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Typical simplicially convex bodies

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Abstract. In this note we describe some geometrical properties that simplicially convex bodies typically enjoy. It is shown, for example, that they are nowhere dense and of measure zero. Moreover, they look at least half-dense from any of their points.

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During their long genesis, before having become convex, the convex bodies went through several interesting intermediate stages. This we learn from work of W. Bonnice and V. Klee [1], and the author [10], [11]. In those intermediate stages, while not yet convex, they were so-called simplicially convex bodies. A simplicially convex body is the union of all simplices of dimension less than some fixed number, with vertices in some fixed compact set. In this note we describe several unexpected properties that many of them enjoy. To do this, we use Baire categories as a main tool.

Generic results about compact sets can be found e.g. in [14], [4], [17]. Such results about starshaped compact sets, which are somewhat more closely related to the simplicially convex bodies, appeared e.g. in [15], [6], [17]. The subject of geometric generic properties in Convexity was treated in many papers, see — to pick only a few examples — [12], [13], [2], [5] or, more recently, [7].

Definitions and notation

A k-dimensional simplex, a *k-simplex*, or just a *simplex* for short, is the convex hull of $k+1$ affinely independent points in $\mathbb{R}^d$.

A set $A \subset \mathbb{R}^d$ is called *ℓ-simplicially convex* if, for some set $A' \subset \mathbb{R}^d$, A is the union $\mathcal{S}_\ell(A')$ of all simplices of dimension less than ℓ with vertices in A'. Every convex set is ℓ-simplicially convex, and every ℓ-simplicially convex set is convex if $\ell > d$. We obviously have

$$A = \mathcal{S}_1(A) \subset \mathcal{S}_2(A) \subset \cdots \subset \mathcal{S}_d(A) \subset \mathcal{S}_{d+1}(A) = \mathrm{conv}A.$$

A set A is said to be *simplicially convex* if it is ℓ-simplicially convex for some ℓ. Such a set is called a *simplicially* (*ℓ-simplicially*) *convex body* in case it is compact. It is easily seen that $\mathcal{S}_\ell(A)$ is compact if A is compact.

We continue here the geometric investigation started in [9]. There, among other things, it was proven that all "holes" of a d-simplicially convex body must be interiors of d-dimensional polytopes. After this somewhat surprising result, we shall provide here further insight on the possible number of "holes" and about their boundaries.

For $M \subset \mathbb{R}^d$, by $\mathrm{int}M$, $\mathrm{bd}M$, $\overline{M}$, $\mathrm{conv}M$ we denote the interior, boundary, closure, and convex hull of M. Let $\mathcal{C}$ be the complete metric space of all compact sets in $\mathbb{R}^d$, with the usual Pompeiu–Hausdorff distance. Any closed half-space H^+ with the origin $\mathbf{0}$ of $\mathbb{R}^d$ on its boundary will be called a $\mathbf{0}$-*half-space.*

We say that *most* (or *typical*) elements of a Baire space enjoy property $\mathbf{P}$ if those elements not enjoying $\mathbf{P}$ form a first category set. Sometimes we shall use the notion of "typical element" even before mentioning $\mathbf{P}$. The property $\mathbf{P}$ is said to be *generic*.

For any point $x \in \mathbb{R}^d$ and set $M \subset \mathbb{R}^d$, consider the subset

$$D_x(M) = \{\|y - x\|^{-1}(y - x) : y \in M \setminus \{x\}\}$$

of the unit sphere S^{d-1} in $\mathbb{R}^d$.

We say that $C \in \mathcal{C}$ *looks dense from* $x \in C$ if for any neighbourhood N of x, the set $D_x(N \cap C)$ is dense in S^{d-1} [17]. In particular, $C \in \mathcal{C}$ is said to *look full from* $x \in C$ if for any neighbourhood N of x, the set $D_x(N \cap C)$ equals S^{d-1}. We also say that $C \in \mathcal{C}$ *looks at least half-dense from* $x \in C$ if there is a $\mathbf{0}$-half-space H^+ such that for any neighbourhood N of x, the set $D_x(N \cap C)$ is dense in $S^{d-1} \cap H^+$ [17]. In particular, $C \in \mathcal{C}$ is said to *look at least half-full from* $x \in C$ if there is a $\mathbf{0}$-half-space H^+ such that for any neighbourhood N of x, the set $D_x(N \cap C)$ includes $S^{d-1} \cap H^+$.

