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Acute Triangulations of Flat Möbius Strips

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Abstract. In this paper we investigate the acute triangulations of flat Möbius strips. We find out that we can always triangulate a flat Möbius strip with at most nine acute triangles, and sometimes (but not always) that many triangles are really needed.

1. Introduction

Historically, the investigation of acute triangulations has one of its origins in a problem proposed by Goldberg in 1960 in the *American Mathematical Monthly* (E1406), which is identical to a problem of Stover reported in the same year by Gardner in his Mathematical Games section of the *Scientific American* (see [4] and [5]). The problem was to cut an obtuse triangle into the least number of smaller triangles, all of them acute.

In the same year, independently, Burago and Zalgaller [2] treated in considerable depth acute triangulations of polygonal complexes, being led to them by the problem of their isometric embedding into $\mathbb{R}^3$. (It happens that, involuntarily, their paper also includes a solution to Goldberg–Stover's problem!)

In 1980 Cassidy and Lord [3] considered acute triangulations of the square. Recently, Maehara investigated acute triangulations of quadrilaterals [9] and arbitrary polygons [10], and Yuan improved the results on polygons [11].

A *triangulation* of a two-dimensional Alexandrov space with curvature bounded below (see [1] for a definition) means a collection of (full) triangles covering the space, such that the intersection of any two triangles is either empty or consists of a vertex or of an edge. A triangle is called *geodesic* if all its edges are *segments*, i.e., shortest paths between the corresponding vertices.

Our interest is focused on triangulations that are *acute*, which means that all triangles are geodesic and all angles involved are smaller than $\pi/2$.

Acute triangulations of all Platonic surfaces, which are the surfaces of the five well-known Platonic solids, were investigated in [6]–[8].

The following problem first raised in [6] is natural, and not easy.

Problem 1. Does there exist a number N such that every compact convex surface in $\mathbb{R}^3$ admits an acute triangulation with at most N triangles?

As remarked in [7], Problem 1 can be transfered to other families of Alexandrov surfaces, with or without boundary.

Besides the platonic surfaces, other surfaces homeomorphic to the sphere have been acutely triangulated: the double triangle by Zamfirescu [14], the double quadrilateral by Yuan and Zamfirescu [13], and the double pentagon by Yuan [12]. We remark here that Problem 1 for the family of all tetrahedral surfaces is surprisingly difficult, and still open.

In this paper we consider acute triangulations of the family $\mathcal{M}$ of all flat Möbius strips. These well-known non-oriented surfaces with boundary are locally isometric to a planar disc or semidisc. We solve Problem 1 for this family $\mathcal{M}$, and establish the best possible corresponding bound N. We chose this family for our investigation because of its importance among non-oriented surfaces, and because no non-oriented surfaces have been considered so far.

Among flat surfaces, besides planar polygons, only the case of flat tori has been mentioned (but not completely settled) so far [6].

Let R be a rectangle of sides 1 and α in the Euclidean plane $\mathbb{R}^2$. If we identify pairs of points symmetric about the center of R and lying on the sides of length 1, we obtain a flat Möbius strip, which will be denoted by M_R.

Let $\mathcal{T}$ be an acute triangulation of M_R. Let $V(\mathcal{T})$ be the set of vertices of $\mathcal{T}$. A vertex $p \in V(\mathcal{T})$ is called a *side vertex* if p lies on the boundary bd M_R of M_R, and an *interior vertex* otherwise. A *side edge* of $\mathcal{T}$ is an edge of $\mathcal{T}$ lying in bd M_R. A *transversal edge* of $\mathcal{T}$ is an edge which is not a side edge, but joins two side vertices of $\mathcal{T}$. For $a, b \in M_R$, let ab denote a segment joining a to b, and let $|ab|$ be the length of ab.

Now, let $R = a_1b_1a_2b_2 \subset \mathbb{R}^2$ with $\|a_1 - b_1\| = 1$; then a_1 and a_2 (respectively, b_1 and b_2) are identical in M_R.

2. Partial Results

We shall see here that, in our constructions, the number of triangles in $\mathcal{T}$ is a decreasing function of α. For large α, five triangles will suffice; for $1 < \alpha < \sqrt{\frac{5}{3}}$, we need eight triangles, while for small α we need one more triangle.

