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Right Triple Convex Completion

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A set M in $\mathbb{R}^d$ is *rt*-convex if every pair of its points is included in a 3-point subset $\{x, y, z\}$ of M satisfying $\angle xyz = \pi/2$.

We find here for various families of sets the minimal number of points necessary to add to the sets in order to render them *rt*-convex.

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Introduction

Given a family $\mathcal{F}$ of sets in a certain space X, a set $M \subset X$ is called $\mathcal{F}$-*convex* if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$. The second author proposed at the 1974 meeting on Convexity in Oberwolfach the investigation of $\mathcal{F}$-convexity.

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Obviously, usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are all examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

Blind, Valette and the second author [1], and also Böröczky Jr. [2], investigated the rectangular convexity, the case when $\mathcal{F}$ contains all non-degenerate rectangles.

More recently, Magazanik and Perles dealt with staircase connectedness, a special kind of polygonal connectedness [3].

In [5] the second author studied the case when $\mathcal{F}$ is the family of all right triangles in a Hilbert space of dimension at least 2.

In [4] we introduced the right triple convexity, for short *rt*-convexity, where $\mathcal{F}$ is the family of all triples $\{x, y, z\}$ such that $\angle xyz = \pi/2$.

Let K be a compact set in $\mathbb{R}^d$, where $d \geq 2$. We call K^* an *rt-convex completion* of K, for short a *completion* of K, if K^* is *rt*-convex, $K^* \supset K$ and $\mathrm{card}(K^* \setminus K)$ is minimal. This minimal number $\gamma(K)$ is then called the *rt-completion number* of K. Of course, $\gamma(K) = 0$ iff K is *rt*-convex.

In this paper we investigate the *rt*-convex completion of convex bodies, of star-shaped sets and of 2-connected, polygonally connected sets in $\mathbb{R}^d$, and of 3-connected sets and of finite sets in $\mathbb{R}^2$.

Definitions, notation and prerequisites

As usual, for $M \subset \mathbb{R}^d$, $\overline{M}$, $\mathrm{int}\, M$ and $\mathrm{bd}\, M$ denote its topological closure, interior and boundary, and $\mathrm{diam}\, M = \sup_{x,y \in M} \|x - y\|$. A 2-point set $\{x, y\} \subset M$ with $\|x - y\| = \mathrm{diam}\, M$ is called a *diametral pair* of M, while the line-segment xy is a *diameter* of M.

Two points in M are said to enjoy *the rt-property in M* if they belong to some set $\{x, y, z\} \subset M$ such that $\angle xyz = \pi/2$.

As already mentioned, a set $M \subset \mathbb{R}^d$ is called *right triple convex*, for short *rt-convex*, if any pair of its points enjoys the *rt*-property. Clearly, this *rt*-convexity generalizes the concept of right convexity introduced in [5]. Open sets are obviously *rt*-convex.

A set $M \subset \mathbb{R}^d$ is called *almost rt-convex* if each pair of points of M, with at most one exception, enjoys the *rt*-property.

A set in $\mathbb{R}^d$ is called *polygonally connected* if any pair of its points can be joined by a polygonal line included in the set.

A continuum, i.e. a compact connected set, C is said to be *n-connected* if for any subset $F \subset C$ with $\mathrm{card} F \leq n - 1$, the set $C \setminus F$ is connected.

For any compact set $K \subset \mathbb{R}^d$, let S_K be the smallest hypersphere containing K in its convex hull, and denote by F_x the set of all points in K farthest from $x \in \mathbb{R}^d$.

For distinct $x, y \in \mathbb{R}^d$, let xy be the line-segment from x to y, and l_{xy} the line including xy. Also, let H_{xy} be the hyperplane through x orthogonal to l_{xy}, and put $W_{xy} = (H_{xy} \cup H_{yx} \cup S_{xy}) \setminus \{x, y\}$.

A *convex body* is a compact convex set in $\mathbb{R}^d$ with non-empty interior. A convex body K is called *strictly convex* if its boundary bd K contains no (non-degenerate) line-segments. It is *smooth* if its boundary bd K is of class C^1.

