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On the critical points of a Riemannian surface

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Abstract. We show that all critical points with respect to a point on a Riemannian surface lie on a subset of the cut locus which is locally a tree and has relatively few endpoints. Moreover, we offer some inequalities involving the length of the set of critical points.

Introduction

Let S be a compact Riemannian (2-dimensional) surface without boundary. For an arbitrary point $x \in S$ we consider the Riemannian distance $\rho_x(y)$ from x to $y \in S$ and the cut locus $C(x)$, defined as the set of all $y \in S$ such that no *segment*, i.e., shortest path, from x to y can be extended as a segment beyond y.

The cut locus was introduced by H. Poincaré, [11]. Further basic properties of the cut locus have been investigated by S. B. Myers [8], [9] and J. H. C. Whitehead [12], and later by many other authors. Among other things, it is well-known that $C(x)$ is connected and locally a tree. For an introduction to the cut locus see, for example [7].

For any set which is locally a tree, a point in the set is called *endpoint* if its deletion does not disconnect any connected neighbourhood (tree, if small) of the point. Let $E(x)$ denote the set of all *endpoints* of $C(x)$. A point $y \in S$ is called *regular* with respect to x (and ρ_x) if some open halfplane of T_yS contains the tangent vectors at y of all segments from y to x. A point $y \in S$ is called *critical* with respect to x (and ρ_x) if it is not regular, i.e., if for any tangent vector τ at y there exists a segment from y to x with direction σ at y such that $\langle \tau, \sigma \rangle \geqslant 0$ (see, for instance, [2], p. 2). For example, every relative maximum of ρ_x and every relative minimum of $\rho_x|_{C(x) \setminus E(x)}$ is a critical point. Let $Q(x)$ be the set of all critical points with respect to x. All these points lie on $C(x)$.

We may encounter uncountably many critical points on one hand (to see this, take the example in [13], p. 320, and modify it appropriately in order to obtain a Riemannian surface) and, on the other, $C(x)$ may be quite large, for example non-triangulable (see Gluck and Singer [3]).

We point out in this paper that in fact $Q(x)$ cannot be too scattered in $C(x)$; more precisely, it must belong to a single handsome tree in $C(x)$ the number of endpoints of which depends only on the positive curvature of S.

The case of a convex surface was treated in [16] without any differentiability assumptions.

Since every farthest point from x on S (an absolute maximum of ρ_x) is also critical, we contribute here to a description H. Steinhaus had asked for (see [1]). For similar work on farthest points in the case of convex surfaces, see [15].

On the number of terminal points

The following well-known result can be found, for instance, in [7].

Lemma. *If the point y of $C(x)$ is a relative minimum of $\rho_x|_{C(x)\setminus E(x)}$, then there are two segments from x to y forming a closed geodesic arc at x, and there is no other segment from x to y.*

For a generalization to Alexandrov spaces, see [14].

Let $S_+ \subset S$ be the subset of those points z in S where the Gaußian curvature $K(z)$ is positive and put, for any Borel set $B \subset S$,

$$K(B) = \int_B K\,ds, \quad K^+(B) = \int_{B \cap S_+} K\,ds, \quad k = K^+(S).$$

We shall prove the following result.

Theorem 1. *All critical points of the surface S with respect to $x \in S$ belong to some set which is locally a tree, lies in $C(x)$ and has less than k/π endpoints.*

Proof. As usual, $\lceil r \rceil$ denotes the smallest integer larger or equal to $r \in \mathbb{R}$.

Consider the union U of all Jordan arcs in $C(x)$ joining critical points. Obviously, all endpoints of U are in $Q(x)$. It will suffice to show that this set U has less than $\lceil k/\pi \rceil$ endpoints.

Assume, on the contrary, that we can find $\lceil k/\pi \rceil$ points among the endpoints of U. Let y be one of these $\lceil k/\pi \rceil$ points. By the definition of a critical point, from x to y there must either exist

(i) two segments with opposite directions at y, or else

(ii) three segments whose directions at y enclose the origin in the interior of their convex hull (in the tangent plane).

Only one, say D, of the domains (i.e., open connected sets) into which these two or three segments divide S meets U. Indeed, the contrary assumption together with the fact that U is connected but y is an endpoint of U implies the existence of a point of U different from x and y on one of these segments, and a contradiction is obtained.

