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Hamiltonicity of Topological Grid Graphs [1]

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Abstract: In this paper we study connectivity and hamiltonicity properties of the topological grid graphs, which are a natural type of planar graphs associated with finite subgraphs of the usual square lattice graph of the plane. The main results are as follows. The shortness coefficient of the family of all topological grid graphs is at most 16/17. Every 3-connected topological grid graph is hamiltonian.
Key Words: grid graph, topological grid graph, hamiltonian graph, shortness coefficient, 3-connectedness
Category: G.2.2

1 Introduction

A *grid graph*, as defined in [Menke 1991] or [Menke et al. 1997], is a graph having as vertex set and edge set all lattice points and edges lying inside or on some (finite) cycle Γ of the infinite square lattice graph in the plane.

A *topological grid graph* is a graph of minimal degree 3, without loops and multiple edges, homeomorphic to some grid graph. For an example, see Fig. 1. The family $\mathcal{T}$ of all topological grid graphs appears in connection with the study of grid graphs. The outer cycle ∂G of a graph $G \in \mathcal{T}$ is the image through the above homeomorphism of the cycle Γ defining the corresponding grid graph.

Every graph in $\mathcal{T}$ has a natural representation with lattice points as vertices and with line segments of length 1 or $\sqrt{2}$ as edges.

We shall use the notion of a *shortness coefficient* of a family $\mathcal{G}$ of graphs, which is the greatest lower bound of all numbers a such that, for some sequence of distinct graphs $G_n \in \mathcal{G}$, and for some number b, $h(G_n) \le a|G_n| + b$ for all n, where $h(G_n)$ means the length of the largest cycle in G_n (compare [Grünbaum and Walther 1973]).

[1] C. S. Calude, G. Stefanescu, and M. Zimand (eds.). *Combinatorics and Related Areas. A Collection of Papers in Honour of the 65th Birthday of Ioan Tomescu.*

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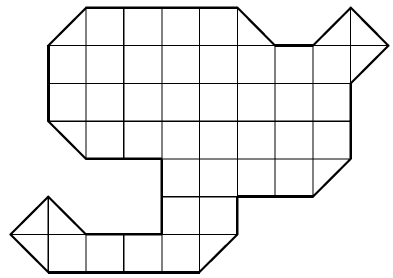

Figure 1: A topological grid graph

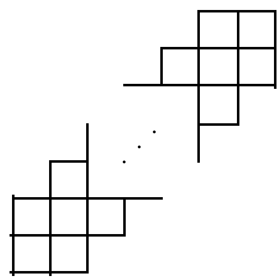

Figure 2: Shortness coefficient 4/5

Some properties of the longest cycles of grid graphs have been given in [Menke 1991] and [Menke et al. 1997]. The grid graphs may well fail to be hamiltonian. In fact, the shortness coefficient of the family of all grid graphs is at most 4/5, as the sequence of graphs described in Fig. 2 shows. We conjectured that it is exactly 4/5. This still open conjecture was recently proved for the subfamily of all convex grid graphs by D. Blankenagel [Blankenagel 1992].

The graphs in Fig. 2 fail to be cyclically 3-connected. This seems to be important to get the shortness coefficient 4/5, but not to make it smaller than 1. Indeed, the graphs depicted in Fig. 3 are cyclically 3-connected and form a family with shortness coefficient 5/6.

The topological grid graphs behave somewhat differently. While they may fail in a similar way to be hamiltonian and their shortness coefficient is less than 1, it will be proven here that in their case cyclic 3-connectedness suffices to insure hamiltonicity. So, for topological grid graphs connectedness properties

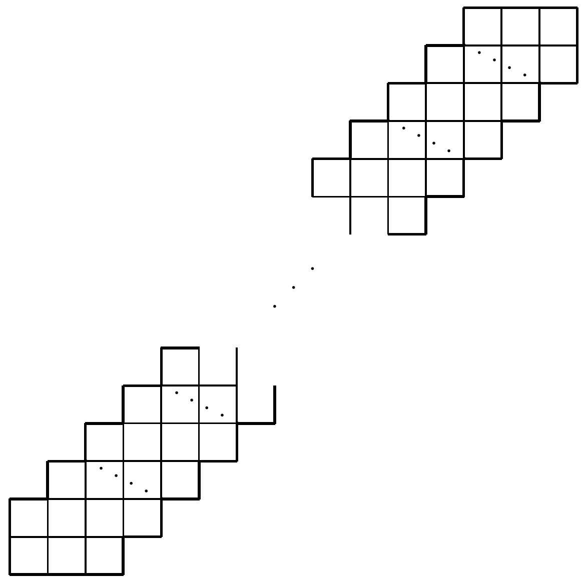

Figure 3: Shortness coefficient 5/6

play a greater role in guaranteeing increased hamiltonicity. Therefore we first establish a result about their connectedness.

