PLANAR LATTICE GRAPHS WITH GALLAI'S PROPERTY

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Introduction. The property of the Petersen graph of being hypohamiltonian is notorious, and known for a long time. The existence of hypotraceable graphs, on the contrary, was discovered much later. Before that, in 1966, Gallai [1] had asked the easier question whether (connected) graphs do exist such that each vertex is missed by some longest path. Such graphs have been discovered by Walther in 1969 [6] before J. Horton found the first hypotraceable graph. In this context appeared Zamfirescu's questions about the existence of (small, if possible minimal) k-connected graphs with the property that for any j vertices there is a longest path avoiding all of them [9]. The questions were also asked for cycles instead of paths, and, on the other hand, specialized to planar graphs.

Since several good examples answering Zamfirescu's questions were published in the following years by Grünbaum [2], Schmitz [5], Zamfirescu [10], [11], Zamfirescu and Zamfirescu [8] and others, an opposite question was asked: Are there large families of graphs without any such examples? B. Menke discovered one such family, the *grid graphs*. These are (finite) subgraphs of the infinte square lattice graph $\mathcal{L}$ in the plane consisting of all vertices and all edges lying on or within a cycle in $\mathcal{L}$. Menke proved that every grid graph has at least 4 vertices lying on every longest cycle ([3], see also [4]).

Motivated by this result, we investigate here other graphs embedded in $\mathcal{L}$ concerning the properties mentioned above. Moreover, we shall extend our investigation to graphs embedded in the (infinite) hexagonal lattice $\mathcal{H}$. We find out that graphs with the mentioned properties do exist (for $j = 1$), although their orders are considerably larger.

Graphs with empty intersection of their longest paths. Let G be a graph homeomorphic to the graph G' in Fig. 1. For each edge of G' the corresponding path of G has a number of vertices of degree 1 or 2 shown on Fig. 1 as well.

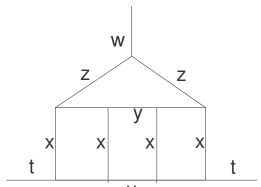

Figure 1

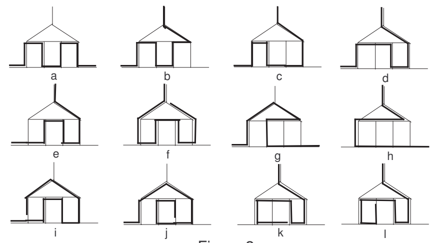

Figure 2

Lemma 1 *The longest paths of* G *have empty intersection if* $2x \geq y + 2z + 1$, $t \geq y + 2z + 1$, $t \geq x + z + 1$, *and* $w = x + t - z$.

Proof. The paths which are candidates for being longest in G are shown in Fig 2. The longest paths of G have empty intersection if the paths of Fig. 2, a), b), c), d) and e) are among the longest ones, i.e. they have equal lengths and the paths of Fig. 2, f), g), h), i), j), k) and l) are not longer. The number of vertices of the paths in Fig. 2, a), b), f), g), h), i), j), k) and l) are, respectively,

$a = 4x + y + 2t + 8$,
$b = 3x + y + z + t + w + 8$,
$f = 4x + y + 2z + w + 9$,
$g = 2x + 2y + 2z + 2t + 9$,
$h = x + 2y + z + t + w + 9$,
$i = 4x + y + 2z + t + 9$,
$j = 4x + 2y + 2z + t + 9$,
$k = 3x + 2y + z + w + 9$,
$l = 4x + y + z + w + 9$.

We easily see that the paths in Fig 2, b), c), d) and e) have equal lengths. Thus, the required conditions become

$w = x + t - z$, $t \geq z + x + 1$, $2x \geq y + 2z + 1$, $2x \geq y + 1$, $t \geq 2z + 1$, $t \geq y + 2z + 1$, $t \geq y + 1$, $t \geq x + 1$.

