Dissecting the square into five congruent parts

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Abstract.

We give an affirmative answer to an old conjecture proposed by Ludwig Danzer: there is a unique dissection of the square into five congruent convex tiles.

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Introduction and notation

In the eighties of the last century, Ludwig Danzer conjectured in several conferences that there is a unique dissection of the square into five congruent parts—see Figure 1. In its most general setting, the conjecture asks the parts to be finite unions of closed topological discs.

Danzer formulated his conjecture for the case that the parts are convex, and for the general case as well. We give here an affirmative answer for the case where the parts are convex.

Dissecting convex and other bodies was a frequent occupation of mankind since prehistorical times. We make no attempt here to evoke those efforts and achievements in arts (like painting and cuisine) and sciences, throughout the millennia. As just one example of relatively recent work, we mention Archimedes' "Ostomachion" [1], because he dissected precisely the square.

For many of the mathematical variants, we recommend Grünbaum and Shephard's authoritative book [3], but have to mention the existence of several other important books and surveys in this area.

Danzer's conjecture can be obviously generalized to one in which dissection into n congruent tiles is required, where n is any prime number not less than 3 (see Problem 4). The case $n = 3$ has been solved by Maltby [5].

For points $p, q \in \mathbb{R}^2$, let pq denote the line-segment from p to q, including p and q, and let $|pq|$ be its length. For $M \subset \mathbb{R}^2$, diam$M$, int$M$, bd$M$, $\mathcal{A}(M)$ denote its diameter, interior, boundary, area, respectively. The convex hull of the finite set $\{a_1, ..., a_n\} \subset \mathbb{R}^2$ will be denoted by $a_1 ... a_n$. The circle with centre x and radius r will be denoted by $C(x, r)$.

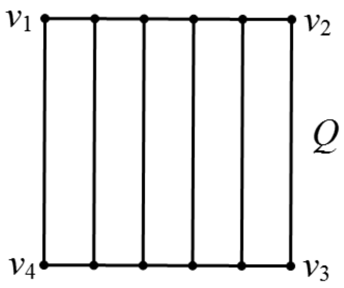

Figure 1

Consider the square $Q = [0,1]^2$.

A compact convex set $K \subset \mathbb{R}^2$ is called here a *tile*, if Q is the union of five congruent copies of K such that any two of them are either disjoint or have just boundary points in common. Throughout the paper, these five tiles will be denoted by $K_1, ..., K_5$. Obviously, K must be a convex polygon. Indeed, since the convex tiles K_i form a tiling of the square, the intersection of two tiles is either empty, or a single point, or a line-segment; so K has a boundary consisting of finitely many line-segments, and hence is a polygon. We will call here this particular dissection a *tiling.*

The boundaries of the five tiles form a graph, which has as vertices the vertices of the tiles and as edges their sides or parts of them, joining those vertices. We use the same term of *tiling* when referring to this graph.

Let $v_1 = (0,1)$, $v_2 = (1,1)$, $v_3 = (1,0)$, $v_4 = (0,0)$, and put $v_5 = v_1$. So Q has vertices v_i and sides $v_i v_{i+1}$ $(i = 1, ..., 4)$. Put $Q^* = \mathrm{bd}Q \setminus \{v_1, v_2, v_3, v_4\}$.

The main steps of our proof of Danzer's conjecture are these: first, we eliminate the possibility that the tiles are triangles. Then we eliminate several topologically different cases of tiling the square Q. Third, we show that some edge of Q must contain no vertex of the tiling, which provides the strong geometric property of the tiles of having a side of length 1. Finally, we are led to the obvious tiling.

Preparation

Lemma 1. *K is not a triangle.*

Proof. This is a direct consequence of Monsky's theorem saying that there is no tiling of the square into an odd number of triangles of equal areas [6]. Although the proof of Monsky's theorem is elegant and not too long, we give here a very simple argument for (the weaker) Lemma 1.

Suppose there exists a tiling of Q into five congruent triangles. The angle sum of the five triangles is 5π. The sum of the angles in the four corners of Q is 2π. Therefore, further vertices must account for precisely 3π—thus, they are at least two and at most three. Choose the points $p = (3/5, 3/5)$, $q = (0, 3/5)$.

Claim 1. The triangle v_1v_2p cannot be a tile. Indeed, suppose it is. We have $\angle v_1pv_2 > \pi/2$, $\angle pv_2v_1 = \pi/4$, $\angle v_2v_1p = \arctan\frac{2}{3}$.

Then another tile must have a vertex at v_1. Its angle there can only measure $\pi/4$ or $\arctan\frac{2}{3}$. Therefore there must be a further tile with vertex at v_1. The remaining angle at v_1 for this tile is at most

$$\frac{\pi}{2} - 2\arctan\frac{2}{3} < \arctan\frac{2}{3},$$

so this is impossible. See Figure 2(a).

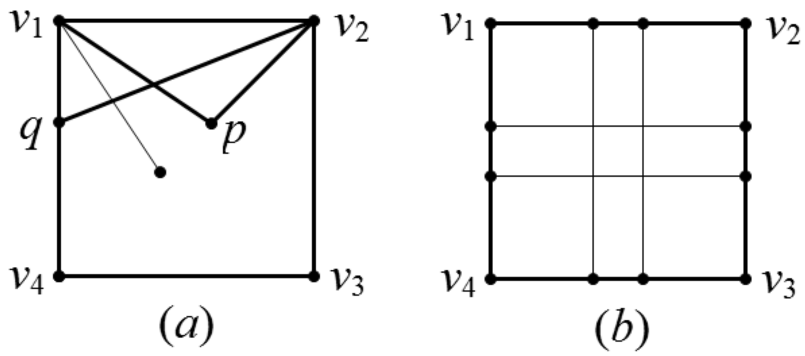

Figure 2

Claim 2. The triangle v_1v_2q cannot be a tile. Indeed, suppose it is. We have $|qv_1| = 2/5$, $|v_1v_2| = 1$, $|v_2q| = \sqrt{29}/5$. Then $|qv_4| = 3/5$. As a line-segment of length $3/5$ is not a union of line-segments of length at least $2/5$ with pairwise disjoint relative interiors, Claim 2 is true.