2 Connectedness

We consider in this section an arbitrary topological grid graph G. Let Δ denote the bounded component of $\mathbb{R}^2 \setminus \mathrm{set}(\partial G)$.

Remark. Let $(v_1, v), (v, v_2) \in E(\partial G)$. It follows directly from the definition that the angle between these two edges (towards Δ) lies between $\pi/2$ and $3\pi/2$. Moreover, this angle equals $\pi/2$ only if $(v_1, v), (v, v_2)$ have length $\sqrt{2}$, and it equals $3\pi/2$ only if $(v_1, v), (v, v_2)$ have length 1.

Lemma 1. *Suppose two consecutive edges (v_1, v), (v, v_2) of length $\sqrt{2}$ in $E(\partial G)$ are perpendicular. Then the common neighbour of v, v_1, v_2 in the lattice graph (i.e., the midpoint of the line segment v_1v_2) belongs to $V(G - \partial G)$.*

Proof. Let v' be the common neighbour of v, v_1, v_2. It follows directly from the fact that G is homeomorphic to a grid graph that the interior int vv' of vv' cannot lie in the unbounded component of the complement of $\mathrm{set}(\partial G)$. Thus, int $vv' \subset \Delta$.

If $v' \notin V(G - \partial G)$ then $v' \in V(\partial G)$ and either a point of degree 2 in G is produced or two edges of length $\sqrt{2}$ parallel to the previous ones meet at v' and belong to ∂G, which contradicts, however, the preceding Remark. Hence $v' \in V(G - \partial G)$.

Lemma 2.

$$G - \partial G \neq \emptyset.$$

Proof. Suppose this is not true. Choose then a graph $G \in \mathcal{T}$ with $G - \partial G = \emptyset$ and with set(∂G) of minimal (Euclidean) length. The points of set$G$ of largest ordinate determine a family of paths in ∂G. If P is one of them then, by Lemma 1, P has more than one vertex. Obviously, the angles of set(∂G) at the endpoints of set$P$ towards Δ measure $3\pi/4$ each and the points $v_1, ..., v_k$ of the lattice, at distance 1 from those of P and right below them, must lie on ∂G. If one of the edges (v_i, v_{i+1}) $(1 \leq i < k)$ is not in $E(\partial G)$ then the removal of v_i, v_{i+1} disconnects G. In this case the removal of the component not containing P results in a topological grid graph G' with $G' - \partial G' = \emptyset$, having a smaller length of set$(\partial G')$, and a contradiction is obtained. So all edges (v_1, v_2), (v_2, v_3),...,(v_{k-1}, v_k) belong to ∂G. The angles of set(∂G) at v_1 and v_k towards Δ measure $5\pi/4$ or $3\pi/2$. Consider the points $w_1, w_2, ..., w_k$ at distance 1 from $v_1, v_2, ..., v_k$ and lying below them. We delete the point w_j $(j = 1$ or $k)$ if $(v_j, w_j) \in E(G)$. The remaining points form a set W. If $W \cap V(G) \neq \emptyset$ then $W \cap V(G)$ determines a family of paths and we pick up again one of them. We repeat the same procedure as many times as possible. We finally arrive at two paths $Q_1, Q_2 \subset \partial G$ of equal lengths, with Q_2 below Q_1 at distance 1, such that the angles of set(∂G) at the endpoints of Q_1 towards Δ measure $3\pi/4$ and those at the endpoints of Q_2 towards Δ measure $5\pi/4$ or $3\pi/2$, and such that no point below Q_2 at distance 1 from some vertex of Q_2 belongs to $V(G)$ except perhaps those under the endpoints of Q_2 in case those vertical edges belong to G. Let $a_i, b_i \in V(\partial G) \setminus V((Q_i)$ be neighbours of the endpoints of Q_i (a_2 under a_1 or under an endpoint of Q_2, b_2 under b_1 or under the other endpoint of Q_2). We replace now the path Q'_i between a_i and b_i satisfying $Q_i \subset Q'_i \subset \partial G$ with the rectilinear path from a_i to b_i (i = 1, 2). Thus we get another graph $G' \in \mathcal{T}$ with $G' - \partial G' = \emptyset$ and smaller length of $\partial G'$. This contradiction achieves the proof.