If K is a convex body, let p_x be the closest point of K from $x \in \mathbb{R}^d$, and put $p_M = \{p_x : x \in M\}$, for $M \subset \mathbb{R}^d$.

Also, let h denote the Pompeiu-Hausdorff distance between compact sets, and d the minimal distance

$$d(x, K) = \min_{y \in K} \|x - y\|, \qquad d(K, K') = \min_{x \in K} d(x, K'),$$

where $K, K' \subset \mathbb{R}^d$ are compact and $x \in \mathbb{R}^d$.

Lemma 1. *A convex body K is rt-convex if and only if* $\mathrm{card}(K \cap S_K) \geq 3$.

Lemma 2. *Each 2-connected, polygonally connected continuum is almost rt-convex. If it is not rt-convex, then the exceptional pair without the rt-property is diametral.*

Lemma 3. *Every 3-connected continuum in $\mathbb{R}^2$ is almost rt-convex.*

These lemmas are Theorem 3 in [4], applied to convex bodies, and Theorems 1 and 5 in [4], respectively.

Convex bodies

Not all compact sets are *rt*-convex. Finite sets may have a rather large *rt*-completion number. Convex bodies are almost *rt*-convex, but not necessarily *rt*-convex either. How large can their *rt*-completion number be? Theorem 7 will give the answer.

Theorem 4. *For any convex body K which is strictly convex, $\gamma(K) \neq 1$.*

Proof. If K is *rt*-convex, $\gamma(K) = 0$. So, suppose K is not *rt*-convex. Let $x \notin K$. It is easily seen that $K \cup \{x\}$ is not *rt*-convex for any choice of x. Indeed, for the point x and the point p_x, there exists no third point $z \in K$ such that xp_xz be a right triangle. Here we used the fact that the boundary of K contains no line-segments. □

Lemma 5. *Let $K \subset \mathbb{R}^d$ be a convex body, $x \notin K$, $y \in K$. If $y \notin \{p_x\} \cup F_x$, then x, y have the rt-property in $K \cup \{x\}$.*

Proof. If $y \in H_{p_x x}$, then $\angle x p_x y = \pi/2$. Otherwise, $H_{p_x x}$ separates x from y, whence $\angle x p_x y > \pi/2$ and $p_x \in \mathrm{int}\,\mathrm{conv}\, S_{xy}$. For all points $z \in \mathrm{conv}\, S_{xy}$,

$$\|x - z\| \leq \|x - y\| < \|x - w\|,$$

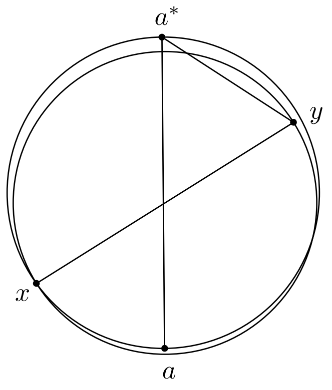

Figure 1.

for any point w of F_x. Hence, S_{xy} separates p_x from w and contains therefore some point of K different from y. Thus, x, y have indeed the *rt*-property in $K \cup \{x\}$. □

Theorem 6. *For any convex body K which is strictly convex, but not rt-convex, $\gamma(K) = 2$.*

Proof. By Lemma 1, $\text{card}(K \cap S_K) = 2$. Let a, a^* be the two points of $K \cap S_K$. Choose $x \in (\text{conv}\, S_K) \setminus K$ and $y \in F_x$. Since $y \in (\text{int conv}\, S_K) \cup \{a, a^*\}$, $\|x-y\| < \|a - a^*\|$. Therefore, $\text{conv}\, S_{xy}$ cannot contain both a, a^*, say $a^* \notin \text{conv}\, S_{xy}$. We have $\angle xya^* < \pi/2$, because $\|x - a^*\| \leq \|x - y\|$. Hence, $ya^* \cap S_{xy} \neq \{y\}$. Since $ya^* \subset K$, x, y have the *rt*-property in $K \cup \{x\}$. See Fig. 1.

