Two Problems on Cages for Discs

Luis Montejano and Tudor Zamfirescu

2010 Mathematics Subject Classfication: 52A15

A *cage* is the 1-skeleton of a polytope in $\mathbb{R}^3$. It is said to *hold* a compact set K disjoint from the cage if no rigid motion can bring K in a position far away without meeting the cage on its way. A compact 2-dimensional ball in $\mathbb{R}^3$ will be called a *disc*.

For any cage G, let $\mathcal{D}(G)$ be the space of all discs held by G, equipped with the Pompeiu-Hausdorff metric.

Let $\mathcal{D}_r(G)$ be the set of all discs in $\mathcal{D}(G)$ of radius at least r. Assume that, for some component $\mathcal{E}$ of $\mathcal{D}_r(G)$ and any number $s > r$, $\mathcal{D}_s(G) \cap \mathcal{E}$ is connected or empty. We call such a component $\mathcal{E}$ an *end-component* of $\mathcal{D}(G)$. If n is the maximal number of pairwise disjoint end-components of $\mathcal{D}(G)$, we say that G *holds n discs.* See Fig. 1.

The investigation of cages holding convex bodies seems to have started in 1959 with a problem by H.S.M. Coxeter [1], later settled by Aberth and Besicovitch. The following results were proved in [2].

Theorem 1 *The regular tetrahedral cage holds 16 discs.*

L. Montejano (✉)
Instituto de Matemáticas, National University of Mexico, Queretaro, Mexico

T. Zamfirescu
Fachbereich Mathematik, Universität Dortmund, 44221 Dortmund, Germany
e-mail: tudor.zamfirescu@mathematik.uni-dortmund.de; tuzamfirescu@googlemail.com

T. Zamfirescu
Institute of Mathematics "Simion Stoilow", Roumanian Academy, Bucharest, Romania

T. Zamfirescu
College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050024, People's Republic of China

K. Adiprasito et al. (eds.), *Convexity and Discrete Geometry Including Graph Theory*, Springer Proceedings in Mathematics & Statistics 148,
DOI 10.1007/978-3-319-28186-5_24

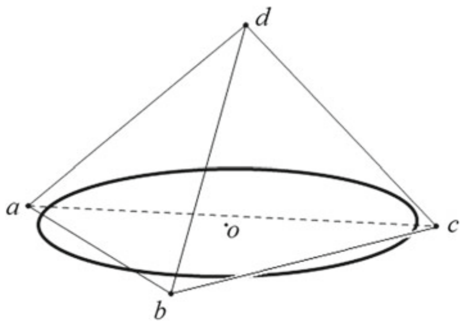

Fig. 1 Example of a disc held by a tetrahedral cage

Theorem 2 *There are tetrahedral cages holding n discs, for every* $n \leq 16$ *except for* $n \in \{7, 9, 11, 13, 14, 15\}$*, and there is no such cage for any other n.*

And now the problems:

1. Does a cage holding 7 discs exist?
2. How many discs can be held by a pentahedral cage?

Acknowledgement The second author's contribution was partly supported by a grant of the Roumanian National Authority for Scientific Research, CNCS–UEFISCDI, project number PN-II-ID-PCE-2011-3-0533.

References

1. H. S. M. Coxeter, *Review 1950*, Math. Reviews, **20** (1959) 322
2. L. Yuan, T. Zamfirescu, *Tetrahedral Cages for Discs*, manuscript