In fact, we "need" eight or nine triangles in the sense that our constructions are made this way. The proof that, for $\alpha < 1$, nine acute triangles are really needed will come only in the last section.

Proposition 1. *If* $\alpha \geq \sqrt{\frac{5}{3}}$, *then* M_R *can be triangulated into five acute triangles, and no smaller acute triangulation is possible.*

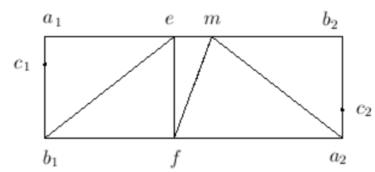

Fig. 1. $\alpha \geq \sqrt{3}$.

Proof. *Case* 1: $\alpha \geq \sqrt{3}$. Let $e, m \in a_1b_2$ in R, such that $\|b_1 - e\| = \|b_2 - e\|$, $\|a_1 - m\| = \|a_2 - m\|$. Clearly, $e \in \text{relint}\, a_1m$ in R. Let f be the orthogonal projection of e on b_1a_2 in R; then $\|a_2 - f\| = \|a_2 - m\|$, which implies that $\angle mfa_2 < \pi/2$. By our construction, it is easy to check that the line-segments a_1b_1, a_1e, b_1e, b_1f, em, ef, mb_2, ma_2 and fa_2 are all segments in M_R. Furthermore, since $\|m - f\| = (\sqrt{\alpha^2+1})/\alpha$, $\|b_1 - f\| = \|m - b_2\| = (\alpha^2 - 1)/2\alpha$ and $\alpha \geq \sqrt{3}$, we have $\|m - f\| \leq \|m - b_2\| + \|b_1 - f\| \leq \|m - c_2\| + \|c_1 - f\|$ for any point $c_i \in a_ib_i$ (c_1 and c_2 are identical), which implies that the line-segment mf is also a segment in M_R. Since $|ma_2| = (\alpha^2 + 1)/2\alpha$, we have $|mf| \leq |ma_2|$, whence $\angle ma_2 f \leq \pi/3$. Thus M_R can be triangulated into five non-obtuse geodesic triangles, as shown in Fig. 1. Now we replace in this triangulation the vertices f and a_2 by two vertices on the side edge fa_2, close to f, respectively a_2. So we obtain an acute triangulation of M_R of size 5.

Case 2: $\sqrt{\frac{5}{3}} \leq \alpha < \sqrt{3}$. Let $e_1 \in a_1b_2$ in R, such that $\|b_1 - e_1\| = \|b_2 - e_1\|$. The Möbius strip M_R is the isosceles trapezoid $e_1b_1e_2b_2 \subset \mathbb{R}^2$, as shown in Fig. 2, where e_1 coincides with e_2 in M_R. Let f be the midpoint of the side edge e_1b_2, and let $g, h \in b_1e_2$ in $\mathbb{R}^2$ be such that $\|e_1 - g\| = \|e_2 - g\| = \|b_1 - h\| = \|b_2 - h\|$. Since $\|b_1 - e_2\| - (\|b_1 - g\| + \|h - e_2\|) = (3 - \alpha^2)/2\alpha > 0$, $g \in \text{relint}\, b_1h$. From our construction, it is easy to check that the line-segments b_1e_1, e_1f, e_1g, b_1g, gh, fb_2, b_2h, $he_2 \subset \mathbb{R}^2$ are all segments in M_R. Since $\|f - g\| = (\sqrt{\alpha^4 + 10\alpha^2 + 9})/4\alpha$, $\|f - b_2\| + \|b_1 - g\| = (5\alpha^2 - 3)/4\alpha$ and $\alpha \geq \sqrt{\frac{5}{3}}$, we have $\|f - g\| \leq \|f - b_2\| + \|b_1 - g\| < \|f - e_1\| + \|e_2 - g\|$. Let m_i be the midpoint of $e_ib_i \subset \mathbb{R}^2$ ($i = 1, 2$). m_1, m_2 coincide in M_R. If $n_1 \in b_1m_1$, then $\|f - b_2\| + \|b_1 - g\| = \|f' - b_1\| + \|b_1 - g\| \leq \|f' - n_1\| + \|n_1 - g\|$, where $f' = f$ in M_R (see Fig. 2). If $n_1 \in e_1m_1$, then $\|f - e_1\| + \|e_2 - g\| = \|f'' - e_2\| + \|e_2 - g\| \leq \|f'' - n_2\| + \|n_2 - g\|$, where $f'' = f$ and $n_2 = n_1$ in M_R (see Fig.