Claim. *There are two points $x \in S_K \setminus K$, $z \in (\text{conv}\, S_K) \setminus K$, such that $x \in H_{p_z z}$ and $z \in H_{p_x x}$.*

Indeed, let $y \in S_K \setminus aa^*$ and $C = H_{p_y y} \cap S_K$. If, for some point $x \in C$, $H_{p_x x}$ meets yp_y in a point z, then for these points x, z the Claim is verified. So, assume now that $H_{p_x x} \cap yp_y = \emptyset$ for all $x \in C$.

Let $s \in l_{yp_y} \setminus K$. Consider the set $V(s) = \{t \in K : ts \cap K \subset \text{bd}\, K\}$. Clearly, $s' \in sp_y$ implies $V(s') \subset V(s)$.

For each $x \in C$, let $i(x)$ denote the intersection point $H_{xp_x} \cap l_{yp_y}$. Since i is continuous, there is a point $x_0 \in C$ for which $i(x_0)$ is closest to y.

Choose now again $x \in C$ arbitrarily. Consider the triangle $\Delta = \text{conv}\{p_x, i(x), p_y\}$ and its plane Π, see Fig. 2. Let $\gamma = \Delta \cap \text{bd}\, K$. Clearly, one of the two supporting lines of $\Pi \cap \text{bd}\, K$ through y touches $\text{bd}\, K$ at a point $v' \in \gamma$. Of course, $v' \in V(y)$. But

$$V(y) \subset V(i(x_0)) \subset p_{\text{conv}\, C}.$$

This implies that some outer normal of K at v' meets $\text{conv}\, C$ in w, say. Then, the points y and w playing the roles of x and z respectively, the Claim is verified.

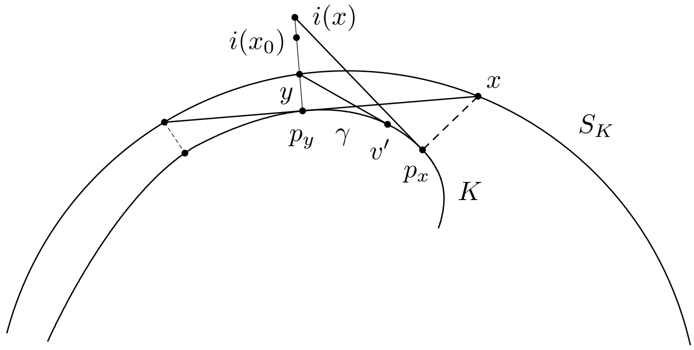

Figure 2.

To finish the proof, we show now that $K \cup \{x, z\}$ is *rt*-convex, the points x and z being delivered by the Claim.

The only pair of points of K without the *rt*-property in K is, by Lemma 2, a, a^*. In $K \cup \{x, z\}$, a, a^* have the *rt*-property, because $\angle axa^* = \pi/2$. For $y \in F_x$, we proved already that x, y have the *rt*-property in $K \cup \{x\}$. For $y \in \{p_x, z\}$, the pair x, y has the *rt*-property in $K \cup \{x, z\}$, since the triangle $\mathrm{conv}\{x, p_x, z\}$ is right. For all other points $y \in K$, the pair x, y enjoys the *rt*-property in $K \cup \{x\}$, by Lemma 5.

For $y \in F_z$, we proved already that z, y have the *rt*-property in $K \cup \{z\}$. The pair z, y has the *rt*-property in $K \cup \{x, z\}$, since $\angle zp_zx = \pi/2$.

For all other points $y \in K$, the pair z, y enjoys the *rt*-property in $K \cup \{z\}$, by Lemma 5.

Hence, $K \cup \{x, z\}$ is *rt*-convex. □

Theorem 7. *If the convex body K is neither rt-convex, nor strictly convex, then* $\gamma(K) = 1$. *Hence, for every convex body K,* $\gamma(K) \leq 2$.

Proof. Assume K is neither *rt*-convex, nor strictly convex. Let aa^* be the (unique) diameter of K.

Let $\sigma \subset \mathrm{bd}\, K$ be a line-segment. Consider a point $x' \in \sigma \setminus \{a, a^*\}$. An arbitrary outer normal at x' to bd K meets S_K at x, say. Then $K \cup \{x\}$ is a completion of K.