Let s_1, s_2 be the segments bounding D and let α_x, α_y be the angles towards $S\setminus D$ determined by s_1 and s_2 at x, respectively y. Then D contains all cycles of $C(x)$ (if any) and by the Gauß–Bonnet formula we have

$$K(S \backslash D) = \alpha_x + \alpha_y.$$

Clearly, $\alpha_x > 0$ (because $s_1 \neq s_2$) and $\alpha_y \geqslant \pi$ (indeed, in Case (i) above $\alpha_y = \pi$ and, in Case (ii), $\alpha_y > \pi$).

For each of the $\lceil k/\pi \rceil$ critical points we obtain a domain analogous to D; let $D_1, D_2, \ldots, D_{\lceil k/\pi \rceil}$ be these domains. The sets $S \backslash D_i$ ($i = 1, \ldots, \lceil k/\pi \rceil$) have pairwise disjoint interiors because the two segments bounding D_i join x with a critical point which is not in $S \backslash D_j$ ($i \neq j$), and no pair of these segments cross each other. Thus, letting

$$M = \bigcup_{i=1}^{\lceil k/\pi \rceil} (S \backslash D_i),$$

we have

$$K(M) = \sum_{i=1}^{\lceil k/\pi \rceil} K(S \backslash D_i) > \lceil k/\pi \rceil \pi \geqslant k.$$

However, this contradicts $K(M) \leqslant K^+(M) \leqslant k$. Hence U has at most $\lceil k/\pi \rceil - 1$ endpoints, q.e.d.

We sketch the construction of examples showing that the bound in Theorem 1 is sharp. To this end we use the non-differentiable example presented in [16], p. 1402.

Take the surface of a regular tetrahedron *abcd* and approximate it by a Riemannian surface S respecting the symmetries, such that the curvature of each region close to a vertex be slightly larger than π. (Thus there must also exist points of negative curvature.) The point of S corresponding to the midpoint m of ab certainly has five critical points, four of them close to a, b, c, d and one close to the midpoint of cd. In this case U has the first four critical points as endpoints, and k/π is slightly larger than 4.

Take now a small ball B with centre in the interior of the facet *abc* and not collinear with m and c, consider the set $abcd \cup \mathrm{conv}(\{m\} \cup B)$, and appropriately approximate (as before) its boundary by a Riemannian surface S'. Then the point of S' corresponding to m has an additional critical point, endpoint of U, behind B, and k/π is slightly larger than 5. Further examples are obtained by adding other balls with centres in the interior of *abc* and in various directions as seen from m.

The endpoints of the set U from the preceding proof will be called *terminal* points of x. Of course, every terminal point is critical. By Theorem 1, every point has less than k/π terminal points.

Theorem 2. *All critical points of the orientable surface S with respect to $x \in S$ belong to some tree lying in $C(x)$ and having less than k/π endpoints outside the cycles of $C(x)$.*

Proof. Consider the set U from the proof of Theorem 1 (which contains all cycles of $C(x)$). Since S is orientable, if $C(x)$ contains the cycle Γ then it must contain at least one more cycle having with Γ a common branching point of $C(x)$. (The number of cycles in $C(x)$ is finite and depends on the genus of S.)

Each cycle must contain some point which is not critical. Indeed, suppose all points of the cycle Γ are critical with respect to x. Then ρ_x is constant on Γ. But then each point $y \in \Gamma$ is a relative minimum of $\rho|_\Gamma$. By the Lemma, there are two segments from x to y forming a closed geodesic arc starting and ending at x, and there is no other segment from x to y. Since at every point of $C(x)$ the number of branches of the tree $C(x) \cap V$—for a sufficiently small neighbourhood V—equals the number of segments from x, there are in our case precisely two branches of $C(x)$ at y, and thus y is not a branching point, and a contradiction is obtained.

Hence we can choose finitely many points in $U \backslash Q(x)$ so that after their deletion the resulting set U^* remains connected but possesses no cycle. Consider the union of all Jordan arcs in U^* joining critical points. This is obviously a tree included in $C(x)$, which includes $Q(x)$. Its endpoints outside the cycles of $C(x)$ coincide with the endpoints of U. By Theorem 1, there are less than k/π such endpoints.