If, out of the at most three additional vertices of the tiling, i lie in $\text{int}Q$, then at most $3 - i$ of them belong to Q^*, and at least $4 - (3 - i) = i + 1$ sides of Q are left without such vertices. So, these sides are sides of triangles of the tiling, and therefore they require the existence of a vertex on each of some $i + 1$ lines among the four lines $x = 2/5$, $x = 3/5$, $y = 2/5$, $y = 3/5$, see Figure 2(b). The points in $\text{bd}Q$ lying on these lines cannot be used, by Claim 2.

Since there is one more line than vertices in $\text{int}Q$, two of the $i + 1$ lines must be served by the same interior vertex (or, for $i = 0$, there is no suitable vertex). This amounts to using a vertex "like" p, which is, however, excluded by Claim 1. $\square$

Lemma 2. *A tiling as in Figure 3(a) is impossible.*

Proof. Indeed, since one of the tiles is a quadrilateral, all must be quadrilaterals, so we must have the collinearities shown in Figure 3(b).

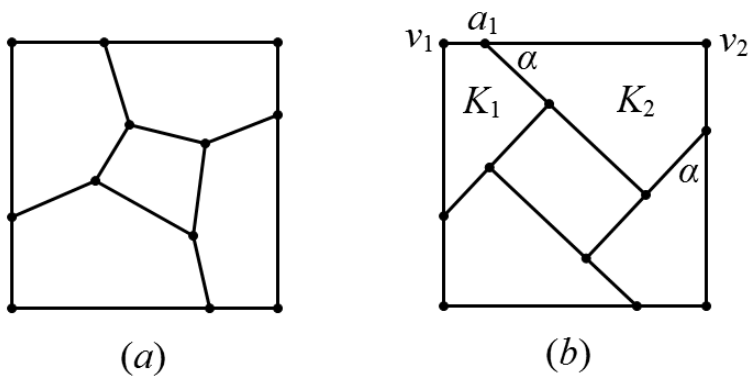

Figure 3

Suppose first that the angle of K_2 at a_1 is $\alpha < \pi/2$. Then K_1 has at a_1 an angle of $\pi - \alpha$, and K has two right angles, a third one measuring α and a fourth one measuring $\pi - \alpha$. This consequently holds for the tiles $K_1, ..., K_4$, whence K_5 is a rectangle, which is false.

Now, suppose that the angle of both K_1 and K_2 at a_1 is $\pi/2$, see Figure 4.

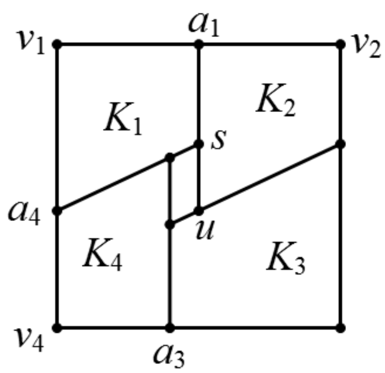

Figure 4

Assume that $\angle a_1 s a_4 \neq \pi/2$. Since K_1 is congruent with K_2, $|v_1 a_1| = |a_1 v_2| = 1/2$. Since K_1 is congruent with K_4 and $\angle s a_4 v_4 \neq \pi/2$, we have $|v_4 a_3| = 1/2$. But this does not allow K_5 to exist!

Assume now that $\angle a_1 s a_4 = \pi/2$. Then the tiles are rectangles. If a, b are their sides, they must satisfy the conditions $a + b = 1$ and $ab = 1/5$. With the solutions, which are unequal, we have congruent tiles $K_1, ..., K_4$ (containing $v_1, ..., v_4$, respectively), but the resulting K_5 is a square (of side-length $b - a$). □

Notice the equiangular solution obtained in the case studied last.

Lemma 3. *A tiling as in Figure 5(a) is impossible.*

Proof. Indeed, all tiles must be quadrilaterals, so the situation is as in Figure 5(b).

Suppose $\alpha = \angle w a_1 v_1 \neq \pi/2$. Then the angles of K measure $\pi/2, \pi/2, \alpha, \pi - \alpha$. Thus, the angle of K_2 at a_2 must be α or $\pi/2$. If it is α, K has two opposite right angles. Hence, in K_1,

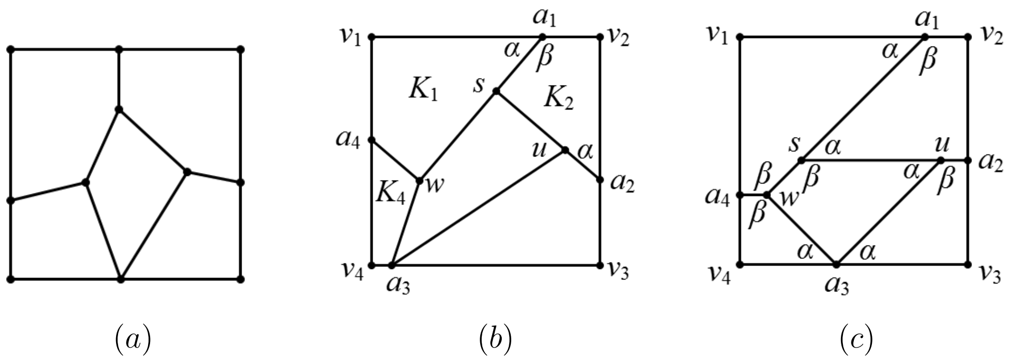

Figure 5

$\angle a_1 w a_4 = \pi/2$, where w is the neighbour of a_4 in int$Q$, and in K_4, $\angle a_3 w a_4 = \pi/2$, which implies that a_1, w, a_3 are collinear, which is wrong (because swa_3u must be a quadrilateral).