Indeed, a, a^* have the *rt*-property in $K \cup \{x\}$; x, x' have the *rt*-property because xx' and σ are orthogonal; x, y have it for any $y \in F_x$, the argument being the same as in the preceding proof. □

Theorem 8. *Let* $\varepsilon > 0$. *If K is strictly convex or a polytope, then a completion* K^* *of K can be chosen such that* $h(K, K^*) < \varepsilon$. *This is not necessarily true for other convex sets.*

Proof. Assume K is not *rt*-convex, otherwise the statement is trivially true. Let again aa^* be the unique diameter of K.

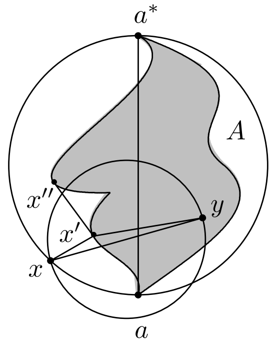

Figure 3.

If K is a polytope, then a is one of its vertices. If we choose σ to have an endpoint in a, to lie in a facet and to be small enough, and then follow the preceding proof, we find a completion K^* with $h(K, K^*) < \varepsilon$.

If K is strictly convex, we use the proof of Theorem 6. The choice of x in $S_K \setminus aa^*$ was arbitrary. Now, we choose x close to a. The second point of the completion, z, which exists according to the Claim, is by construction close to a, too. Thus, for $\|x - a\|$ small enough, $h(K, K \cup \{x, z\}) < \varepsilon$.

To see why the statement is not true for all convex sets, consider a smooth convex body K which is not strictly convex, such that $\operatorname{card}(K \cap S_K) = 2$ and all the boundary line-segments are far away from $\{a, a^*\} = K \cap S_K$ (for example, the Minkowski sum of a ball and a line-segment). Take $\varepsilon = d(S_K, \overline{\operatorname{bd} K \setminus \operatorname{ext} K}) > 0$. Then $\gamma(K) = 1$, and the additional point x must be chosen so that p_x lies in a boundary line-segment, otherwise x, p_x don't enjoy the *rt*-property. As xaa^* must also be a right triangle, this implies that $x \in W_{aa^*}$. Under these circumstances,

$$h(K, K \cup \{x\}) = \|x - p_x\| \geq d(S_K, \overline{\operatorname{bd} K \setminus \operatorname{ext} K}) = \varepsilon. \qquad \square$$

2-connected sets

We investigated in [4] the *rt*-convexity of 2-connected, polygonally connected sets in $\mathbb{R}^d$ and of planar 3-connected sets. In this section we add information about their *rt*-completion number.

It will be seen that it is at most 1, with just one exceptional situation.

Lemma 9. *Let $K \in \mathbb{R}^d$ be a 2-connected, not rt-convex continuum, such that* $\operatorname{card}(K \cap S_K) = 2$, *and* $\operatorname{conv} K$ *is not strictly convex. If $d = 2$, then $\gamma(K) = 1$. If $d \geq 3$ and K is polygonally connected, then again $\gamma(K) = 1$.*

Proof. Let $d \geq 2$ be arbitrary and $K \cap S_K = \{a, a^*\}$.

Let $x \in S_K \setminus \{a, a^*\}$ and $y \in K$. Since $y \in (\operatorname{int}\operatorname{conv} S_K) \cup \{a, a^*\}$ and $x \in S_K \setminus \{a, a^*\}$, $\|x - y\| < \|a - a^*\|$. Therefore, the points a, a^* cannot be both in

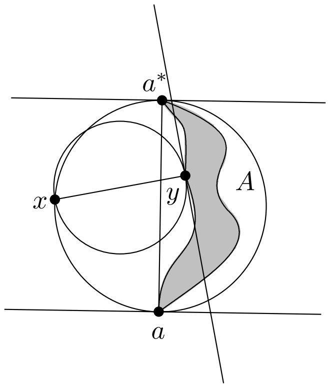

Figure 4.

conv S_{xy}. Suppose w.l.o.g. that $a^* \notin$ conv S_{xy}. Also, observe that a, a^* have the *rt*-property in $K \cup \{x\}$.

Since conv K is not strictly convex, there are two distinct points $x', x'' \in K$ with $x'x'' \subset$ bd conv K. See Fig. 3.