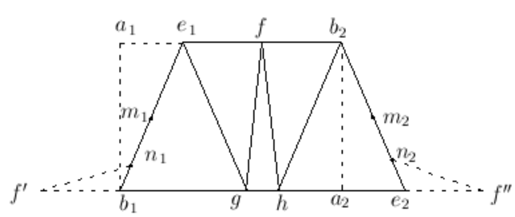

Fig. 2. $\sqrt{\frac{5}{3}} \leq \alpha < \sqrt{3}$.

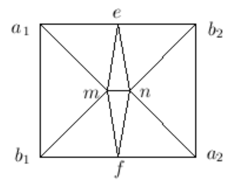

Fig. 3. $1 < \alpha < \sqrt{\frac{5}{3}}$.

2). Hence $fg \subset R$ is a segment in M_R. Analogously, $fh \subset R$ is also a segment in M_R. Furthermore, since $\sqrt{\frac{5}{3}} \le \alpha < \sqrt{3}$, we have $\tan \frac{1}{2}\angle gfh = (3-\alpha^2)/4\alpha < 1$ and $\tan \frac{1}{2}\angle b_1 e_1 g = (\alpha^2 - 1)/2\alpha < 1$. Thus, all triangles in Fig. 2 are acute. Hence we obtained an acute triangulation of M_R with size 5.

Now let $\mathcal{T}$ be an arbitrary acute triangulation of M_R with t triangles. We regard $\mathcal{T}$ as a planar graph embedded on M_R. If $\mathcal{T}$ has at least one interior vertex, then clearly $t \ge 5$. If $\mathcal{T}$ has no interior vertex, then we assume that it has s side vertices. Notice that every side vertex has degree at least 4, so $s \ge 5$. Now denote the number of edges of $\mathcal{T}$ by e. Since $3t + s = 2e = \sum_{x \in V(\mathcal{T})} d(x) \ge 4s$, we have $t \ge s \ge 5$.

The proof is complete. □

Proposition 2. *If* $1 < \alpha < \sqrt{\frac{5}{3}}$, *then* M_R *can be triangulated into eight acute triangles.*

Proof. Inside of R, let e, f be the midpoints of a_1b_2, a_2b_1 respectively, and let m, n be the two points such that both a_1b_1m and a_2b_2n are right isosceles triangles. Since $\alpha > 1$, these triangles are disjoint. Thus M_R is triangulated into eight non-obtuse triangles as shown in Fig. 3. Now we can slightly slide m and n towards each other, such that all the eight triangles become acute. □

Proposition 3. *If* $\alpha \le 1$, *then* M_R *admits an acute triangulation with nine triangles.*

Proof. There are two cases to consider.

Case 1: $\alpha < 1$. In R, let $m_i \in a_ib_i$ $(i = 1, 2)$ be a point which is close to the midpoint of a_ib_i and satisfies $\|a_i - m_i\| < \|b_i - m_i\|$ (m_1, m_2 are identical in M_R). Let l denote the perpendicular bisector of b_1b_2. Let $e, f \in l$ such that, in R, m_1e, m_2f, b_1a_2 are parallel. Clearly e and f are symmetric with respect to the center of R. Now, still in R, let $n \in a_1b_2$ be a point close to b_2 such that $\|n - m_2\| < \|n - m_1\|$ and nef is an acute triangle. Since $\alpha < 1$, we obtain a non-obtuse triangulation of M_R with size 9, as shown in Fig. 4. Now, again in $\mathbb{R}^2$, we take the vertex $f + \varepsilon(a_2 - b_2)$ instead of f and the vertex $m_2 + \eta(a_1 - b_2)$ instead of m_2, with ε and η chosen so small that all triangles become acute.