Hence $\angle v_2 a_2 s = \pi/2$, $\angle a_1 s a_2 = \alpha$, $\angle a_1 w a_4 = \pi - \alpha$, $\angle v_1 a_4 w = \pi/2$, $\angle a_4 w a_3 = \pi - \alpha$, $\angle w a_3 v_4 = \alpha$, $\angle v_3 a_3 u = \alpha$, $\angle a_3 u s = \alpha$.

Therefore $\angle s w a_3 = \angle w a_3 u = \pi/2$, which holds only if $2\alpha = \pi/2$.

Thus $\alpha = \pi/4$ leads to the nice equiangular tiling of Figure 5(c). But it is easily seen that $|a_1 s| \neq |a_1 w|$, and the tiles cannot be congruent. □

Lemma 4. *A tiling as in Figure 6 is not possible* (*including the case* $a_1 = a_1'$).

Proof. Suppose w.l.o.g. that $|v_4 a_3| \geq 1/2$. Since diamK_4 = diam$K_5 \geq |a_1 a_3| \geq 1$, we have $|a_3 a_4| \geq 1$ or $|v_4 s| \geq 1$ or $|a_3 s| \geq |a_1 a_3|$. Notice that the diameter of K_4 cannot be realized by $a_4 s$, because otherwise $|a_4 s| \geq |v_4 s|$ implies diam$K_1 \geq |v_1 s| > |a_4 s| =$ diamK_4.

Claim. $|a_3 v_4| > \sqrt{3}/2$.

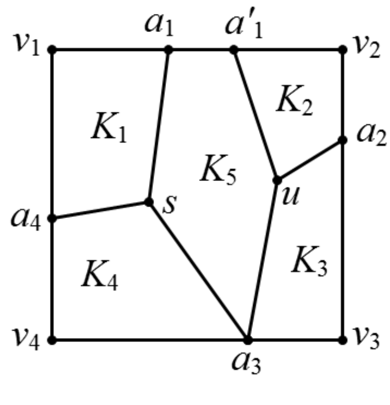

Figure 6

Indeed, assume $|a_3 v_4| \leq \sqrt{3}/2$.

In the case $|a_3 a_4| \geq 1$, we have $|a_4 v_4| = \sqrt{|a_3 a_4|^2 - |a_3 v_4|^2} \geq 1/2$. It is elementary to calculate that, for $1/2 \leq |a_3 v_4| \leq \sqrt{3}/2$, we have $\mathcal{A}(a_3 a_4 v_4) \geq \sqrt{3}/8 > 1/5$, which is absurd.

In the case $|v_4 s| \geq 1$, the point s lies on $C(v_4, 1)$ or outside it. Let s' be the intersection of $C(v_4, 1) \cap Q$ with the line through s parallel to $v_4 v_1$.

Consider the points $\beta = (3/5, 4/5), \gamma = (\sqrt{3}/2, 1/2), m = (1/2, 0)$, see Figure 7.

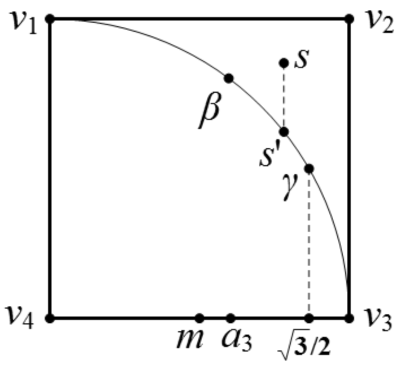

Figure 7

If s' belongs to the (relative) interior of the arc $\widetilde{v_1\beta}$ of $C(v_4, 1) \cap Q$ from v_1 to β, then $\mathcal{A}(a_3sv_4) \geq \mathcal{A}(a_3s'v_4) > \mathcal{A}(m\beta v_4) = 1/5$, which is impossible.

If s' belongs to the arc $\widetilde{\beta\gamma}$ of $C(v_4, 1) \cap Q$, then

$$\mathcal{A}(K_1 \cup K_4) \geq \mathcal{A}(v_1sa_3v_4) = \mathcal{A}(v_1sv_4) + \mathcal{A}(a_3sv_4) \geq \mathcal{A}(v_1\beta v_4) + \mathcal{A}(m\gamma v_4)$$
$$= \frac{1}{2} \cdot \frac{3}{5} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{17}{40} > \frac{2}{5},$$

which is false.

If s' belongs to the arc $\widetilde{\gamma v_3}$ of $C(v_4, 1) \cap Q$, we either have $\angle sa_3v_4 > \pi/2$, or $\angle sa_3v_4 \leq \pi/2$. Consider the point $s^* \in v_1v_2$ such that $s \in s^*a_3$.

If $\angle sa_3v_4 > \pi/2$, we have

$$\mathcal{A}\left(K_2 \cup K_3 \cup K_5\right) \leq \mathcal{A}\left(a_3v_3v_2s^*\right) = \frac{|a_3v_3| + |s^*v_2|}{2} \leq \frac{\frac{1}{2} + 1 - \frac{\sqrt{3}}{2}}{2} < \frac{3}{5}.$$

If $\angle sa_3v_4 \leq \pi/2$, consider the point $\gamma^* \in v_1v_2$ such that $\gamma \in \gamma^*a_3$. We have

$$\mathcal{A}(K_2 \cup K_3 \cup K_5) \leq \mathcal{A}(a_3v_3v_2s^*) \leq \mathcal{A}(a_3v_3v_2\gamma^*) = 1 - \frac{\sqrt{3}}{2} < \frac{3}{5}.$$

In both situations we obtained contradictions.