Assume that the outer normal of bd conv K at x' orthogonal to a supporting hyperplane Ξ of conv K including $x'x''$ meets $S_{aa^*} \setminus \{a, a^*\}$. Let x be this intersection point, and $y \in K$.

We saw that $a^* \notin$ conv S_{xy}. On the other hand, x and y are separated by $H_{x'x}$, so $\angle xx'y > \pi/2$, and $x' \in$ int conv S_{xy}. Thus, S_{xy} separates x' from a^*, and $S_{xy} \cap K \neq \{y\}$, because K is 2-connected, whence x, y have the *rt*-property in $K \cup \{x\}$.

Assume now that both outer normals of bd conv K at x' and x'' orthogonal to Ξ miss $S_{aa^*} \setminus \{a, a^*\}$. This can only happen if $\{x', x''\} = \{a, a^*\}$. Then K is included in the closure A of one half of $D =$ int conv S_{aa^*} determined by Ξ. Let the 2-plane $\Pi \supset aa^*$ be orthogonal to Ξ.

Now, x will be taken arbitrarily in $\Pi \cap S_{aa^*} \setminus A$. We show that $K \cup \{x\}$ is *rt*-convex. Consider a point y of K. We remember that $a^* \notin$ conv S_{xy}.

If $K \cap$ conv $S_{xy} \neq \{y\}$, then another point of K lies on S_{xy} and sees $\{x, y\}$ under a right angle, or S_{xy} separates two points of K, an interior point of conv S_{xy} and a^*, which again implies that a further point of K must lie on S_{xy}, whence x, y have the *rt*-property in $K \cup \{x\}$.

If $K \cap$ conv $S_{xy} = \{y\}$, consider H_{yx}. This hyperplane does not separate x from both a, a^*, otherwise

$$\|x - y\| < d(x, aa^*) = d(x, \Xi) \leq d(x, K),$$

which is false. It follows that H_{yx} either meets $\{a, a^*\}$, or separates a from a^*, or does not meet conv$\{x, a, a^*\}$. In the first two situations, $H_{yx} \cap K \setminus \{y\} \neq \emptyset$. Let now $d = 2$. In this case, if $H_{yx} \cap$ conv$\{x, a, a^*\} = \emptyset$, then the line H_{yx} separates a from a^* in $\overline{D} \setminus$ conv S_{xy}. Thus, once more the 2-connectedness of K implies

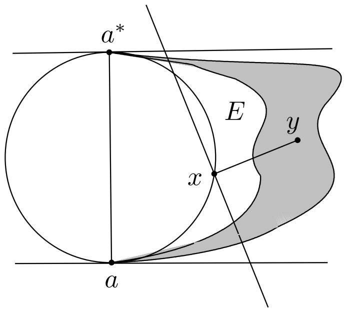

Figure 5.

that $K \cap H_{yx} \setminus \{y\}$ contains some point z, whence $\angle xyz = \pi/2$. See Fig. 4. Finally, let $d \geq 3$. If $H_{yx} \cap \operatorname{conv}\{x, a, a^*\} = \emptyset$, then any polygonal line from a to y meets W_{xy}, hence again $K \cap H_{yx} \setminus \{y\} \neq \emptyset$ and x, y have the *rt*-property in $K \cup \{x\}$. □

Theorem 10. *Let $K \in \mathbb{R}^d$ be a 2-connected, polygonally connected, not rt-convex continuum. Then $\gamma(K) = 2$ if* $\operatorname{conv} K$ *is strictly convex, and $\gamma(K) = 1$ otherwise.*

Proof. By Lemma 2, K is almost *rt*-convex, and the exceptional pair is diametral, so $\operatorname{card}(K \cap S_K) = 2$. If $\operatorname{conv} K$ is strictly convex, then $\operatorname{bd}\operatorname{conv} K \subset K$, and we closely follow the proof of Theorem 6, leading to $\gamma(K) = 2$. If $\operatorname{conv} K$ is not strictly convex, then, by Lemma 9, $\gamma(K) = 1$. □

Theorem 11. *Let $K \in \mathbb{R}^2$ be a 3-connected, not rt-convex continuum. Then $\gamma(K) = 2$ if* $\operatorname{conv} K$ *is strictly convex, and $\gamma(K) = 1$ otherwise.*

Proof. By Lemma 3, K has a single pair of points a, a^* without the *rt*-property.