For a convex body K, we denote by $\mathrm{ext}K$ its set of extreme points, as usual. For a simplicially convex body K, any bounded component of $\mathbb{R}^d \setminus K$ will be called a *hole* of K.

For a d-simplicially convex body K, we call $\mathrm{bdconv}K$ the *surface* of K. Clearly, $K \supset \mathrm{extconv}K$. If $K = \mathcal{S}_d(C)$, then $C \supset \mathrm{extconv}K$ too, and $\mathrm{conv}C = \mathrm{conv}K$, whence the surface of K also equals $\mathrm{bdconv}C$. The surface of K is included in K, because each of its points lies by Carathéodory's Theorem in a d-simplex Δ with vertices in C, more precisely in a $(d-1)$-dimensional face of Δ, because $\mathrm{int}\Delta \subset \mathrm{intconv}K$.

The open ball in $\mathbb{R}^d$ of centre x and radius r is denoted by $B(x, r)$. Similarly, the open ball in $\mathcal{C}$ of centre C and radius r is denoted by $\mathcal{B}(C, r)$.

Auxiliary results

We shall make use of the following five lemmas. The first four are known results, obtained by A. Wieacker and the author.

Lemma 1 ([9]). *Each hole of a d-simplicially convex body K is the interior of a (d-dimensional) polytope, whose boundary is included in K.*

Note that holes of other simplicially convex bodies might not be interiors of polytopes. The boundary of a bounded circular cone in $\mathbb{R}^3$, which is 2-simplicially convex, may serve as an example.

Lemma 2 ([17]). *Most elements of $\mathcal{C}$ look at least half-dense from any of their points.*

Lemma 3 ([8]). *A typical set $C \in \mathcal{C}$ contains no $d+1$ affinely dependent points.*

Recall that for any compact convex set K with $\int K \neq \emptyset$, if $x \in \mathrm{bd}K$, then there exists a closed half-space $H^+ \supset K$ with $x \in \mathrm{bd}H^+$; H^+ is called *supporting half-space* of K at x. If the supporting half-space at each point of $\mathrm{bd}K$ is unique, then K is said to be *smooth.*

The following corollary of Lemma 2 was found by Wieacker long before [17] appeared.

Lemma 4 ([8]). *For most $C \in \mathcal{C}$, $\mathrm{conv}C$ is smooth.*

A *facet* of a hole H means a $(d-1)$-dimensional face of $\overline{H}$.

Lemma 5. *For typical $C \in \mathcal{C}$, each facet of any hole of $\mathcal{S}_d(C)$ lies in a single $(d-1)$-simplex with vertices in C.*

Proof. Let C be typical and F be a facet of a hole of $\mathcal{S}_d(C)$. Let x be in the relative interior of F. If x lies in two different $(d-1)$-simplices with vertices in C, then these are both included in the hyperplane H of F, and it follows that there exist at least $d+1$ points in $C \cap H$, which contradicts Lemma 3.

Results

If C is finite, then $\mathcal{S}_d(C)$ has finitely many holes, and equals the union of their boundaries. Therefore it is nowhere dense and of zero (d-dimensional) Lebesgue measure. Is this true for many C's?

Theorem 1. *For most $C \in \mathcal{C}$, $\mathcal{S}_d(C)$ is nowhere dense and has measure zero.*

Proof. Let $\mathcal{C}^*$, $\mathcal{C}_\varepsilon$ be the subsets of $\mathcal{C}$ consisting of all compact sets C such that $\mathcal{S}_d(C)$ includes some (non-degenerate) ball or has non-zero outer measure, respectively includes a ball of radius ε or has outer measure at least ε. Then $\mathcal{C}^* = \bigcup_{n=1}^{\infty} \mathcal{C}_{1/n}$, and $\mathcal{C}^*$ is precisely the subset of $\mathcal{C}$ of all compact sets C for which $\mathcal{S}_d(C)$ is not nowhere dense or is of positive outer measure. We prove the theorem by showing that each $\mathcal{C}_{1/n}$ is nowhere dense.

