Libertas Mathematica (new series)
Volume 35(2015), No. 2, 43-49

Escaping from a cage

Tudor Zamfirescu

Abstract: This paper is about how to escape from a cage if you are a convex body.

Keywords: Cage, convex body, holding, n-move

MSC2010: 53C45, 53C22

Dedicated to Professors Nicolae Dinculeanu and Solomon Marcus on the occasion of their 90th birthday.

1 Introduction

A *cage* is the 1-skeleton of a polytope in $\mathbb{R}^3$. A cage G is said to *hold* a compact set K if no rigid motion can bring K in a position far away without meeting G on its way.

Cages are widely used to hold chicken, other birds, lions, other mammals, including homo sapiens.

In 1959, Coxeter [3] asked to find cages holding the unit ball and having total length as small as possible. This was settled by Besicovitch [2] and Aberth [1].

Several more articles have been written about life in cages. They mainly investigated the ways in which a tetrahedral or a pentahedral cage may hold a disc (see [5]), or how small can be the length of a cage holding a given convex body (see [4]).

Our story is about a convex body, which is caught in a cage, from which it wants to escape. How can it manage this? As we shall see, it must famish a little, but then two moves will suffice.

A *convex body* is a 3-dimensional compact convex set in $\mathbb{R}^3$. In the space $\mathcal{K}$ of all convex bodies in $\mathbb{R}^3$ endowed with the usual Pompeiu-Hausdorff metric,

$$\{\lambda A + (1-\lambda)B : \lambda \in [0,1]\}$$

is called a *segment* joining A and B. A path in $\mathcal{K}$ consisting of n consecutive segments will be called an *n-move*.

Let G be a cage; we say that a convex body K *migrates from A to B*, if A and B are translates of K, A is held by G, B is not held by G and there exists a path in $\mathcal{K}$ from A to B such that every element of the path is disjoint from G. The convex body B is considered *outside* G if $B \cap \mathrm{conv}G = \emptyset$.

For distinct $x, y \in \mathbb{R}^3$, let $\overline{xy}$ be the line through x, y, xy the line-segment from x to y, and H_{xy} the hyperplane through $(x+y)/2$ orthogonal to $\overline{xy}$.

For $M_1, M_2 \subset \mathbb{R}^d$, let $d(M_1, M_2) = \inf\{d(x,y) : x \in M_1,\ y \in M_2\}$ denote the *distance* between M_1 and M_2. (The metric of $\mathcal{K}$ is not this distance.)

As usual, for $M \subset \mathbb{R}^d$ with $d \geq 2$, $\mathrm{bd}M$ denotes its (relative) boundary, $\mathrm{diam}M = \sup_{x,y \in M} \|x-y\|$ is its *diameter*, and the *convex hull* $\mathrm{conv}M$ is the intersection of all convex sets including M.

2 Migration of convex bodies

Our first result helps migrants who just want to flee.

Theorem 2.1. *If a convex body A is held by the cage G, it can migrate through a 2-move to a translate B of A outside G, keeping constant its diameter on the way.*

Proof. Let $x, y \in A$ realize the diameter δ of A.

Choose $x', y' \in \mathbb{R}^3$ such that $xyy'x'$ be a rectangle disjoint from G and $\overline{x'y'} \cap G = \emptyset$.

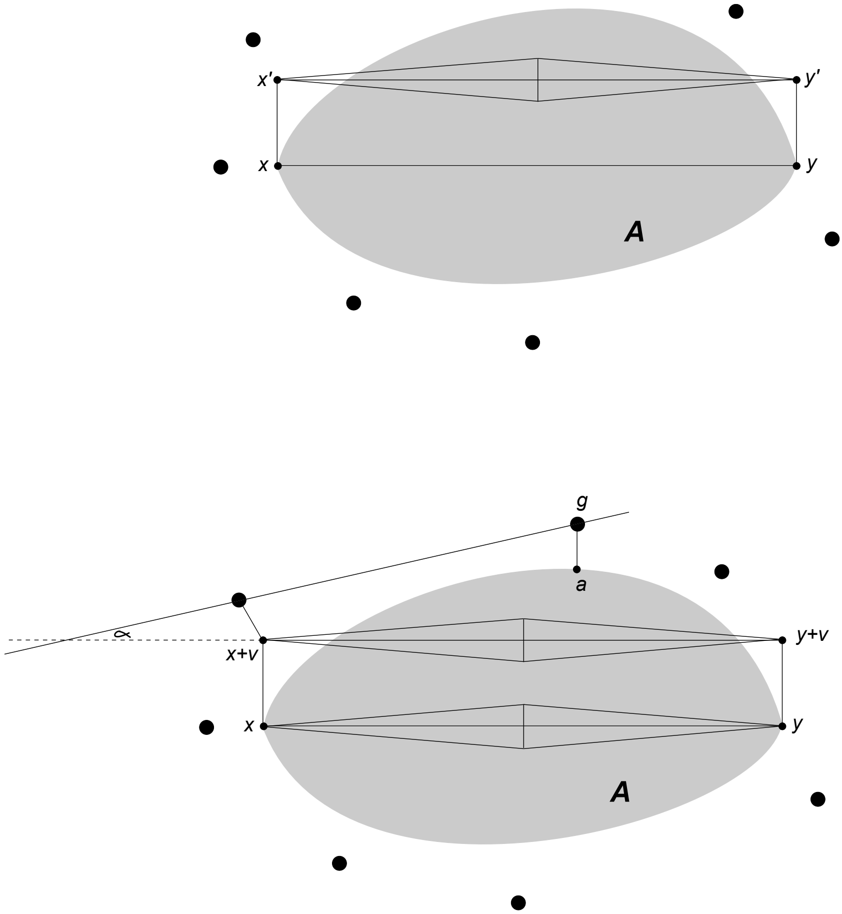
Let ε be the distance from $\overline{x'y'}$ to G. Consider the circle C of radius $\varepsilon/2$ and centre $(x'+y')/2$ in the plane $H_{x'y'}$. Let $D = \mathrm{conv}(\{x', y'\} \cup C)$. Clearly, no convex body $\lambda A + (1-\lambda)D$ meets G when $\lambda \in [0,1]$, if ε is small enough. See Figure 1.