In the last case, $|a_3s| \geq |a_1a_3|$, from inspecting the triangle a_1sa_3 it follows that $\angle a_1sa_3 < \pi/2$. We saw already that $|a_3a_4| < 1$. Hence $|a_3s| > |a_3a_4|$, whence, similarly, $\angle a_4sa_3 < \pi/2$. But then, in K_1, $\angle a_1sa_4 > \pi$, which contradicts the convexity of K_1. So the Claim is completely verified.

We continue the proof. Since $\mathcal{A}(K_2 \cup K_3) = 2/5$, we must have $\mathcal{A}(v_2v_3a_3a_1') \geq 2/5$, which together with $|a_3v_4| > \sqrt{3}/2$ implies $|v_2a_1'| > \frac{\sqrt{3}}{2} - \frac{1}{5}$.

Then

$$\operatorname{diam} K_5 \geq |a_3a_1'| = \sqrt{1 + (|v_2a_1'| - |v_3a_3|)^2}.$$

As $|v_2a_1'| - |v_3a_3| > \sqrt{3} - \frac{6}{5} > \frac{1}{2}$, we have $\operatorname{diam} K > \sqrt{5}/2$.

Since $u \in v_2v_3a_3a_1'$, $|v_2u| < \sqrt{5}/2$. This implies that diam$K_2 = |a_2a_1'| > \sqrt{5}/2$. This yields $\mathcal{A}(v_2a_2a_1') \geq 1/4$, and consequently $\mathcal{A}(K_2) \geq 1/4$, which is wrong. □

Lemma 5. *A tiling as in Figure 8 is not possible.*

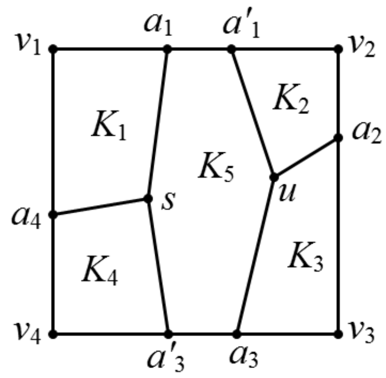

Figure 8

Proof. Indeed, since K_5 must be a quadrilateral, a_1, s, a_3' must be collinear, and a_1', u, a_3 must be collinear, too. Thus, K has two opposite sides of length at least 1. But this is impossible for K_1, as both $|v_1a_1|$ and $|v_1a_4|$ are less than 1. □

Result

Here we prove the result of this paper, which confirms Danzer's conjecture for convex tiles.

Theorem. *The tiling of the square with five congruent convex tiles shown in Figure 1 is unique.*

Proof. In the whole proof we use the fact that the tiles are not triangles, by Lemma 1.

Suppose that each side of Q contains some vertex different from the v_i's.

There can only be at most four vertices interior to Q. We argue combinatorially.

Let e be the number of interior edges, i that of the interior vertices, and b that of boundary vertices. Each face has at least four sides, each interior vertex has degree at least 3. By counting in the standard two ways the double of the number of interior edges, we get

$$2e \geq 20 - b, \qquad 2e \geq 3i + b - 4.$$

By Euler's formula,

$$(i + b) - (e + b) + 6 = 2.$$

We obtain $2 \leq i \leq 4$ for $b = 8$. The case $i = 4$ appears in the situation of Figure 3(a), eliminated by Lemma 2. We also obtain $1 \leq i \leq 2$ for $b = 10$, realized in Figure 8, and treated by Lemma 5. Otherwise we have $2 \leq i \leq 3$.

Case I. Each side $v_i v_{i+1}$ contains exactly one vertex a_i ($i = 1, ..., 4; v_5 = v_1$) in its relative interior.

If no edge starts at some v_i, then there is a tile with no edge on bd$Q$, contrary to Lemmas 2, 3, 4.

Hence, assume an edge $v_1 s$ exists. Now, asking that more than one interior edge starts at the same v_i or a_i leads to no solution (respecting Lemma 1).

Denote by u the neighbour of a_1 different from v_1 and v_2.

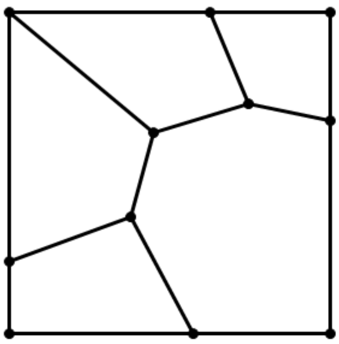

Figure 9

Let $\theta = \angle a_1 v_1 s$.

Taking only into account that the tiles are not triangles, we are led to the situation illustrated in Figure 9. As all tiles must then be quadrilaterals, there are in fact only two possibilities, depicted in Figure 10. The following proof works for both possibilities.

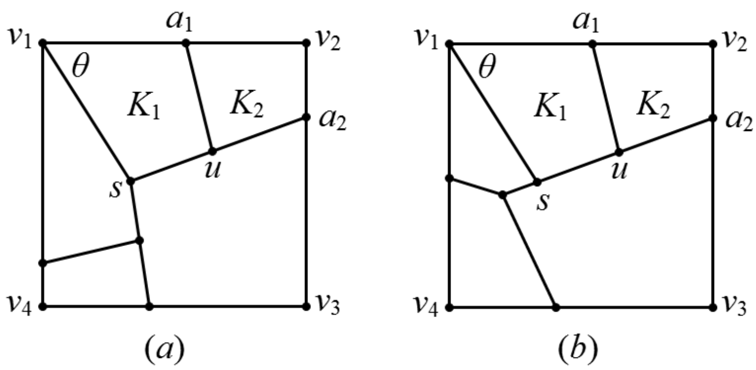

Figure 10

Since the angle at v_2 is right, K has a right angle and the angle $\theta < \pi/2$.

