x \in S$ with $q(x)$ terminal points, the following inequality holds:

$$\lambda Q(x) < \frac{(k - \pi q(x))L_k(r_x)}{4\pi}.$$

**Proof.** Let $y \in Q(x)$ be a terminal point of $x$. Like in the proof of Theorem 1 we find a domain $D_y$ with the union of two segments from $x$ to $y$ as boundary, such that $Q(x) \setminus \{y\} \subset D_y$ and $K(S \setminus D_y) > \pi$.

Let $\Delta$ be the union of the sets $S \setminus D_y$, for all terminal points $y$ of $x$. Then $K(\Delta) > q(x)\pi$.

For any arc $\Lambda \subset Q(x)$ and any point $y \in \Delta$, we have a loop at $x$ with midpoint $y$. Let $y_1, y_2$ be the endpoints of $\Lambda$. As in the proof of Theorem 3, the loops $L_1, L_2$ at $x$ through $y_1, y_2$ determine two angles $\beta'_\Lambda, \beta''_\Lambda$ at $x$. Let $\Phi_\Lambda$ be the component of $S \setminus (L_1 \cup L_2)$ containing $\Lambda \setminus \{y_1, y_2\}$. Then $K(\Phi_\Lambda) = \beta'_\Lambda + \beta''_\Lambda$, and

$$\beta_\Lambda = \min\{\beta'_\Lambda, \beta''_\Lambda\} \leq K(\Phi_\Lambda)/2.$$  

By Toponogov’s theorem,

$$\lambda \Lambda \leq \frac{\beta_\Lambda L_k}{2\pi} \leq \frac{K(\Phi_\Lambda)L_k(r_x)}{4\pi}.$$  

Summing over all maximal arcs $\Lambda \subset Q(x)$, we get

$$\lambda Q(x) \leq \sum_\Lambda \frac{K(\Phi_\Lambda)L_k(r_x)}{4\pi} = \frac{K(\Phi)L_k(r_x)}{4\pi},$$

where $\Phi = \bigcup_\Lambda \Phi_\Lambda$. Moreover,

$$K(\Phi) \leq K^+(\Phi) \leq k - K^+(\Delta) \leq k - K(\Delta) < k - q(x)\pi.$$  

The inequality of the theorem now follows.

It is easily seen that Theorem 4 provides a smaller upper bound than Theorem 3 if (and only if) $k < (q(x) + 2)\pi$.

**Corollary 3.** For every point $x$ of the orientable surface $S$ with $q(x) = \lceil k/\pi \rceil - 1$,

$$\lambda Q(x) < \frac{L_k(r_x)}{4}.$$
In the special case of a surface of minimal positive curvature, we obtain the following result.

**Corollary 4.** For every point \( x \) of an orientable surface of minimal positive curvature with \( q(x) = 3 \),

\[
\lambda Q(x) < \frac{L_x(r_x)}{4}.
\]

Applied to convex surfaces, Corollary 4 yields the following non-trivial inequality.

**Corollary 5.** For every point \( x \) of the convex surface \( S \) with \( q(x) = 3 \),

\[
\lambda Q(x) < \frac{\pi r_x}{2}.
\]

To illustrate Corollary 5 we present the following example, which is itself not a Riemannian surface, but a limit of such surfaces; our conclusions can easily be transferred to these.

Consider the equilateral triangle \( abc \) with centre \( o \) and circumradius 1. Also, consider the point \( d \) symmetric to \( o \) with respect to the line \( ab \), and the parabola of focal point \( o \) and directrix \( bd \). Let \( P_{ab} \) be the closed convex set bounded by this parabola. With \( P_{ab} \) and the other 5 analogous convex sets we construct the compact convex set

\[ K = \text{conv}((P_{ab} \cap P_{ac}) \cup (P_{ca} \cap P_{cb}) \cup (P_{bc} \cap P_{ba})) \cap abc. \]

This set is not strictly convex, having on each side of \( abc \) a line-segment the endpoints of which trisect the side. Let \( a_1 \) be the boundary point of \( K \) on \( oa \), let \( a_2 \) be the midpoint of \( oa \), and analogously consider the points \( b_1, c_1, b_2, c_2 \).

Let \( S \) be the doubly covered set \( K \) and let \( o' \) be the point corresponding to \( o \) on the other side of \( S \). Then, on \( S \), we have \( C(o') = a_1o \cup b_1o \cup c_1o, F_{o'} = \{ o \}, \) and \( Q(o') = a_1a_2 \cup b_1b_2 \cup c_1c_2 \cup \{ o \} \). Also, \( r_{o'} = 1 \), while

\[
\lambda Q(o') = 3\lambda(a_1a_2) = \frac{3}{2} (\sqrt{3} - 1) = 1.098076 \ldots
\]

By Corollary 5,

\[
\lambda Q(o') < \frac{\pi}{2} = 1.570796 \ldots
\]

Thus, this example leaves open the question whether the bound in Corollary 5 is best possible.
References


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