Lemma 3. *If $G - \partial G$ is connected then each vertex of ∂G has a neighbour in $G - \partial G$.*

Proof. By Lemma 2, $G - \partial G \neq \emptyset$. Suppose $G - \partial G$ is connected but some point $v \in V(\partial G)$ has no neighbour in $G - \partial G$. By Lemma 1, v has a neighbour in ∂G at distance 1 or both neighbours are at distance $\sqrt{2}$ from v and collinear with v.

Two edges (u, v), (v, w) belong to ∂G. Besides u and w there exists another neighbour v' of v in G. Then $v' \in V(\partial G)$. We may suppose without loss of generality that v' lies below v and $V(G - \partial G)$ lies in the component of $\Delta - vv'$ containing points of the plane close to the line segment vv', on its left side.

Suppose v is chosen such that the path from v to v' on ∂G, which together with (v, v') surrounds $G - \partial G$, has maximal length.

There are four cases to treat, depicted in Fig. 4 a), b), c), d), according to the position of the edge (v, w). For each case there may be several subcases according to the position of the edge $(v', w') \in E(\partial G)$.

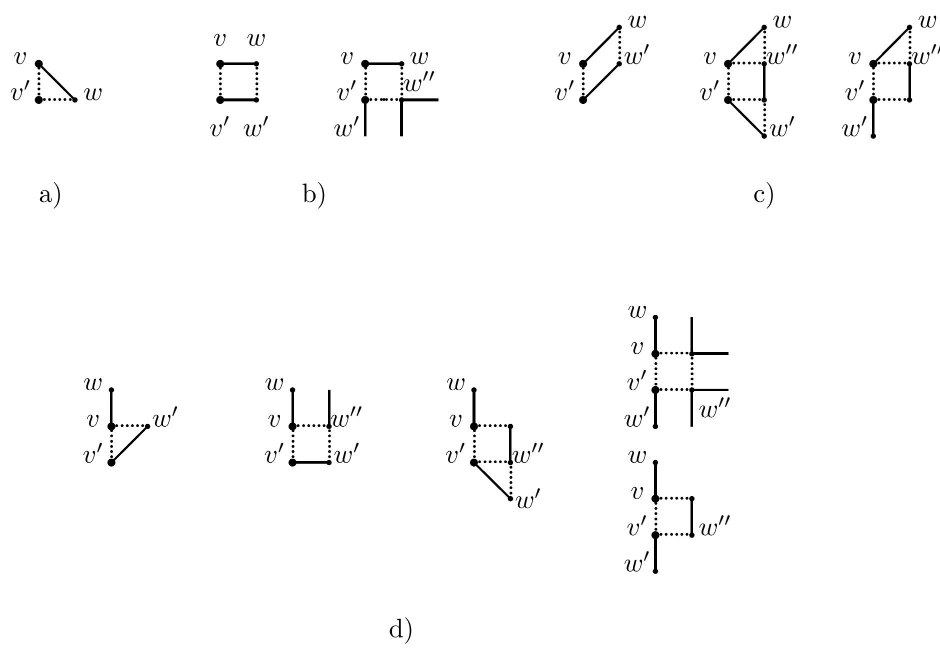

Figure 4: Four cases

In each subcase we are able to find another pair of points instead of v, v', connected by a longer path in ∂G which, together with the edge between its endpoints, surrounds $G - \partial G$. In situation a) the pair becomes w, v'. In situation b) the new pair is w, w' or w, w'' according to whether the edge (v', w') is horizontal or vertical. In situation c) the pair will be w, w' for the first or w, w'' for the second and third possibility. In situation d) the pair becomes v, w' in the first case, w'', w' in the second and third case, and w'', v' in the last case.

This is a contradiction and the proof is finished.