If $\angle sua_1 = \pi/2$ then $K_1 = v_1 sua_1$ has no opposite right angles. But $K_2 = a_1 ua_2 v_2$ has such angles, those in u and v_2, which is absurd.

If $\angle ua_1v_1 = \pi/2$ then K_2 has two neighbouring right angles. The same must have K_1, too, so its angle at u must be right, which implies that K_2 is a rectangle, which is false under our present hypotheses.

If $\angle v_1su = \pi/2$, then K has two adjacent angles measuring $\pi/2$ and θ. Moreover, $\angle ua_2v_2 = \pi - \theta > \pi/2$. Then K_2 must have its adjacent angles measuring $\pi/2$ and θ either

(i) at v_2 and a_1, or

(ii) at a_1 and u.

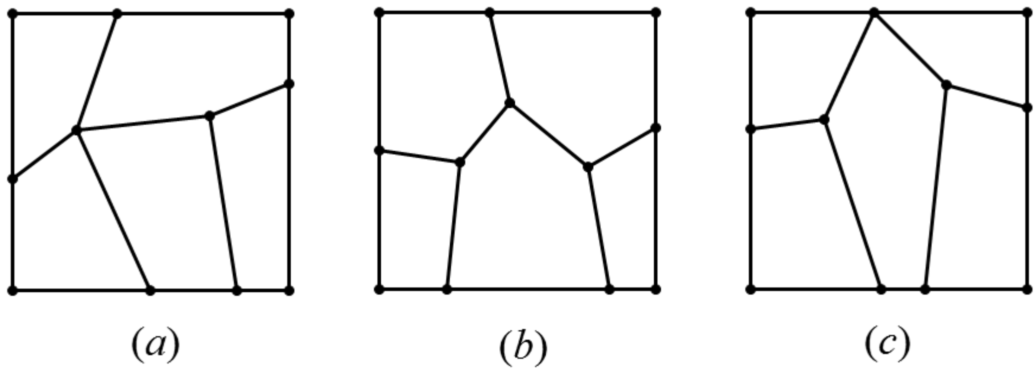

Figure 11

In case (i), v_1s and a_1u are parallel, $\angle a_1ua_2 = \pi/2$, so K_1 has adjacent right angles, while K_2 has not.

In case (ii), K_2 has adjacent right angles, and K_1 not.

Case II. Each side v_iv_{i+1} contains exactly one vertex a_i in its relative interior, except for one side, say v_3v_4, which contains two.

Since there are no triangles, we have only three possibilities, displayed in Figure 11. Lemma 4 forbids the possibility in Figure 11(c). The existence of quadrilaterals implies that all tiles are quadrilaterals. This implies collinearities in both cases of Figures 11$(a), (b)$, see Figure 12.

We treat first the case of Figure 12(a), see Figure 13.

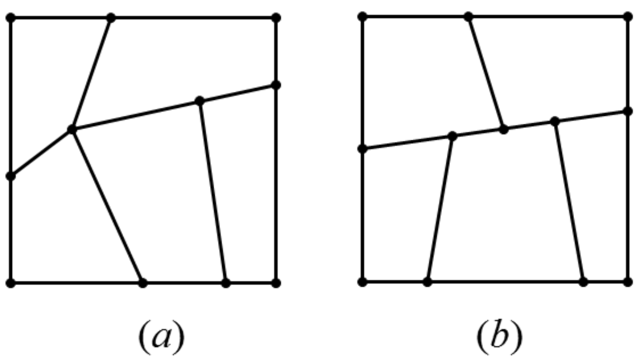

Figure 12

Case 1. $\angle sa_2v_2 = \pi/2$.

If the tiles are rectangles, then a_2, s, a_4 are on the line $y = 3/5$, as $\mathcal{A}(K_1) + \mathcal{A}(K_2) = 2/5$. Since $\mathcal{A}(K_1) = 1/5$, $|a_4s| = 1/2$. But K_4 has a side $|a_4v_4| = 3/5$, and we have a contradiction.

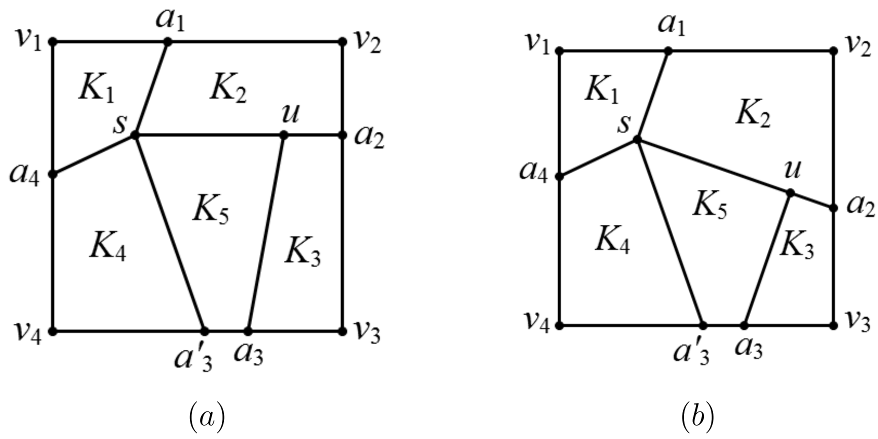

Figure 13

If the tiles are not rectangles, let $\angle v_1a_1s = \alpha \neq \pi/2$. See Figure 13(a). Then, in K_1, $\angle v_1a_4s = \pi/2$. Thus, a_2, s, a_4 are again collinear, and $|v_1a_4| = 2/5$. The tile K_4 has right angles at a_4 and v_4, but $|v_4a_4| = 3/5$, and we obtain again a contradiction.