In [4], J. Itoh introduced and studied the essential cut locus (compare with our set U employed in Theorems 1 and 2). Also, knowing Theorem 2 of this paper from the author, Itoh provided strengthened variants of it in [6].

For surfaces embedded in $\mathbb{R}^3$ the following concept is a generalization of convexity from genus 0 to arbitrary genus. S has *minimal positive curvature* if $k = 4\pi$. Of course, a surface in $\mathbb{R}^3$ homeomorphic to S^2 has minimal positive curvature if and only if it is convex. Theorems 1 and 2 have the following immediate corollary.

Corollary 1. *All critical points of the surface $S \subset \mathbb{R}^3$ of minimal positive curvature, with respect to $x \in S$, belong to some tree lying in $C(x)$ and having at most* 3 *endpoints outside the cycles of $C(x)$.*

In particular, if S is convex, they belong to some tree lying in $C(x)$ and having at most 3 *endpoints.*

In the convex case, this also follows from Theorem 4 in [16]; by that theorem, if no differentiability of S is assumed there exists an exceptional case in which $Q(x)$ does not belong to any tree with 3 endpoints (but to one with 4) lying in $C(x)$. The exceptional case is that of a tetrahedron with curvature π at every vertex.

On the measure of the set of critical points

Otsu and Shioya showed that the cut locus has 2-dimensional Hausdorff measure 0 on any Alexandrov surface [10]. But, on such surfaces, the length of the cut locus can be infinite (see [5], [16]).

In our case of a compact Riemannian surface, the cut locus has dimension at most 1, and has finite length [5]. However this length may be very large. How large can the length of $Q(x)$ be? Let λA denote the length of A. Also, let r_x denote the radius of S at x, i.e., $r_x = \max_{y \in S} \rho_x(y)$, and put $\kappa = \min_{x \in S} K(x)$.

Let $L_\kappa(r)$ be the length of the intrinsic circle of radius r on the simply connected Riemannian surface of constant curvature κ.

Theorem 3. *For any point* $x \in S$,

$$\lambda Q(x) \leqslant \frac{L_\kappa(r_x)}{2},$$

with strict inequality if S is orientable.

Proof. For any arc $\Lambda \subset Q(x)$ and any point $y \in \Lambda$ we have a loop at x with midpoint y. Let y_1, y_2 be the endpoints of Λ. The loops at x through y_1, y_2 determine two angles β'_Λ, β''_Λ at x. Let $\beta_\Lambda = \min\{\beta'_\Lambda, \beta''_\Lambda\}$. By Toponogov's comparison theorem (hinge version), $\lambda\Lambda \leqslant \beta_\Lambda L_\kappa(r_x)/2\pi$.

If $Q(x)$ includes no cycle of $C(x)$, summing over all (pairwise disjoint) maximal arcs $\Lambda \subset Q(x)$ gives

$$\lambda Q(x) \leqslant \sum_\Lambda \frac{\beta_\Lambda L_\kappa(r_x)}{2\pi} \leqslant \sum_\Lambda \frac{(\beta'_\Lambda + \beta''_\Lambda) L_\kappa(r_x)}{4\pi} \leqslant \frac{L_\kappa(r_x)}{2},$$

since $\sum_\Lambda (\beta'_\Lambda + \beta''_\Lambda) \leqslant 2\pi$.

If $Q(x)$ does include an entire cycle of $C(x)$, this cycle equals $C(x)$ (see the proof of Theorem 2), so S is a projective plane and, analogously,

$$\lambda Q(x) \leqslant \frac{L_\kappa(r_x)}{2}.$$

Assume now S is orientable and $\sum_\Lambda (\beta'_\Lambda + \beta''_\Lambda) = 2\pi$. Then there are only two angles measuring β'_Λ, β''_Λ corresponding to a single arc Λ. Since S is orientable, the sides of one of them cannot separate those of the other. So the loop at x through an endpoint of Λ makes a non-zero angle at x of interior disjoint from the above two angles, which yields $\sum_\Lambda (\beta'_\Lambda + \beta''_\Lambda) < 2\pi$, in contradiction with our assumption.

It is interesting to note the following result concerning the set F_x of all absolute maxima of $\rho(x)$ in the convex case. Although F_x is usually much smaller than $Q(x)$, no better estimate can be obtained.