Lemma 4. *If $G - \partial G$ is connected then, for any two points $x, y \in V(G - \partial G)$,* *there is a path from x to y in $G - \partial G$, three paths from x to ∂G and three paths from y to ∂G, all seven paiwise disjoint (except for the points x, y as endpoints) and lying (except possibly for one vertex) in $G - \partial G$, or there is a path from x to y in $G - \partial G$, another one from x to y, two paths from x to ∂G and two paths from y to ∂G, all six pairwise disjoint (except for the points x, y as endpoints) and lying (except possibly for one vertex) in $G - \partial G$.*

Proof. Consider a shortest path P joining x and y in $G - \partial G$. Now go away from x rectilinearly in the four possible directions. In precisely three directions we meet ∂G before meeting $P - x$ (or meet only ∂G), otherwise P would not have minimal length. Let x_1, x_2, x_3 be those points on ∂G, and let y_1, y_2, y_3 be analogously defined, starting from y.

At most once it may happen that a line segment xx_i meets a line segment yy_j. Indeed, if this happened twice, then P would lie in the rectangle determined by the four line segments; but then $P - x$ would necessarily meet one of the two involved line segments xx_i, a contradiction.

Thus, if some xx_k meets some yy_l at z, then P, the path Q having set $Q = xz \cup zy$, the two rectilinear paths from x to x_i $(i \neq k)$ and the two rectilinear paths from y to y_j $(j \neq l)$ satisfy the conditions of the lemma.

If no xx_i meets any yy_j, then P, the three rectilinear paths from x to x_1, x_2, x_3 and the three rectilinear paths from y to y_1, y_2, y_3 are the paths we look for.

Theorem 1. *For a topological grid graph G the following assertions are equivalent.*

(i) *G is 3-connected,*

(ii) *G is cyclically 3-connected,*

(iii) *$G - \partial G$ is connected.*

Proof. (i) implies (ii) by definition.

We show now that (ii) implies (iii). Suppose $G - \partial G$ has at least two components G_1, G_2. Every point of the lattice at distance 1 from $V(G_i)$ belongs to $V(\partial G)$ $(i = 1, 2)$. However, not every edge of length 1 or $\sqrt{2}$ joining two such points belongs to $E(\partial G)$, otherwise ∂G would contain smaller cycles. The removal of the endpoints of such an edge decomposes G in two components, each of which has several cycles, and this contradicts the cyclic 3-connectedness of G.

Finally, let us prove that (i) follows from (iii). Let $x, y \in V(G)$. We shall find three internally disjoint paths from x to y. First, suppose $x, y \in V(G - \partial G)$. By Lemma 4, there is a path P from x to y in $G - \partial G$, two paths P'_i from x to $x_i \in V(\partial G)$ and two paths P''_j from y to $y_j \in V(\partial G)$ $(i, j = 1, 2)$, all five pairwise disjoint (except for the points x and y as endpoints). There are two disjoint paths Q_1, Q_2 in ∂G from some x_i to some y_j, say from x_1 to y_1 and from x_2 to y_2. Then P, $P'_1 \cup Q_1 \cup P''_1$ and $P'_2 \cup Q_2 \cup P''_2$ are pairwise internally disjoint.

Now, let $x \in V(G - \partial G)$, $y \in \partial G$. By Lemma 3, y has a neighbour $y' \in V(G - \partial G)$. By Lemma 4 there is a path P' from x to y' in $G - \partial G$ and another two paths P_1, P_2 from x to $x_1, x_2 \in \partial G$ respectively, all three pairwise disjoint (except for the point x as an endpoint). Let P be P' plus the vertex y and the edge (y, y'). Let $Q_1, Q_2 \subset \partial G$ be internally disjoint paths from y to x_1, x_2 respectively. Then P, $P_1 \cup Q_1$, $P_2 \cup Q_2$ are internally disjoint and join x to y.

Finally, if $x, y \in \partial G$, consider the two paths P_1, P_2 from x to y on ∂G. By Lemma 3, x has a neighbour $x' \in V(G - \partial G)$ and y has a neighbour $y' \in V(G - \partial G)$. Since $G - \partial G$ is connected there is a (maybe degenerate) path from x' to y' in $G - \partial G$, which can obviously be extended to a path from x to y internally disjoint from both P_1 and P_2.

Hence G is 3-connected.

3 Hamiltonian properties

As we already mentioned, the family $\mathcal{T}$ of all topological grid graphs, like that of all grid graphs, has shortness coefficient less than 1.

Theorem 2. *The shortness coefficient of* $\mathcal{T}$ *is at most* 16/17.

Proof. It suffices to present a sequence of graphs in $\mathcal{T}$ with shortness coefficient 16/17. Such a sequence is illustrated in Fig. 5.