Let C_0 be compact. Let F be a finite set approximating C_0. Since $\mathcal{S}_d(F)$ is a finite union of $(d-1)$-simplices, it contains no ball and has measure zero. For some small ε, $\mathcal{S}_d(F + B(\mathbf{0}, \varepsilon))$ too contains no ball of radius $1/n$ and has measure less than $1/n$. So, for every $C \in \mathcal{B}(F, \varepsilon)$, we have $C \notin \mathcal{C}_{1/n}$. Hence, $\mathcal{C}_{1/n}$ is nowhere dense, and the proof is finished.

Corollary. *Let $1 \le \ell \le d$. For most $C \in \mathcal{C}$, $\mathcal{S}_\ell(C)$ is nowhere dense and has measure zero.*

For $d = 1$, this is well-known. (One finds stronger results in [3], [4], [14].)

Theorem 2. *For most $C \in \mathcal{C}$, $\mathcal{S}_d(C)$ has infinitely many holes.*

Proof. Let $\mathcal{C}^* = \{C \in \mathcal{C} : \mathcal{S}_d(C)$ has finitely many holes$\}$ and let $\mathcal{C}_n = \{C \in \mathcal{C} : \mathcal{S}_d(C)$ has at most n holes$\}$. Again, $\mathcal{C}^* = \bigcup_{n=1}^{\infty} \mathcal{C}_{1/n}$, and we shall show that $\mathcal{C}_n$ is nowhere dense.

For $C \in \mathcal{C}$, take a finite set F in general position (i.e. without any $d+1$ affinely dependent points in it) approximating C. If $\mathrm{card}F$ is sufficiently large, then $\mathcal{S}_d(F)$ has more than n holes. The combinatorial structure of $\mathcal{S}_d(F)$ — including the components of its complement — does not change if the points of F are gently moved. More precisely, for some $\varepsilon > 0$, if the distance from the old position of any point in F to its new position is less than ε, then the number of holes remains exactly the same. That is, $\mathcal{S}_d(F')$ has the same number of holes as F if $\mathrm{card}F' = \mathrm{card}F$ and $F' \in \mathcal{B}(F, \varepsilon)$. Now, replacing the point x of F by any nonempty set in $B(x, \varepsilon)$ cannot decrease the number of holes. Hence $\mathcal{B}(F, \varepsilon) \cap \mathcal{C}_n = \emptyset$, which ends the proof.

By Theorem 1, for most $C \in \mathcal{C}$, $\mathcal{S}_d(C)$ coincides with its boundary. However, the following holds.

Theorem 3. *For most $C \in \mathcal{C}$, most points of $\mathcal{S}_d(C)$ lie neither on the boundary of any hole, nor on the surface of $\mathcal{S}_d(C)$.*

Proof. Let $C \in \mathcal{C}$. Of course, the family of all holes of $\mathcal{S}_d(C)$ is at most countable. Since, by Lemma 1, the boundary $\mathrm{bd}H$ of every hole H consists of finitely many $(d-1)$-dimensional polytopes, to prove the theorem it will suffice to show that every such polytope and the surface S of $\mathcal{S}_d(C)$ are nowhere dense in $\mathcal{S}_d(C)$, in case C is typical.

Now, let C be typical, and let P be such a $(d-1)$-dimensional polytope, a facet of a hole H. Let $p \in P \cup S$ and $\varepsilon > 0$. We have to find a ball in $\mathcal{S}_d(C)$, included in $B(p, \varepsilon)$ and disjoint from $P \cup S$.

Let $s_1, s_2, \ldots, s_d \in C$ satisfy $p \in \mathcal{S}_d(\{s_1, s_2, \ldots, s_d\})$. For small ε, consider the balls $B(s_1, \varepsilon), B(s_2, \varepsilon), \ldots, B(s_d, \varepsilon)$. In case $p \in P$, let Σ be the open half-space including H with $P \subset \mathrm{bd}\Sigma$. In case $p \in S$, let Σ be the open half-space disjoint from $\mathrm{conv}\mathcal{S}_d(C) = \mathrm{conv}C$, with $p \in \mathrm{bd}\Sigma$. The latter is unique, because $\mathrm{conv}C$ is smooth by Lemma 4. (If $p \in P \cap S$, one would define Σ either way, but in fact $P \cap S = \emptyset$ by Theorem 7.) For small ε, no $B(s_i, \varepsilon) \cap \Sigma$ meets C, otherwise $\mathcal{S}_d(C)$ would contain points in H or outside $\mathrm{conv}C$. By Lemma 2, the interior of each $B(s_i, \varepsilon) \setminus \Sigma$ meets C. Thus, $\mathcal{S}_d(C)$ meets