Case 2. $\angle sa_2v_2 = \alpha \neq \pi/2$.

Then, in K_3, $\angle v_3a_3u = \alpha$, $\angle v_3a_2u = \pi - \alpha$, $\angle a_2ua_3 = \pi/2$, and K_3 has two opposite right angles. See Figure 13(b). Further, $\angle sa_3'a_3 = \pi/2$, whence K_4 has two adjacent right angles, and no opposite such angles, absurd.

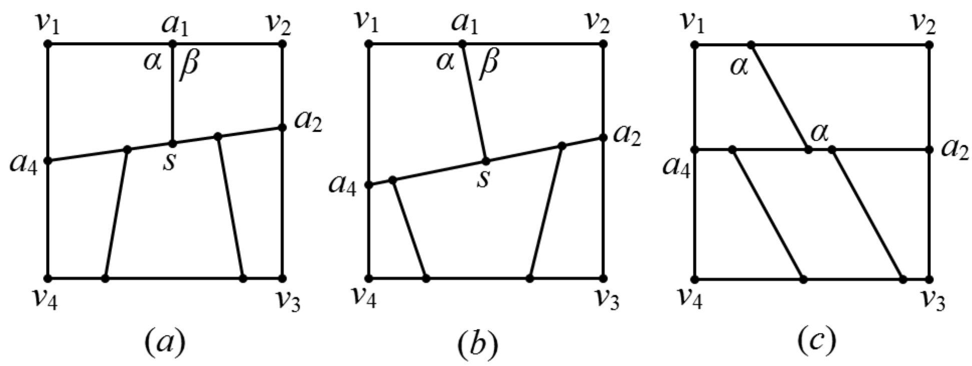

Figure 14

We now treat the case of Figure 12(b).

Let s be the neighbour of a_1 in int$Q$. Assume first that $\alpha = \beta = \pi/2$, where $\alpha = \angle sa_1v_1$ and $\beta = \angle sa_1v_2$. See Figure 14(a). If $a_4sa_1v_1$ and $a_2sa_1v_2$ are not rectangles, then the length of sa_1 lies between the lengths of a_4v_1 and a_2v_2, and the two tiles are not congruent. Hence K is a rectangle.

Then $a_4 = (0, 3/5)$, since the three tiles meeting v_3v_4 make up together 3/5 of the whole area. Thus, they have sides of length 3/5, 1/3. The remaining tiles have a side of length 2/5. A contradiction is reached.

Assume now w.l.o.g. that $\alpha > \beta$. Then $\angle sa_2v_2 = \alpha$ or $\angle a_1sa_2 = \alpha$.

In the first case, it follows that $\angle v_1a_4s = \beta$ and the congruence of the tiles $a_4sa_1v_1$ and $a_2sa_1v_2$ yields $|a_4v_1| = |a_1v_2|$, $|a_1v_1| = |a_2v_2|$, $|a_4s| = |a_1s|$, $|a_1s| = |a_2s|$, and $\angle a_1sa_4 = \angle a_1sa_2 = \pi/2$. See Figure 14(b). Thus, s is the centre of Q, and the tiles $a_4sa_1v_1$ and $a_2sa_1v_2$ occupy an area of 1/2, which is too much.

In the second case, the line-segment a_2a_4 is parallel to v_1v_2, see Figure 14(c). The tile containing a_4v_4 and the tile containing a_2v_3 have each two right angles and two angles equal to α and β. This leaves the third tile under a_2a_4 without any right angle, which gives a contradiction.

Case III. Each side v_iv_{i+1} contains exactly one vertex a_i in its interior, except for two opposite sides, each of which contains two.

This situation of six vertices in Q^*, i.e. $b = 10$, appears indeed in Figure 8 and is solved by Lemma 5. It is easily seen that there are no further possibilities with six vertices in Q^*, admitting at least one of them on each side of Q and respecting Lemma 1.

We know now that at least one side of Q, say v_1v_4, has no vertex in its (relative) interior. Hence, a side of K has length 1. Let K_4 be the tile including v_1v_4.

Assume first that K_4 has two acute angles, β at v_1 and α at v_4, not both $\pi/4$. W.l.o.g. $\alpha \leq \beta$. Thus $\alpha < \pi/4$. Indeed, if $\alpha \geq \pi/4$, then $\beta > \pi/4$, and some tile has at v_1 an angle $\gamma < \pi/4$, whence K has all angles α, β, γ with $\alpha + \beta + \gamma < \pi$, which is not possible. Then all other angles of K_4 are obtuse.

Another tile with an edge on v_1v_2, say K_1, has a vertex at v_1, too. Its angle at v_1 is at most $\frac{\pi}{2} - \beta$. This angle is necessarily α if $\beta \geq \pi/4$, but can also be β if $\beta < \pi/4$. We consider the case that this angle is α, the other case being analogous.

All sides of K_4 but one, v_1v_4, have length less than 1. If v_1v_2 is not the side of K_1 of length 1, then K_1 has a side v_1s of length 1, with $s \in \mathrm{int}Q$. Its angle at s must be β. See Figure 15.