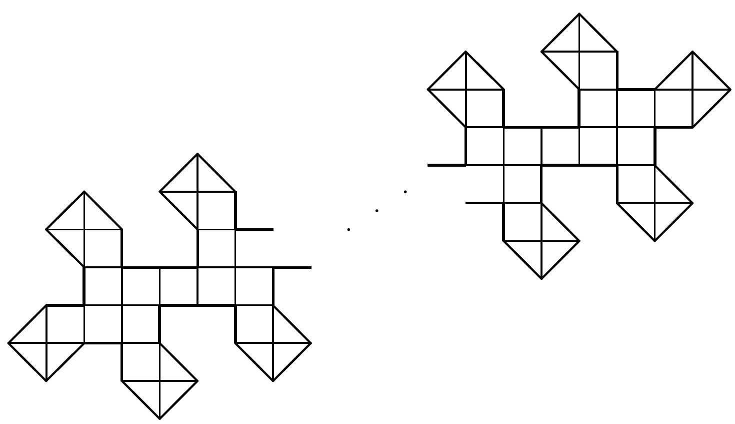

Figure 5: Shortness coefficient 16/17

The graphs used in the preceding proof are not cyclically 3-connected. Unlike the grid graphs, all cyclically 3-connected topological grid graphs are hamiltonian. By Theorem 1 this is equivalent to saying that all 3-connected graphs in $\mathcal{T}$

are hamiltonian. To show this we make use of the following result, first observed by D. Nelson.

Lemma 5. *The deletion of any vertex from a 4-connected planar graph results in a hamiltonian graph.*

This is Theorem 1.7.7 in [Voss 1991] and follows from Tutte's well-known result from [Tutte 1956].

Theorem 3. *Every 3-connected topological grid graph is hamiltonian.*

Proof. Let G be a 3-connected topological grid graph. By Theorem 1, $G-\partial G$ is connected. Let v be a new vertex outside of ∂G, i.e., in the complement of the closure of Δ. Adding to G the vertex v and all edges joining v with the vertices of ∂G yields a graph G^*. This graph is obviously planar and we show that it is 4-connected.

Let x, y be two arbitrary points in G^*. There are 5 essentially different cases to treat.

Case 1. $x, y \in V(G-\partial G)$. We apply Lemma 4 and find

(i) a path P from x to y in $G-\partial G$, three paths P_i' from x to $x_i \in V(\partial G)$ and three paths P_j'' from y to $y_j \in V(\partial G)$ $(i,j=1,2,3)$, or

(ii) two paths P_1, P_2 from x to y with $P_1 \subset G-\partial G$ and $P_2 \cap \partial G$ consisting of one vertex at most, two paths P_i' from x to $x_i \in V(\partial G)$ and two paths P_j'' from y to $y_j \in V(\partial G)$ $(i,j=1,2)$.

All these paths enjoy the properties required in Lemma 4.

In the situation (i), because of the planarity of G, we may suppose without loss of generality that $x_1, x_2, x_3,\ y_3, y_2, y_1$ lie in this order on ∂G. Let Q_1, Q_3 be the disjoint paths joining x_1 to y_1 and x_3 to y_3. Let R be the path with vertices x_2, v, y_2. Then x and y are joint by the four internally disjoint paths P, $P_1' \cup Q_1 \cup P_1''$, $P_2' \cup R \cup P_2''$, $P_3' \cup Q_3 \cup P_3''$.

In the situation (ii), we may suppose that $x_1, x_2,\ y_2, y_1$ are in this order on ∂G. Then let $Q_i \subset \partial G$ be paths from x_i to y_i $(i=1,2)$ so that $Q_1 \cap Q_2 = \emptyset$. At most one of them, say Q_2, contains $P_2 \cap \partial G$. Let R be the path with vertices x_2, v, y_2. Then the paths P_1, P_2, $P_1' \cup Q_1 \cup P_1''$, $P_2' \cup R \cup P_2''$ verify our requirements.

Case 2. $x \in V(G-\partial G), y \in V(\partial G)$. By Lemma 3, there exists a neighbour $y' \in V(G-\partial G)$ of y. Let P be a shortest path from x to y' in $G-\partial G$. Then there are three rectilinear paths P_i from x to $x_i \in V(\partial G)$ $(i=1,2)$ having with P only the endpoint x in common. We may suppose that $x_1, x_2,\ x_3, y$ lie in this order on ∂G. Let $Q_i \subset \partial G$ be paths from x_i to y $(i=1,3)$ so that Q_1 and Q_3 have only the endpoint y in common. Let R be the path with vertices x_2, v, y. Let P' be the path whose points are those of P and y. Then x and y can be joined by the following four internally disjoint paths: P', $P_1 \cup Q_1$, $P_2 \cup R$, $P_3 \cup Q_3$.