The first, second, third and sixth condition above are those in the statement. The others follow. □

Theorem 1 *There exists a graph of order* 46 *embedded in* $\mathcal{L}$ *such that its longest paths have empty intersection.*

Proof. If $y = 1, z = 2, x = 3, t = 6$ and $w = 7$, then the conditions in Lemma 1 are verified. Thus the corresponding graph G has the stated property. Fig. 3 reveals an embedding of G in $\mathcal{L}$. □

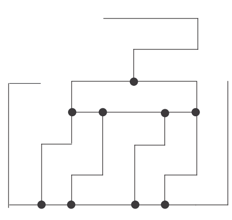
Figure 3

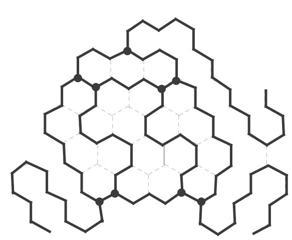
Figure 4

Theorem 2 *The hexagonal lattice* $\mathcal{H}$ *has a subgraph of order* 94 *such that its longest paths have empty intersection.*

Proof. The conditions of lemma 1 are also satisfied if we take $y = 3, z = 3, x = 8, t = 12$ and $w = 17$, and the resulting graph G has order 94. Fig. 4 shows an embedding of G in $\mathcal{H}$. □

It can be noticed that our examples so far were not 2-connected. As in the original questions from [9], one can demand the graph with empty intersection of its longest paths to have higher connectivity. Very high it cannot be, because every finite subgraph of $\mathcal{L}$ or $\mathcal{H}$ must have vertices of degree at most 2.

Consider the graph H homeomorphic to the graph H' in Fig. 5, where each path of H corresponding to an edge of H', has a number of vertices of degree 2 shown on Fig. 5 as well. Another graph homeomorphic to H' was used in [10].

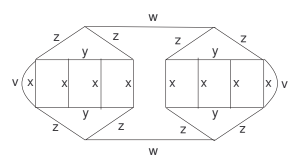

Figure 5

Lemma 2 *Let* $x \geq v$*. The longest paths of* H *have empty intersection if the following conditions are fulfilled.*

(i) $v \geq y + 2z + 1$,

(ii) $x + v = y + 2z + w + 1$.

The proof goes along the same lines as the one of Lemma 1, and is therefore omitted.

Theorem 3 *In* $\mathcal{L}$ *there exists a 2-connected subgraph of order* 126*, whose longest paths have empty intersection.*

Proof. We use a particular case of Lemma 2. By setting $y = 1, z = 2, w = 8$, and $x = v = 7$, we get a graph H of order 126 and with the desired property. One of its embedding in $\mathcal{L}$ is shown in Fig. 6. □

The graph H and Lemma 2 cannot be used for an embedding in $\mathcal{H}$, because H has vertices of degree 4. Therefore, we consider the graph K' in Fig. 8 (left side), which has three subgraphs isomorphic to G', and the graph K homeomorphic to the graph K', where x, y, z, t and w are numbers of vertices of degree 2, as before. □

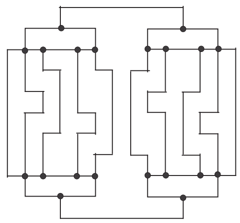

Figure 6

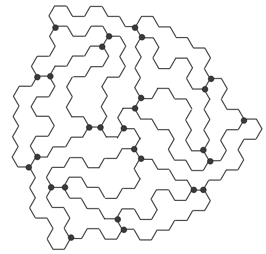

Figure 7

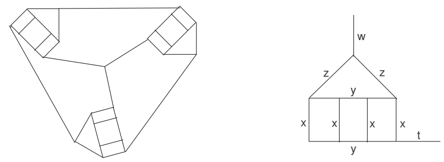

Figure 8

Lemma 3 *The longest paths of K have empty intersection if $y \geq 1$ and $x = y + z \geq w = y + 2t + 1$.*

The proof, similar to the proof of Lemma 1, is omitted.