Since K_1 has at least four sides, the third and the fourth side (not v_1s and not on v_1v_2) are common edges with other two tiles, K_2 and K_3. Since $\alpha < \pi/4$ and $\alpha + \beta \leq \pi/2$, we have $K_1 \cup K_2 \cup K_3 \subset X$, where $X = v_1v_2v_3 \cup v_3t't$. Here, $t \in v_1v_3$ has distance 1 from v_1, and t' is its orthogonal projection onto v_3v_4. But

$$\mathcal{A}(X) = \frac{1}{2} + \left(\sqrt{2} - 1\right)^2 \cdot \frac{1}{2} \cdot \frac{1}{2} < \frac{3}{5},$$

which is false. Hence K_1 has a vertex at v_2, and an angle β there. Thus, a tile K_2 has an angle α at v_2 and, analogously, an angle β at v_3, while a tile K_3 has an angle α at v_3 and an angle β at v_4.

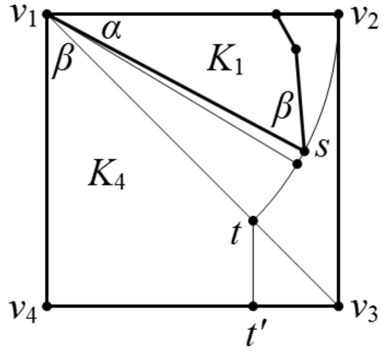

Figure 15

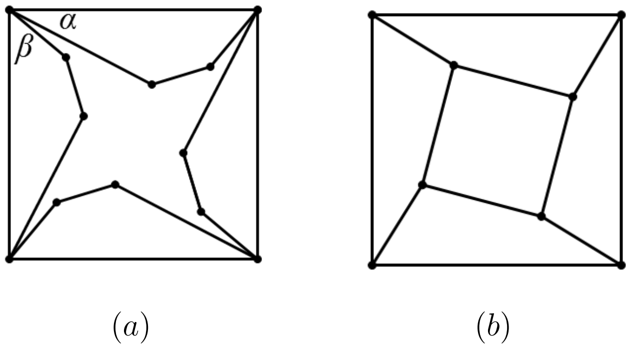

Figure 16

If $\alpha + \beta < \pi/2$, we are led to the existence of a huge non-convex tile, see Figure 16(a). So, $\alpha + \beta = \pi/2$. Since the tile in int$Q$ is convex, it must be a quadrilateral, see Figure 16(b). Since K_1, K_2, K_3, K_4 are congruent, K_5 is a square, which is impossible.

Assume now that K has a side of length 1 and both incident angles measure $\pi/4$, or one of them measures $\pi/4$ and the other $\pi/2$.

Suppose that both K_4 and K_1 have at v_1 the angle $\pi/4$. See Figure 17.

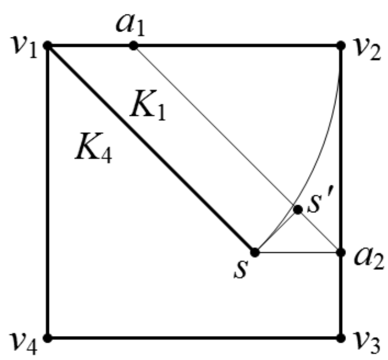

Figure 17

Assume K_1 has not v_1v_2 as a side. Then it has a side v_1s of length 1, and another side $v_1a_1 \subset v_1v_2$. A third tile K_2 must then have a vertex at a_1. If $\angle v_2a_1s \geq \pi/2$, then $\mathcal{A}(v_1sa_1) \geq 1/4$, absurd. Hence $\angle v_2a_1s < \pi/2$. Consequently, K_2 must have at a_1 an angle of $\pi/4$. If $a_2 \in v_2v_3$ is chosen such that $\angle a_2a_1v_2 = \pi/4$, we must have $|a_1a_2| \geq 1$, because $\mathrm{diam}K_2 \geq 1$. It follows that $|v_1a_1| \leq 1 - \frac{1}{\sqrt{2}}$. If s' denotes the orthogonal projection of s onto a_1a_2, we have $K_1 \subset v_1ss'a_1$. We calculate

$$\mathcal{A}(v_1ss'a_1) \leq \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} - \frac{1}{4}\left(1 - \frac{1}{\sqrt{2}}\right)^2 = \frac{6\sqrt{2}-7}{8} < \frac{1}{5},$$

and obtain a contradiction.

Hence K_1 has v_1v_2 as a side. If K has two angles measuring $\pi/4$ each, then all other angles are obtuse, and we are led to the tiling of Figure 18(a), which displays a rhombus as a tile. This is impossible. Hence, K has, incident to a side of length 1, two angles, one measuring $\pi/4$ and the other $\pi/2$. Any other angle of K is larger than $\pi/4$. Thus, the tiles K_4 and K_1 are like in Figure 18(b). If v_1s is their common edge, then $s \in v_1v_3$.

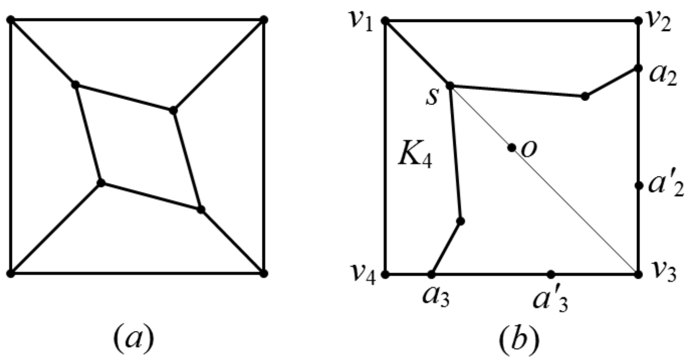

Figure 18

Suppose a tile K_2 has a right angle at v_3. Then, according to the congruence between K_2 and K_4, the vertex v_3 either corresponds to v_4 or to a_3. In both cases the side of K_2 corresponding to v_1v_4 joins a point of v_3v_4 with a point of v_1v_2, or a point of v_2v_3 with a point of v_1v_4,which is impossible.