Case 3. $x \in V(G-\partial G), y = v$. Let P_i be the four rectilinear paths from x

to ∂G ($i = 1, ..., 4$). Their endpoints different from x are neighbours of v, so the paths P_i can be extended to four internally disjoint paths from x to v.

Case 4. $x, y \in V(\partial G)$. By Lemma 3, x and y have neighbours in $G - \partial G$. Since $G - \partial G$ is connected, there is a path P joining x to y whose interior points lie in $G - \partial G$. Moreover x and y are the endpoints of two paths Q_1, Q_2 in ∂G. Let R be the path of vertices x, v, y. Then P, Q_1, Q_2, R are the four paths we are looking for.

Case 5. $x \in V(\partial G), y = v$. By Lemma 3, x has a neighbour x' in $G - \partial G$. The rectilinear path P starting at x and going through x' meets again ∂G at x'', say, where it ends. Obviously x'' is not a neighbour of x. Let x_1, x_2 be the neighbours of x on ∂G. Then the paths with vertex sets $\{x, v\}$, $\{x, x_1, v\}$, $\{x, x_2, v\}$, $V(P) \cup \{v\}$ satisfy our conditions.

Hence G^* is 4-connected and, by Lemma 5, the graph $G^* - v = G$ is hamiltonian. This achieves the proof of Theorem 3.

The condition of 3-connectedness is equivalent by Theorem 1 to the connectedness of $G - \partial G$. Theorem 3 can be strengthened in the following way.

Theorem 4. *If G is a topological grid graph and $G - \partial G$ has at most two components, then G is hamiltonian.*

Proof. We first remark that the choice of the pair of adjacent vertices $u_1, u_2 \in \partial G$ at the end of the preceding proof was arbitrary.

Now, let G_1, G_2 be the components of $G - \partial G$ (if there is just one component, we use Theorem 3). Obviously, the vertices of the infinite square lattice at distance 1 from $V(G_i)$ determine a cycle C_i ($i = 1, 2$). Then $\partial G \cap C_i$ is a path P_i with the endpoints v_i, v_i' at Euclidean distance 1 or $\sqrt{2}$ from each other ($i = 1, 2$).

If v_1, v_1', v_2, v_2' are all distinct, then they determine four paths on ∂G, namely P_1, P_2 and other two paths, P, P', with endpoints v_1, v_2 and v_1', v_2', say. If, for example, $v_1' = v_2'$, then, with the above notation, P' reduces to a single point. Finally, if $\{v_1, v_1'\} = \{v_2, v_2'\}$ then both P, P' reduce to single points.

In the first case (v_1, v_1', v_2, v_2' distinct), let C be the cycle determined by P, P' and the edges (v_1, v_1'), (v_2, v_2'). In the second case (three of the points v_1, v_1', v_2, v_2' are distinct), let C be the cycle determined by P and the edges (v_1, v_1'), (v_2, v_2').

In the first two cases, C_i and C have the common edge (v_i, v_i') ($i = 1, 2$). In the third case, C_1 and C_2 have the common edge (v_1, v_1').

By Theorem 3 together with our remark at the beginning of this proof, the graph spanned by $G_i \cup C_i$ has a hamiltonian cycle H_i containing the edge (v_i, v_i'). Then, in the first two cases, $H_1 \cup C \cup H_2$ minus the edges (v_1, v_1'), (v_2, v_2') is a hamiltonian cycle in G. In the third case, $H_1 \cup H_2$ minus the edge (v_1, v_1') is a hamiltonian cycle in G.

This construction of a hamiltonian cycle can be viewed as an illustration of the characterization of hamiltonian graphs given in [Zamfirescu 1974].

The smallest example of a nonhamiltonian graph in $\mathcal{T}$ we were able to produce is a member G_1 of the family shown in Fig. 5 and $G_1 - \partial G_1$ has 5 components. This remark and Theorem 4 lead to the following.

Open problem. Does there exist a nonhamiltonian graph $G \in \mathcal{T}$ such that $G - \partial G$ has 4 or even 3 components?

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