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Lattice graphs with Gallai's property

by
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Dedicated to the memory of Nicolae Popescu (1937-2010)
on the occasion of his 75th anniversary

Abstract

We investigate graphs with the property that all longest paths or all longest cycles have empty intersection. In this paper, we find such graphs as subgraphs of cubic lattices.

Key Words: Longest paths, longest cycles, Gallai's property, cubic lattices.
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1 Introduction

The problem we treat goes back to the following question of T. Gallai [1] from 1966. Do there exist connected graphs such that every vertex is missed by some longest path? Soon an example of such a graph has been constructed by H. Walther [8]. Walther's example had connectivity 1.

In [10] it was asked about examples with higher connectivity. Such graphs have been subsequently found, up to connectivity number 3 ([2], [11], [12]). The problem whether 4-connected graphs with the property asked by Gallai (with respect to paths or cycles) do or do not exist is still unsolved.

In this paper, all graphs are connected.

There exist large classes of graphs without Gallai's property. S. Klavžar and M. Petkovšek proved that split graphs and cacti are among them [3].

B. Menke [4] considered the following class of graphs. In the usual infinite planar square lattice $\mathcal{L}^2$ consider a finite cycle and all vertices and edges lying on or inside that cycle. Such graphs are called *grid graphs*, and Menke proved that no grid graph has Gallai's property. See also [5].

Motivated by Menke's negative result, F. Nadeem, A. Shabbir and T. Zamfirescu considered the family of all graphs embeddable in $\mathcal{L}^2$, and found that Gallai's question again receives a positive answer. However these graphs have considerably larger order than those which do not satisfy the required embeddability condition. Our goal here is to provide examples of graphs with Gallai's property embeddable in higher-dimensional (cubic) lattices, and of smaller order.

Our investigation has also relevance in applications. Suppose several processing units are interlinked as parts of a lattice network. Some of them forming a chain of maximal length are used to solve a certain task. To get a self-stable fault-tolerant system, it is necessary that in case of failure of any unit or link, another chain of same length, not containing the faulty unit or link, can replace the chain originally used. This is exactly the case investigated here.

We denote by $\mathcal{L}^n$ the n-dimensional cubic lattice in $\mathbb{R}^n$.

2 Embeddings with empty intersection of all longest paths

The smallest known example of a graph, the longest paths of which have empty intersection, without any restrictions, was independently exhibited by Voss and Walther [9], and Zamfirescu [11].

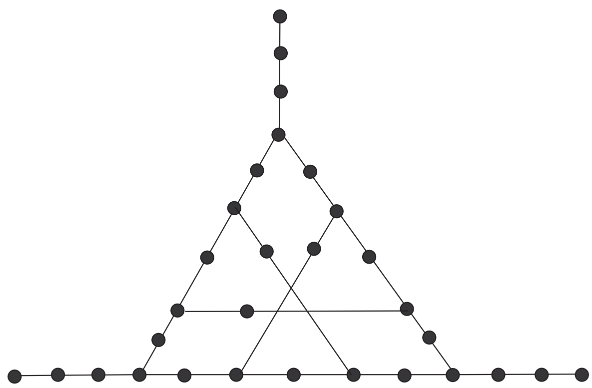

Figure 1

Trying to use it, we found the graph of Fig. 1, homeomorphic to the above graph. It also enjoys the stated property and, moreover, is embeddable in the 3-dimensional cubic lattice. It has order 30. However, another graph, Schmitz' graph of Fig. 2, the smallest known planar graph with Gallai's property with respect to paths, is better suitable, having smaller order [7]. While Schmitz' graph [7] is not embeddable in $\mathcal{L}^2$, it admits an embedding in $\mathcal{L}^3$!

But why is it not embeddable in $\mathcal{L}^2$? One of the regions into which it decomposes the plane is the hexagon with c, d, h, g among its vertices (see Fig. 2). This admits a unique obvious realization in $\mathcal{L}^2$. There is no place inside it for any other vertex. Another region is the hexagon with d, e, f, g, h among its vertices. Also this one admits the same kind of realization. But the path from h to i must be accommodated in one of these regions! Which is impossible.

Theorem 1. *There exists a graph of order* 17*, embeddable in* $\mathcal{L}^3$ *and having empty intersection of its longest paths.*

Proof: Consider Schmitz' graph G in Fig. 2. The order of G is 17, and for every vertex there exists some longest path of length 12 avoiding it [7].

Fig. 3 shows an embedding of G in $\mathcal{L}^3$. ∎

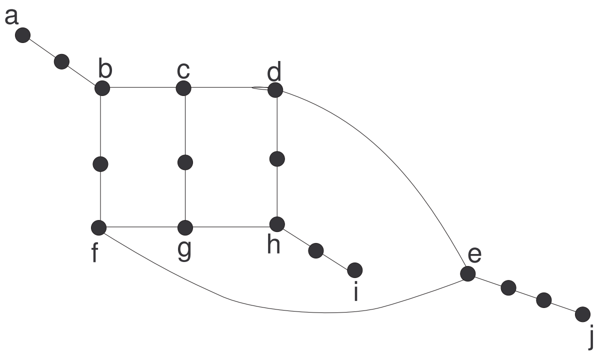

Figure 2

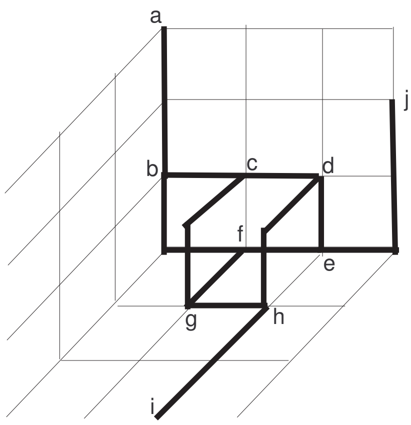

Figure 3

Compare Theorem 1 with the result in [6] about the existence of a graph with the same property, but embeddable in $\mathcal{L}^2$. That graph has 46 vertices.