Let D be the bounded component of the complement of $\overline{W_{aa^*}}$, and E, E' its two unbounded components located between H_{aa^*} and H_{a^*a}. Then $K \setminus \{a, a^*\}$ lies in one of these three components.

If $K \setminus \{a, a^*\} \subset D$, we have $\gamma(K) = 1$ for not strictly convex $\operatorname{conv} K$, by Lemma 9, and $\gamma(K) = 2$ for strictly convex $\operatorname{conv} K$, along the lines of Theorem 6.

So, suppose $K \setminus \{a, a^*\} \subset E$, see Fig. 5.

Take arbitrarily a point $x \in S_K \cap \operatorname{bd} E \setminus \{a, a^*\}$. Any line through x obviously separates a from a^* in $\overline{E}$. Thus, it must meet K. Hence, for any point $y \in K$, $H_{xy} \cap K \neq \emptyset$, and so x, y have the *rt*-property in $K \cup \{x\}$.

By symmetry, the case $K \setminus \{a, a^*\} \subset E'$ is also solved. □

Starshaped sets

In [4] we discussed the *rt*-convexity of starshaped sets. We saw that the most common starshaped sets with a single-point kernel are not *rt*-convex, while in

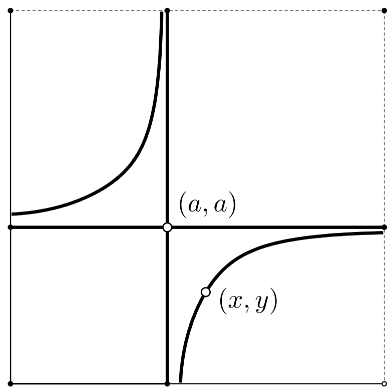

Figure 6.

the sense of Baire categories most of them are. We shall now investigate their completion.

Let $\mathcal{K}$ be the space of all compact starshaped sets in $\mathbb{R}^d$. For $K \in \mathcal{K}$, let $\ker K$ denote its kernel.

For $K \in \mathcal{K}$, let the set $\operatorname{ex} K$ of *extremities* of K be the set of all points $x \in K$ such that $kx \subset ky \subset K$ and $k \in \ker K$ imply $y = x$.

We have seen in [4] (Corollary 3) that starshaped sets with finitely many extremities are not *rt*-convex. In fact, if they have finite length, they already cannot be *rt*-convex, for that reason [4].

Theorem 12. *If $K \in \mathcal{K}$ has a single-point kernel and $\operatorname{ex} K$ is finite, then, for any completion K^* of K, $K^* \setminus K$ is uncountable.*

Proof. Let $\{k\} = \ker K$ and choose $x_0 \in K$ with $\|x_0 - k\|$ maximal. Since $\operatorname{ex} K$ is finite, the orthogonal projection of $(K \setminus kx_0) \cup \{k\}$ on l_{kx_0} is a line-segment not containing x_0. Hence, there is a line-segment $x_0y_0 \subset K$ such that $W_{xy} \cap K = \emptyset$ for any pair of distinct points $x, y \in x_0y_0$. This means that any completion K^* of K must contain a point in W_{xy} for every choice of x and y. Identify x_0y_0 with the real interval I.

We have to prove that $Z = K^* \setminus K$ cannot be countable. Assume it is. Each point $z \in Z$ has a distance b to $l_{x_0y_0}$ and let $a = \|x_0 - \tilde{z}\|$, where $\tilde{z}$ is the orthogonal projection of z onto $l_{x_0y_0}$.

For each point $z \in Z$, the set of pairs (x, y) *covered by* z, i.e. for which $z \in W_{xy}$, either satisfies $x = a$ or $y = a$ or

$$y = a + \frac{b^2}{a - x},$$

while $(x, y) \in I^2 \setminus \Delta$, where $\Delta = \{(\xi, \eta) \in I^2 : \xi = \eta\}$. This is a (disconnected) piece of hyperbola plus two line-segments, a nowhere dense set in I^2. See Fig. 6. So, the set of pairs covered by the whole countable set Z is of first Baire category. Thus, it cannot equal $I^2 \setminus \Delta$, which is obviously of second category (in I^2). □

Finite sets

Let $\mathcal{E}$ be the family of all finite sets in $\mathbb{R}^2$.