$B(p,\varepsilon) \setminus \overline{\Sigma}$. Any point of $\mathcal{S}_d(C) \cap B(p,\varepsilon) \setminus \overline{\Sigma}$ is the centre of a ball in $\mathcal{S}_d(C)$ disjoint from $P \cup S$. This ends the proof.

In [17] it is shown that most $C \in \mathcal{C}$ look dense from most of their points. This property is inherited in a stronger form by the d-simplicially convex bodies.

Theorem 4. *For a typical $C \in \mathcal{C}$, $\mathcal{S}_d(C)$ looks full from most of its points.*

Proof. Take a typical $C \in \mathcal{C}$ and a typical $x \in \mathcal{S}_d(C)$. Suppose $\mathcal{S}_d(C)$ does not look full from x. Then, for some line-segment xy, $\mathcal{S}_d(C) \cap xy = \{x\}$. The component of $\mathbb{R}^d \setminus \mathcal{S}_d(C)$ including $xy \setminus \{x\}$ either is a hole or is the complement of $\mathrm{conv}C$. Then x is on the boundary of a hole or on the surface, contradicting Theorem 3.

Theorem 5. *For most $C \in \mathcal{C}$, no point of C lies on the boundary of any hole of $\mathcal{S}_d(C)$.*

Proof. Let C be typical, and suppose $x \in C \cap \mathrm{bd}H$, where H is a hole of $\mathcal{S}_d(C)$. By Lemma 2, C looks half-dense from x, i.e. there exists a $\mathbf{0}$-half-space H^+ such that $D_x(C \cap N)$ is dense in $S^{d-1} \cap H^+$ for any neighbourhood N of x. Then the half-space $H^- = -H^+ + x$ must be supporting $\overline{H}$. Some facet of $\overline{H}$ is not in $\mathrm{bd}H^-$. By Lemma 5, this facet is included in a unique simplex $\mathcal{S}_d(\{s_1, s_2, \ldots, s_d\})$ with $s_1, s_2, \ldots, s_d \in C$. Then some vertex s_i of $\mathcal{S}_d(\{s_1, s_2, \ldots, s_d\})$ lies in $\mathrm{int}H^-$. It follows that, for some neighbourhood N and point $y \in C \cap (H^+ + x) \cap N$, the line-segment ys_i meets H. A contradiction is reached.

Theorem 6. *If $C \in \mathcal{C}$ is typical then $\mathcal{S}_\ell(C)$ looks at least half-dense from any of its points* $(1 \le \ell \le d)$.

Proof. For $\ell = 1$, we apply Lemma 2. Now let $2 \le \ell \le d$. Let $C \in \mathcal{C}$ be typical and consider an arbitrary point $x \in \mathcal{S}_\ell(C)$. Take $s_1, s_2, \ldots, s_\ell \in C$ such that $x \in \mathcal{S}_\ell(\{s_1, s_2, \ldots, s_\ell\})$. By Lemma 2, C looks at least half-dense from s_1. Then, for some $\mathbf{0}$-half-space H^+ and for any $\varepsilon > 0$, the set $D_{s_1}(C \cap B(s_1, \varepsilon))$ is dense in $S^{d-1} \cap H^+$. We have $x = \Sigma_{i=1}^{\ell} \lambda_i s_i$ for suitable non-negative coefficients λ_i with $\Sigma_{i=1}^{\ell} \lambda_i = 1$.