3 Embeddings with empty intersection of all longest cycles

In the preceding section we dealt with graphs enjoying Gallai's property regarding longest paths. Now we consider graphs in which every vertex is avoided by some longest cycle.

Theorem 2. *There exists a graph of order* 40*, embeddable in* $\mathcal{L}^3$ *and having empty intersection of its longest cycles.*

Proof: Consider the graph shown in Fig. 4. This graph is obtained from Petersen's graph by adding three vertices on each of its edges lying on the cycle $abcdea$ and on each edge lying on $fghijf$. This means, for example in case of ab, that we delete the edge ab, add three vertices x_1, x_2, x_3 and add the edges ax_1, x_1x_2, x_2x_3, and x_3b. Now the resulting graph G is bipartite, it has order 40 and its longest cycles have 30 vertices.

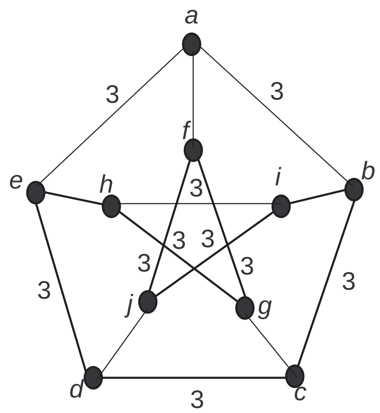

Figure 4

We verify now that every vertex of G is avoided by some longest cycle. To avoid the vertex a or an "interior" vertex of ab or hi we use the cycle in Fig. 5.

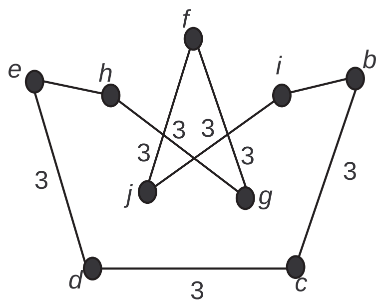

Figure 5

By symmetry, every vertex of G is avoided by some cycle of length 30.
In Fig. 6 we show an embedding of G in $\mathcal{L}^3$. □

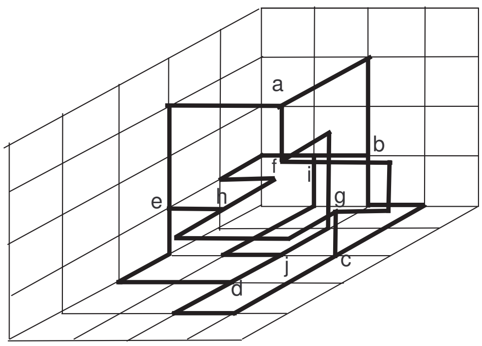

Figure 6

Compare Theorem 2 to Wegner's graph with the same property, but embeddable in $\mathcal{L}^2$ (see [4]). That graph has 95 vertices.

Until now we were considering graphs embeddable in $\mathcal{L}^3$. Can smaller examples be found in higher-dimensional lattices?

Theorem 3. *There exists a graph of order* 20*, embeddable in* $\mathcal{L}^4$ *and having empty intersection of its longest cycles.*

Proof: Consider Petersen's graph, and put one vertex on each edge of the cycle *abcdea* and on each edge of $fghijf$, see Fig. 7. The resulting graph of order 20, whose longest cycles have 16 vertices, is bipartite and embeddable in the 4-dimensional lattice $\mathcal{L}^4$. This graph contains a longest cycles corresponding to the cycle shown in Fig. 5, and hence possesses the empty intersection property.

Fig. 8 shows an embedding of this graph in $\mathcal{L}^4$. □

Compare Theorem 3 with Theorem 2, which provides a graph of twice larger order.

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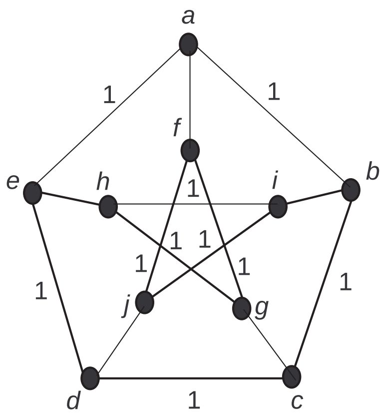

Figure 7

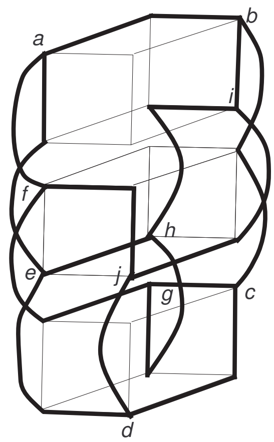

Figure 8

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