Corollary 2. *For any point x on the convex surface S the following inequality holds*:

$$\lambda F_x < \pi r_x.$$

This was essentially already proven in [13]. An example in [13], suitably modified, shows that the above upper bound is best possible.

If, however, x has 3 terminal points we expect a much lower upper bound for λF_x. This suggests a fruitful interplay with the previous section.

Suppose we know the number of terminal points of the point $x \in S$. Then we can be more precise concerning the length of $Q(x)$.

Theorem 4. *Let S be orientable. For any point $x \in S$ with $q(x)$ terminal points, the following inequality holds*:

$$\lambda Q(x) < \frac{(k - \pi q(x))L_\kappa(r_x)}{4\pi}.$$

Proof. Let $y \in Q(x)$ be a terminal point of x. Like in the proof of Theorem 1 we find a domain D_y with the union of two segments from x to y as boundary, such that $Q(x)\backslash\{y\} \subset D_y$ and $K(S\backslash D_y) > \pi$.

Let Δ be the union of the sets $S\backslash D_y$, for all terminal points y of x. Then $K(\Delta) > q(x)\pi$.

For any arc $\Lambda \subset Q(x)$ and any point $y \in \Lambda$ we have a loop at x with midpoint y. Let y_1, y_2 be the endpoints of Λ. As in the proof of Theorem 3, the loops L_1, L_2 at x through y_1, y_2 determine two angles β'_Λ, β''_Λ at x. Let Φ_Λ be the component of $S\backslash(L_1 \cup L_2)$ containing $\Lambda\backslash\{y_1, y_2\}$. Then $K(\Phi_\Lambda) = \beta'_\Lambda + \beta''_\Lambda$, and

$$\beta_\Lambda = \min\{\beta'_\Lambda, \beta''_\Lambda\} \leqslant K(\Phi_\Lambda)/2.$$

By Toponogov's theorem,

$$\lambda\Lambda \leqslant \frac{\beta_\Lambda L_\kappa}{2\pi} \leqslant \frac{K(\Phi_\Lambda)L_\kappa(r_x)}{4\pi}.$$

Summing over all maximal arcs $\Lambda \subset Q(x)$, we get

$$\lambda Q(x) \leqslant \sum_\Lambda \frac{K(\Phi_\Lambda)L_\kappa(r_x)}{4\pi} = \frac{K(\Phi)L_\kappa(r_x)}{4\pi},$$

where $\Phi = \bigcup_\Lambda \Phi_\Lambda$. Moreover,

$$K(\Phi) \leqslant K^+(\Phi) \leqslant k - K^+(\Delta) \leqslant k - K(\Delta) < k - q(x)\pi.$$

The inequality of the theorem now follows.

It is easily seen that Theorem 4 provides a smaller upper bound than Theorem 3 if (and only if) $k < (q(x) + 2)\pi$.

Corollary 3. *For every point x of the orientable surface S with $q(x) = \lceil k/\pi \rceil - 1$,*

$$\lambda Q(x) < \frac{L_\kappa(r_x)}{4}.$$

In the special case of a surface of minimal positive curvature, we obtain the following result.

Corollary 4. *For every point x of an orientable surface of minimal positive curvature with $q(x) = 3$,*

$$\lambda Q(x) < \frac{L_\kappa(r_x)}{4}.$$

Applied to convex surfaces, Corollary 4 yields the following non-trivial inequality.

Corollary 5. *For every point x of the convex surface S with $q(x) = 3$,*

$$\lambda Q(x) < \frac{\pi r_x}{2}.$$

To illustrate Corollary 5 we present the following example, which is itself not a Riemannian surface, but a limit of such surfaces; our conclusions can easily be transferred to these.

Consider the equilateral triangle abc with centre o and circumradius 1. Also, consider the point d symmetric to o with respect to the line $\overline{ab}$, and the parabola of focal point o and directrix $\overline{bd}$. Let P_{ab} be the closed convex set bounded by this parabola. With P_{ab} and the other 5 analogous convex sets we construct the compact convex set

$$K = \operatorname{conv}(((P_{ab} \cap P_{ac}) \cup (P_{ca} \cap P_{cb}) \cup (P_{bc} \cap P_{ba})) \cap abc).$$

This set is not strictly convex, having on each side of abc a line-segment the endpoints of which trisect the side. Let a_1 be the boundary point of K on oa, let a_2 be the midpoint of oa, and analogously consider the points b_1, c_1, b_2, c_2.