Theorem 4 *The lattice $\mathcal{H}$ contains a subgraph of order 244 whose longest paths have empty intersection.*

*Proof***.** It is clear that K with $y = z = 5$, $x = w = 10$ and $t = 2$ satisfies the conditions of Lemma 3. It has order 244 and is embeddable in $\mathcal{H}$, as one can verify by inspecting Fig. 7. □

Conjecture 1 *The orders 46, 94, 126 and 244 of the graphs presented in Theorems $1 - 4$ are minimal.*

Graphs with empty intersection of their longest cycles. When dealing with longest cycles, the adequate frame is that of 2-connected graphs.

Let L be a graph homeomorphic to the graph L' in Fig. 9. For each edge of L', the corresponding path of L has a number of vertices of degree 2 shown on Fig. 9 as well. For $x = z = 0$ and $y = 1$, the graph L was found by Thomassen (see [10]) and is to date the smallest known example of a planar 2-connected graph with empty intersection of its longest cycles.

Lemma 4 *The longest cycles of L have empty intersection if $x = z$ and $2y \geq 3x + 1$.*

Lemma 4, the proof of which is left to the reader, can be used to prove G. Wegner's result (see [3], p. 202) establishing the existence of a graph of order 95 embedded in $\mathcal{L}$ such that its longest cycles have empty intersection.

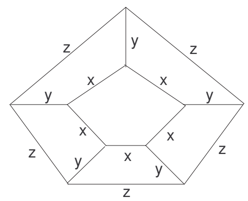

Figure 9

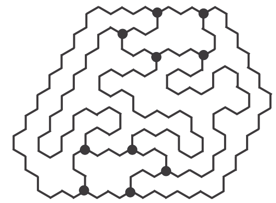

Figure 10

Theorem 5 *There exists a subgraph of* $\mathcal{H}$ *of order* 170 *such that its longest cycles have empty intersection.*

Proof. Consider the graph L of Lemma 4 with $x = 3$ and $y = 26$. Since the conditions of Lemma 4 are verified, the longest cycles of L have empty intersection. Fig. 10 reveals an embedding of L in $\mathcal{H}$. □

Conjecture 2 *The orders* 95 *of Wegner's example and* 170 *of the graph presented in Theorems* 5 *are minimal.*

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References

1. T. Gallai, Problem 4, in: *Theory of Graphs*, *Proc. Tihany* 1966 (ed: P. Erdös & G. Katona), Academic Press, New York, 1968, 362.
2. B. Grünbaum, Vertices missed by longest paths or circuits, *J. Comb. Theory A* **17** (1974) 31-38.
3. B. Menke, Longest cycles in grid graphs, *Studia Sci. Math. Hung.* **36** (2000) 201-230.
4. B. Menke, C. Zamfirescu, T. Zamfirescu, Intersection of longest cycles in grid graphs, *J. Graph Theory* **25** (1997) 37-52.
5. W. Schmitz, Über längste Wege und Kreise in Graphen, *Rend. Sem. Mat. Univ. Padova* **53** (1975) 97-103.
6. H. Walther, Über die Nichtexistenz eines Knotenpunktes, durch den alle längsten Wege eines Graphen gehen, *J. Comb. Theory* **6** (1969) 1-6.
7. H. Walther, H.-J. Voss, *Über Kreise in Graphen*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1974.
8. C. T. Zamfirescu, T. I. Zamfirescu, A planar hypohamiltonian graph with 48 vertices, *J. Graph Theory* **55**(4) (2007) 338-342.
9. T. Zamfirescu, A two-connected planar graph without concurrent longest paths, *J. Combin. Theory* B **13** (1972) 116-121.
10. T. Zamfirescu, On longest paths and circuits in graphs, *Math. Scand.* **38** (1976) 211-239.
11. T. Zamfirescu, Intersecting longest paths or cycles: A short survey, *Analele Univ. Craiova*, *Seria Mat. Inf.* **28** (2001) 1-9.