Case 2: $\alpha = 1$. Let c_1 be the point on the side b_1a_2 of R such that $\|b_1 - c_1\| = \frac{1}{4}$. The Möbius strip is the isosceles trapezoid $a_1c_1a_2c_2 \subset \mathbb{R}^2$, where c_1 and c_2 are identical in

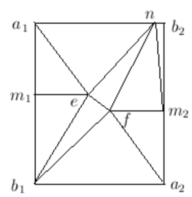

Fig. 4. $\alpha < 1$.

M_R. Let m_i $(i = 1, 2)$ be the midpoint of $a_i c_i$ (m_1 and m_2 are identical in M_R) and let f be the midpoint of c_1c_2 (in $\mathbb{R}^2$). Now denote by n the orthogonal projection of f on a_1c_2 (in $\mathbb{R}^2$). For arbitrary points $a, b, c \in \mathbb{R}^2$, let l^c_{ab} denote the line passing through the point c and perpendicular to ab. Suppose that $l^{m_1}_{a_1c_1} \cap l^f_{c_1n} = \{e\}$. Thus, M_R admits a geodesic triangulation with size 9, as shown in Fig. 5. Clearly, the triangles c_1fa_2, fa_2m_2 and nm_2c_2 are acute. We may assume that $c_1 = \mathbf{0}$, c_1a_2 is the x-axis and $l^{c_1}_{c_1a_2}$ is the y-axis. Denote by p the orthogonal projection of f on c_1n. By elementary calculations, we establish that $p = (\frac{3}{10}, \frac{3}{5})$ and $e = (\frac{7}{24}, \frac{29}{48})$, which implies that both angles nef and c_1ef are acute. Furthermore, it is easy to check that

$$|a_1e|^2 + |en|^2 = \frac{26^2 + 2 \cdot 19^2 + 10^2}{48^2} > \frac{3^2}{4^2} = |a_1n|^2,$$

and hence the angle a_1en is acute. Now, we first replace f by $f' = f + \varepsilon(b_1 - a_2)$ for small ε, such that the angle $nf'm_2$ becomes acute, and c_1f' is still a segment. Then we replace m_2 by $m_2' = m_2 + \eta(m_1 - e)$ with a small η, such that both angles $a_1m_2'e$ and $c_1m_2'e$ become acute. □

3. Main Result

Theorem. *Any flat Möbius strip can be triangulated with at most nine acute triangles, and this is the best possible bound.*

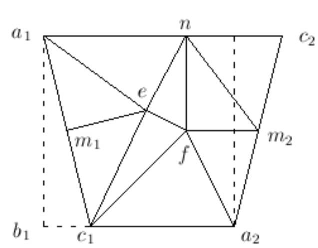

Fig. 5. $\alpha = 1$.

Proof. Propositions 1–3 imply the first part of the statement. Now we shall show that, if $\alpha < 1$, at least nine triangles are indeed needed for any acute triangulation.

Let $\mathcal{T}$ be an acute triangulation of a Möbius strip M_R with $\alpha < 1$, having i interior vertices, s side vertices and t triangles. Trivially $s \geq 3$. Since $\alpha < 1$, there is no transversal edge in $\mathcal{T}$. Hence $i \geq 1$.

If $i = 1$, then $s \geq 5$. Since each side vertex has degree at least 4, there must be at least one transversal edge emanating from it, which is impossible.

If $i = 2$, then $s \geq 4$. If $s = 4$, then both interior vertices have degree 5, and all side vertices have degree 4. So we have $3t + 4 = 2e = 26$, which is impossible. If $s \geq 5$, then by $3t + s = 2e \geq 10 + 4s$ we have $3t \geq 10 + 3s \geq 25$, whence $t \geq 9$.

If $i = 3$ and $s = 3$, then all the vertices of $\mathcal{T}$ have degree 5. So $3t + 3 = 2e = 30$ and thus $t = 9$.

If $i = 3$ and $s \geq 4$, then $3t + s = 2e \geq 15 + 4s$ implies that $3t \geq 15 + 3s \geq 27$ and $t \geq 9$.

If $i \geq 4$ and $s \geq 3$, then from $3t + s = 2e \geq 5i + 4s$ we can conclude that $3t \geq 5i + 3s \geq 29$, whence $t > 9$. □

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