The homothety h with centre $(\Sigma_{i=2}^{\ell}\lambda_i)^{-1}\Sigma_{i=2}^{\ell}\lambda_i s_i$ and ratio λ_1 transforms s_1 into x. It also transforms every point p of $B(s_1,\varepsilon)$ into a point $h(p)$ of $B(x, \lambda_1\varepsilon)$ such that $h(p) - x = \lambda_1(p - s_1)$. Thus $D_x((\lambda_1(C - s_1) + x) \cap B(x, \lambda_1\varepsilon))$ is dense in $S^{d-1} \cap H^+$. Since $\lambda_1(C - s_1) + x$ is included in $\mathcal{S}_\ell(C)$, the latter looks at least half-dense from x.

Theorem 7. *For most $C \in \mathcal{C}$, no pair of holes of $\mathcal{S}_d(C)$ have common boundary points, and no boundary point of any hole of $\mathcal{S}_d(C)$ lies on the surface of $\mathcal{S}_d(C)$.*

Proof. To prove the first part, suppose on the contrary that H and G are two holes of $\mathcal{S}_d(C)$, where C is typical, and $x \in \mathrm{bd}H \cap \mathrm{bd}G$. Consider a hyperplane Π through x separating $\mathrm{int}H$ from $\mathrm{int}G$. Remember that both $\overline{H}$ and $\overline{G}$ are polytopes, by Lemma 1. Rotate Π if necessary, until it contains a facet F of $\overline{H}$ or $\overline{G}$, say $\overline{H}$. By Lemma 5, F is included in a unique $(d-1)$-dimensional simplex S with vertices in C. So $x \in S$. By Lemma 2, C looks at least half-dense from $s_1 \in V$, where V is the set of vertices of

S. Hence, for some $\mathbf{0}$-half-space H^+ and any $\varepsilon > 0$, the set $D_{s_1}(C \cap B(x\varepsilon))$ is dense in $S^{d-1} \cap H^+$. Now, as $\mathcal{S}_d((C \cap (H^+ + x) \cap B(x, \varepsilon)) \cup V)$ is disjoint from H, the hyperplanes $\mathrm{bd}H^+$ and Π must be parallel and the exterior normal unit vector of $\overline{H}$ at any relatively interior point of F must lie in H^+. But then

$$G \cap \mathcal{S}_d((C \cap (H^+ + x) \cap B(x, \varepsilon)) \cup V) \neq \emptyset,$$

and a contradiction is obtained (since $C \cap (H^+ + x) \cap B(x, \varepsilon) \subset C$ and G is a hole).

For the second part of the statement, suppose on the contrary that H is a hole of $\mathcal{S}_d(C)$, S is the surface of $\mathcal{S}_d(C)$, and $x \in S \cap \mathrm{bd}H$. By Theorem 6, $\mathcal{S}_d(C)$ looks at least half-dense from x. Hence, for some $\mathbf{0}$-half-space H^+ and any $\varepsilon > 0$, the set $D_x(\mathcal{S}_d(C) \cap B(x, \varepsilon))$ is dense in $S^{d-1} \cap H^+$. Since $x \in S$, H^+ must be a translate of the unique (by Lemma 4) half-space supporting $\mathrm{conv}\mathcal{S}_d(C)$ at x. Since $\overline{H}$ is a polytope, $x \in \mathrm{bd}H$ and $H \subset H^+$, it follows that $\mathcal{S}_d(C)$ meets H, and a contradiction is obtained.

Theorem 8. *If $C \in \mathcal{C}$ is typical then $\mathcal{S}_d(C)$ looks at least half-full from any of its points.*

Proof. Let $x \in \mathcal{S}_d(C)$. If x neither is a boundary point for any hole, nor belongs to the surface S of $\mathcal{S}_d(C)$, then $\mathcal{S}_d(C)$ looks full from x, because for any $y \neq x$, if $xy \setminus \{x\}$ lies in a component of the complement of $\mathcal{S}_d(C)$, then x is in the boundary of a hole or in S. Suppose now x is a boundary point of a hole H. Consider a supporting half-space H^+ of $\overline{H}$ at x. Then, for any point $y \notin \{x\} \cup \mathrm{int}H^+$, the set $xy \setminus \{x\}$ meets $\mathcal{S}_d(C)$. Indeed, if this is not the case, then $xy \setminus \{x\}$ lies in a component of the complement of $\mathcal{S}_d(C)$, whence x is in the boundary of some hole different from H, or belongs to S; this contradicts Theorem 7. The same reasoning applies when $x \in S$.

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