Theorem 13. *For any set $A \in \mathcal{E}$ with* $\operatorname{card} A = n \geq 3$, *we have* $\gamma(A) \leq \frac{n(n-3)}{2} + 2$.

Proof. If A is included in a line L, take $a \in A$ and consider a line L' parallel to L and the orthogonal projection A' of $A \setminus \{a\}$ onto L'. Then obviously $A \cup A'$ is *rt*-convex and $\operatorname{card} A' = n - 1 \leq \frac{n(n-3)}{2} + 2$.

If A is not included in any line, let ab be a side of the polygon conv$A$, and let L be now the line containing ab.

Consider the at most $n-1$ lines through the points of A, parallel to (or coinciding with) L. Also, consider all $n-2$ line-segments c_1c_1', c_2c_2', ... $c_{n-2}c_{n-2}'$ orthogonal to L, from the points $c_1, c_2, ..., c_{n-2} \in A \setminus \{a, b\}$ to points $c_1', c_2', ..., c_{n-2}' \in L$. Consider $c_1, c_2, ..., c_{n-2}$ ordered such that the distance from c_i to L is a non-decreasing function of i.

Let B' be the set of intersection points of all these lines and line-segments.

Obviously, c_1c_1' has only its endpoints in B'. In general, c_ic_i' contributes $i+1$ points in B'. So, B' has at most $\sum_{i=1}^{n-2}(i+1) = (n-2)(n+1)/2$ points. All pairs of points in $B = B' \cup \{a, b\} \supset A$ but possibly $\{a, b\}$ enjoy the *rt*-property.

So, by adding the point a^* completing the rectangle $c_1c_1'aa^*$, we obtain the *rt*-convex set $B \cup \{a^*\}$ of cardinality

$$\frac{(n-2)(n+1)}{2} + 3 = \frac{n(n-1)}{2} + 2.$$

Hence $\gamma(A) \leq \frac{n(n-3)}{2} + 2$. □

Let $\gamma(n) = \max\{\gamma(A) : \operatorname{card} A = n\}$.

Corollary 14. *For $n \geq 3$, $\gamma(n) \leq \frac{n(n-3)}{2} + 2$.*

How tight is this inequality?

We show that $\gamma(3) = 2$.

Let $A = \{a, b, c\}$. We first find an *rt*-convex set $B \supset A$ with card$B = 5$.

Let conv$\{b, c, d, e\}$ be a rectangle, such that – in case a, b, c are not collinear – a, d, e are collinear. Then $B = \{a, b, c, d, e\}$ is as desired.

Now we show that no set $B \supset A$ with less than 5 points is *rt*-convex, if A is suitably chosen.

Let abc be an isosceles triangle with $\angle abc = 2\pi/3$. Assume there exists an rt-convex set $B = \{a, b, c, z\}$.

The point z must lie in $W_{ab} \cap W_{bc} \cap W_{ca}$. But $W_{ab} \cap W_{bc}$ consists of four points, $(a+c)/2$, $a' \in H_{ab} \cap H_{bc}$, $c' \in H_{cb} \cap H_{ba}$, $b' \in H_{ab} \cap H_{cb}$. Obviously, $(a+c)/2 \notin W_{ca}$. The other points do not belong to $H_{ac} \cup H_{ca}$. The points a' and c' belong to S_{ca} only if $\cos 2\alpha = (\sqrt{5} - 1)/2$, where α is one of the two equal angles of the triangle abc; but we chose the triangle such that $\alpha = \pi/6$, and the preceding condition is not satisfied. The point b' belongs to S_{ca} only if $2\alpha = \pi/2$, and this does not hold either. Hence

$$W_{ab} \cap W_{bc} \cap W_{ca} = \emptyset$$

and B does not exist. This ends the proof.

For arbitrary n, we only offer the following.

Open Problem 15. *Prove (or disprove) that the inequality in Corollary* 7 *is best possible.*

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