Now take $x''y'' \subset \overline{x'y'}$ congruent to xy, far away. Consider $B = A + x'' - x$. Then B has diameter δ, like every convex body $\lambda A + (1-\lambda)D$ and $\lambda D + (1-\lambda)B$, for any $\lambda \in [0,1]$.

Observe that the set

$$\{\lambda u + (1-\lambda)v : u \in D, v \in B, \lambda \in [0,1]\}$$

is disjoint from G if $x''y''$ was chosen far enough on $\overline{x'y'}$. Thus, each $\lambda D + (1-\lambda)B$ is disjoint from G, too.



Figures 1 (top) and 2 (bottom)

Hence, the 2-move

$$\{\lambda A + (1-\lambda)D\}_{\lambda\in[0,1]} \cup \{\lambda D + (1-\lambda)B\}_{\lambda\in[0,1]}$$

is a valid migration path. $\square$

Many migrants want to choose themselves their destination.

Theorem 2.2. *If the convex body B outside a cage G is a translate of the convex body A held by G, then A can migrate to B through a 3-move, keeping constant its diameter on the way.*

Proof. Let again $x, y \in A$ realize the diameter δ of A. We have $B = A + w$, for some $w \in \mathbb{R}^3$. Put

$$\Xi = \bigcup_{\lambda > 0} (B + \lambda(x - y)).$$

Now, consider again the proof of Theorem 2.1, until the choice of $x''y''$ on $\overline{x'y'}$. At that point, we choose $x''y''$ such that y' separates x' from x'' on $\overline{x'y'}$ if $\Xi \cap G \neq \emptyset$, and such that x' separates y' from y'' otherwise.

In this way we guarantee the free way from the convex body B constructed in the previous proof, which shall now be denoted by B', and the convex body B from our statement. Indeed,

$$\Big(\bigcup_{\lambda \in [0,1]} (\lambda B + (1 - \lambda)B')\Big) \cap G = \emptyset$$

if $x''y''$ is chosen far enough on $\overline{x'y'}$.

Now, the 3-move

$$\{\lambda A + (1 - \lambda)D\}_{\lambda \in [0,1]} \cup \{\lambda D + (1 - \lambda)B'\}_{\lambda \in [0,1]} \cup \{\lambda B' + (1 - \lambda)B\}_{\lambda \in [0,1]}$$

is a valid migration path. □

3 Migration on shorter paths

In the previous cases, the guaranteed migration paths could be quite long. As usual, shorter paths are preferable for migrants. The next theorem takes care of this aspect.

Theorem 3.1. *If a convex body A is held by the cage G and is close to G, then it can migrate through a 4-move of length at most close to $4\,\mathrm{diam} A$ to a translate B of A outside G, keeping constant its diameter on the way.*

Proof. Let $x, y \in A$ realize the diameter δ of A. Assume the distance η from A to G, realized by $a \in A$ and $g \in G$, to be small. Either

(i) g belongs to an edge of G parallel to $\overline{xy}$, or

(ii) there is a supporting plane of conv$G$ at g not parallel to $\overline{xy}$; in this case let α be their angle.

Case (ii). Consider the orthogonal projection G' of G onto H_{xy}.

For any $\varepsilon > 0$, there exists an $\varepsilon' > 0$ such that, for any point $z \in H_{xy}$, there is a disc Δ of radius ε and centre z', with $\Delta \cap G' = \emptyset$ and $\|z - z'\| < \varepsilon'$. The number ε' can be chosen arbitrarily small, if ε is small enough.

Let $\varepsilon > 0$ be smaller than η and than wid$A$. Consider the circle C of radius ε and centre $(x+y)/2$ in the plane H_{xy}. Let $D = \mathrm{conv}(\{x, y\} \cup C)$. Clearly, no convex body $\lambda A + (1-\lambda)D$ meets G when $\lambda \in [0, 1]$.