Let S be the doubly covered set K and let o' be the point corresponding to o, on the other side of S. Then, on S, we have $C(o') = a_1 o \cup b_1 o \cup c_1 o$, $F_{o'} = \{o\}$, and $Q(o') = a_1 a_2 \cup b_1 b_2 \cup c_1 c_2 \cup \{o\}$. Also, $r_{o'} = 1$, while

$$\lambda Q(o') = 3\lambda(a_1 a_2) = \frac{3}{2}(\sqrt{3} - 1) = 1.098076\ldots$$

By Corollary 5,

$$\lambda Q(o') < \frac{\pi}{2} = 1.570796\ldots$$

Thus, this example leaves open the question whether the bound in Corollary 5 is best possible.

References

[1] H. T. Croft, K. J. Falconer, R. K. Guy, *Unsolved problems in geometry*. Springer 1991. MR1107516 (92c:52001) Zbl 0748.52001

[2] P. De Bartolomeis, F. Tricerri, editors, *Geometric topology: recent developments*. Springer 1991. MR1168041 (92m:53001) Zbl 0746.00065

[3] H. Gluck, D. Singer, Scattering of geodesic fields. I. *Ann. of Math.* (2) **108** (1978), 347–372. MR506991 (80c:53046) Zbl 0399.58011

[4] J. Itoh, Some considerations on the cut locus of a Riemannian manifold. In: *Geometry of geodesics and related topics* (*Tokyo*, 1982), volume 3 of *Adv. Stud. Pure Math.*, 29–46, North-Holland 1984. MR758646 (86a:53048) Zbl 0544.53036

[5] J. Itoh, The length of a cut locus on a surface and Ambrose's problem. *J. Differential Geom.* **43** (1996), 642–651. MR1412679 (97i:53038) Zbl 0865.53031

[6] J. Itoh, Essential cut locus on a surface. In: *Proc. Fifth Pacific Rim Geometry Conf.* (*Sendai*, 2000), volume 20 of *Tohoku Math. Publ.*, 53–59, Tohoku Univ., Sendai 2001. MR1864886 (2002i:53054) Zbl 1063.53046

[7] S. Kobayashi, On conjugate and cut loci. In: *Global differential geometry*, volume 27 of *MAA Stud. Math.*, 140–169, 1989. Zbl 0683.53043

[8] S. B. Myers, Connections between differential geometry and topology. I. Simply connected surfaces. *Duke Math. J.* **1** (1935), 376–391. Zbl 0012.27502

[9] S. B. Myers, Connections between differential geometry and topology. II. Closed surfaces. *Duke Math. J.* **2** (1936), 95–102. Zbl 0013.32201

[10] Y. Otsu, T. Shioya, The Riemannian structure of Alexandrov spaces. *J. Differential Geom.* **39** (1994), 629–658. MR1274133 (95e:53062) Zbl 0808.53061

[11] H. Poincaré, Sur les lignes géodésiques des surfaces convexes. *Trans. Amer. Math. Soc.* **6** (1905), 237–274. MR1500710 Zbl 36.0669.01

[12] J. H. C. Whitehead, On the covering of a complete space by the geodesics through a point. *Ann. of Math.* (2) **36** (1935), 679–704. MR1503245 Zbl 0012.27802

[13] T. Zamfirescu, On some questions about convex surfaces. *Math. Nachr.* **172** (1995), 313–324. MR1330637 (96e:52004) Zbl 0833.53004

[14] T. Zamfirescu, Closed geodesic arcs in Aleksandrov spaces. *Rend. Circ. Mat. Palermo* (2) *Suppl.* no. **50** (1997), 425–430. MR1603026 (99b:53063) Zbl 0889.53027

[15] T. Zamfirescu, Farthest points on convex surfaces. *Math. Z.* **226** (1997), 623–630. MR1484714 (98m:52003) Zbl 0896.52005

[16] T. Zamfirescu, Extreme points of the distance function on convex surfaces. *Trans. Amer. Math. Soc.* **350** (1998), 1395–1406. MR1458314 (98i:52005) Zbl 0896.52006

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