Hence, we must have two angles measuring $\pi/4$ at v_3, belonging to two tiles, say K_2, K_3.

As the angle $\pi/4$ is adjacent to a side of length 1, such a side must be included in v_1v_3. This yields $|v_1s| \leq \sqrt{2} - 1$. Hence, one of the tiles K_2, K_3, say K_3, has a side $v_3a_3' \subset v_3v_4$ with $|v_3a_3'| = |v_1s| \leq \sqrt{2} - 1 < \frac{1}{2}$, while K_2 has a side $v_3a_2' \subset v_3v_2$ of the same length. The remaining line-segments a_2a_2' and a_3a_3' must be sides of a tile K_5, which cannot be convex, since $a_2'a_3'$ meets $\mathrm{int}K_2$.

Hence, both angles of K_4 at v_1 and v_4 are right. K has now at least two right angles, and therefore at most one acute angle. Let v_4, a_3 be the vertices of K_4 on v_3v_4. Since $\mathcal{A}(K_4) < 1/2$, $\angle v_1a_3v_4 > \pi/4$, whence the angle of K_4 at a_3 is also larger than $\pi/4$. If such an angle is accommodated at v_2 or v_3, then we have there another angle, smaller than $\pi/4$, but such an angle is not available. Hence, there is another tile, K_2, with v_2v_3 as a side and with right angles at v_2, v_3.

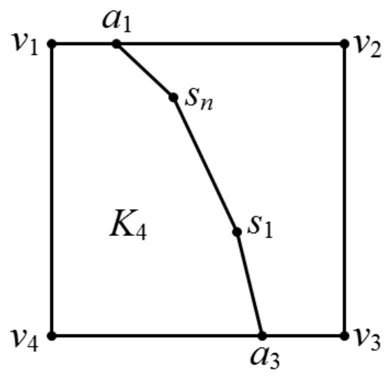

Figure 19

Let $K_4 = v_1v_4a_3s_1s_2...s_na_1$ with $a_1 \in v_1v_2$, $a_3 \in v_3v_4$. See Figure 19. Assume w.l.o.g. that $\angle v_4a_3s_1 \leq \angle v_1a_1s_n$.

If K_4 is not a quadrilateral, then

$$\angle v_1a_1s_n + \angle v_4a_3s_1 > \pi.$$

If $\angle v_4a_3s_1 < \pi/2$, then K has exactly one acute angle. Some tile different from K_4 must have at a_1 an angle measuring at most $\pi - \angle v_1a_1s_n$, which is smaller than $\angle v_4a_3s_1$, and a contradiction is obtained.

If $\angle v_4a_3s_1 \geq \pi/2$, then K_4 has no acute angle, but $\pi - \angle v_1a_1s_n < \pi/2$, and some tile must have an acute angle at a_1, absurd.

Hence, K is a quadrilateral with angles $\pi/2, \pi/2, \alpha, \pi - \alpha$, where w.l.o.g. $\alpha \leq \pi/2$.

If $\angle a_1a_3v_4 = \alpha < \pi/2$, then some tile must have at a_1 the angle α, because $\angle v_2a_1a_3 = \alpha$ and the other angles of K are not acute. As $|a_1v_2| < 1 < |a_1a_3|$, only one possibility exists for K_1, and $K_4 \cup K_1$ is a rectangle. Analogously, $K_2 \cup K_3$ is another rectangle, and consequently K_5 is a rectangle, absurd. Hence, $\alpha = \pi/2$ and we get the tiling of Figure 1. □

Epilogue

Our proof of Danzer's conjecture did not separate combinatorial from geometric tools. It intended to use the whole power of the strong requirement asked to be fulfilled, in order to obtain a reasonably short proof.

We would like to mention that we started our investigation by using Euler's formula and other combinatorial arguments, reaching the conclusion that the tiles must be triangles or quadrilaterals. However, that type of argument did not help further. Geometric tools became necessary, and the new arguments made the previous combinatorial insight almost redundant; so dropping it completely shortened the paper.

Problem 1. Does every dissection of the square into five similar convex tiles use right isosceles triangles or rectangles as tiles?

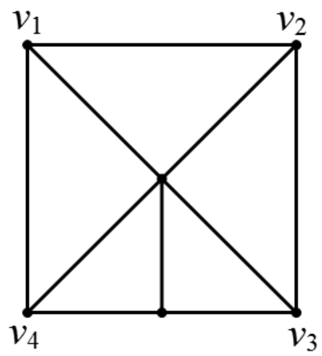

Figure 20

It is easily seen that Figure 1 does not show the only such tiling using rectangles. For example, one of the rectangles, of increased diameter, can be horizontal, the others, of diminished diameter, vertical. Similarly, Figure 20 does not show the only dissection with five right isosceles triangles.

Problem 2. Does every dissection of the square into five equiangular convex polygons use only angles measuring $\pi/4$, $\pi/2$, $3\pi/4$?

Here, two polygons are *equiangular*, if there exists a bijection between their vertex sets respecting the order of the vertices, such that the angles at corresponding vertices be equal.

Problem 3. Find all dissections of the square into five equiangular non-rectangular convex polygons.

Häggkvist, Lindberg, and Lindström [4] estimated the number of dissections of the square into n rectangles of equal areas, thus answering a question by Ihringer in Moser's work [7], see also [2].

Problem 4. Is every dissection of the square into n congruent convex tiles necessarily the "standard" one (i.e. analogous to Figure 1) if $n \geq 3$ is a prime number?

Maltby [5] solved Problem 4 for the case $n = 3$. We solved it for the case $n = 5$ in the present paper, so it remains open for $n \geq 7$.

Besides, Danzer's conjecture in more general settings (see the first section) remains open.

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