We translate now D in direction $a - a'$, where a' is the projection of a onto $\overline{xy}$, and stop just when the distance from G becomes η. The translation vector v verifies $\|v\| \leq \|a - a'\| \leq$ wid$A$. See Figure 2. Close to v, more precisely at distance at most ε' from v, we find a vector v' such that $D + v'$, translated in direction $x - y$ or $y - x$, say $x - y$, can escape from G. Our construction guarantees that the new translation of $D + v'$ can be performed by a vector u with $\|u\|$ close to δ, such that $(D + v' + u) \cap \mathrm{conv}G = \emptyset$.

Indeed, if $\{w\} = \overline{(x+v')(y+v')} \cap \mathrm{bdconv}G$, then $\|x + v' - w\| \leq (\eta + \varepsilon)/\tan\alpha$. Note that possibly $w \in (x+v')(y+v')$. So, u can be chosen to be slightly longer than

$$\|y + v' - w\| \leq \delta + \frac{\eta + \varepsilon}{\tan\alpha}.$$

Now take a plane Π separating $D' = D + v' + u$ from G. In the plane including the line $\overline{(x+v')(x+v'+u)}$ and orthogonal to Π, consider a square $Q = (x + v' + u)(y + v' + u)y'x'$ disjoint from the open half-space determined by Π and containing G. The final position will be the convex body $B = A + x' - x$. The segment $\{\lambda D' + (1-\lambda)B\}_{\lambda\in[0,1]}$ obviously lies entirely outside G.

Thus, the 4-move

$$\{\lambda A + (1-\lambda)D\}_{\lambda\in[0,1]} \cup \{\lambda D + (1-\lambda)(D+v')\}_{\lambda\in[0,1]} \cup$$

$$\cup \{\lambda(D+v') + (1-\lambda)D'\}_{\lambda\in[0,1]} \cup \{\lambda D' + (1-\lambda)B\}_{\lambda\in[0,1]}$$

is a valid migration path and has length close to 4δ, at most.

Case (i). If the orthogonal projection on $\overline{xy}$ of an endpoint e of the edge of G to which g belongs lies on xy, then some supporting plane of conv$G$ at e is not parallel to $\overline{xy}$, it makes an angle α with it, and the proof can proceed as in Case (ii).

Otherwise, the edge of G has length more than δ. Consider one of the two faces of conv$G$ with that edge on its boundary. Then we construct the same kind of double cone D, but choosing ε differently. More exactly, ε will be chosen such that C can be translated through the above face of conv$G$, the direction of the movement being orthogonal to $\overline{xy}$. Note that $\overline{xy}$ cannot lie in the plane of the face, because G holds A.

In this manner, the whole convex body D can be brought outside conv$G$ by a translation vector v of length at most wid$A + \eta + \varepsilon$.

Now, as in Case ii, $D' = D + v$ can be joint by a segment of length δ with a translate B of A, the entire segment lying outside conv$G$.

Thus, the 3-move

$$\{\lambda A + (1-\lambda)D\}_{\lambda\in[0,1]} \cup \{\lambda D + (1-\lambda)D'\}_{\lambda\in[0,1]} \cup \{\lambda D' + (1-\lambda)B\}_{\lambda\in[0,1]}$$

is a valid migration path and has length close to 3δ, at most. $\square$

Conjecture. If a convex body A is held by the cage G and is close to G, then it can migrate through a 3-move of length close to 3diam$A$ to a translate of A outside G, keeping constant its diameter on the way.

Acknowledgements. The author expresses his thanks to Dr. A. Shabbir, who helped preparing this paper. He also thankfully acknowledges the financial support by the High-end Foreign Experts Recruitment Program of People's Republic of China. His work was also partly supported by a grant of the Roumanian National Authority for Scientific Research, CNCS–UEFISCDI, project number PN-II-ID-PCE-2011-3-0533.

References

[1] O. Aberth, An isoperimetric inequality for polyhedra and its application to an extremal problem, *Proc. London Math. Soc.* **13** (1963) 322–336.

[2] A. S. Besicovitch, A cage to hold a unit-sphere, Proc. Sympos. Pure Math., vol. VII, AMS, Providence, RI, (1963) 19–20.

[3] H. S. M. Coxeter, Review 1950, *Math. Reviews* **20** (1959) 322.

[4] A. Fruchard, T. Zamfirescu, Short cages holding convex bodies, to appear.

[5] L. Yuan, T. Zamfirescu, Cages holding discs, to appear.

T. Zamfirescu
Fachbereich Mathematik,
Universität Dortmund,
44221 Dortmund, Germany
and
Institute of Mathematics "Simion Stoilow",
Roumanian Academy,
Bucharest, Roumania
and
College of Mathematics and Information Science,
Hebei Normal University,
050024 Shijiazhuang, P.R. China
E-mail: `tudor.zamfirescu@mathematik.uni-dortmund.de`