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SELFISHNESS OF CONVEX BODIES

LIPING YUAN AND TUDOR ZAMFIRESCU

ABSTRACT. Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$-*convex* if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$.

We call a family $\mathcal{F}$ of compact sets *complete* if $\mathcal{F}$ contains all compact $\mathcal{F}$-convex sets. A single convex body K will be called *selfish*, if the family of all convex bodies similar to K (resulting from an isometry and a dilation) is complete. We investigate here the selfishness of convex bodies.

1. INTRODUCTION

Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$-*convex* if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$.

The second author proposed at the 1974 meeting on Convexity in Oberwolfach the investigation of $\mathcal{F}$-convexity, for various families $\mathcal{F}$. Obviously, usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are all examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

Blind, Valette and the second author [1], and also Böröczky Jr [2], investigated the rectangular convexity, the case when $\mathcal{F}$ contains all non-degenerate rectangles.

Magazanik and Perles dealt with staircase connectedness, a special kind of polygonal connectedness [5].

In [9] the second author studied the case when $\mathcal{F}$ is the family of all right triangles in a Hilbert space of dimension at least 2.

In [7] we generalized the type of convexity studied in [9] and introduced the right triple convexity, where $\mathcal{F}$ is the family of all triples $\{x, y, z\}$ such that $\angle xyz = \pi/2$. See also [6].

In this paper, we call a family $\mathcal{F}$ of compact sets *complete* if $\mathcal{F}$ contains all compact $\mathcal{F}$-convex sets. A single convex body K will be called *selfish*, if the family $\mathcal{F}_K$ of all convex bodies similar to K (resulting from an isometry followed by a homothety, i.e. dilation/contraction) is complete. We investigate here the selfishness of convex bodies.

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For distinct $x, y \in \mathbb{R}^d$, let $\overline{xy}$ be the line through x, y and xy the line-segment from x to y. If the lines or line-segments L, L' are parallel, we write $L \| L'$.

As usual, for $M \subset \mathbb{R}^d$ with $d \geq 2$, $\mathrm{cl}M$ denotes its topological closure, $\mathrm{bd}M$ its boundary, and $\mathrm{diam}M = \sup_{x,y \in M} \|x - y\|$. A 2-point set $\{x, y\} \subset M$ with $\|x - y\| = \mathrm{diam}M$ is called a *diametral pair* of M, while xy is a *diameter* of M.

For any compact set $C \subset \mathbb{R}^d$, let S_C be the smallest hypersphere containing C in its convex hull.

A d-dimensional compact convex set in $\mathbb{R}^d$ is called a *convex body*. Let $\mathcal{K}$ be the space of all convex bodies in $\mathbb{R}^d$, endowed with the Pompeiu-Hausdorff metric. A convex body K is called *long* if $\mathrm{card}(K \cap S_K) = 2$.

The unit ball is denoted by B, and $\mathrm{bd}B = S$.

It is immediately seen that the (circular) disc in $\mathbb{R}^2$ is selfish, as well as every ball in $\mathbb{R}^d$. A little work is needed to show that the square is selfish too.

Theorem 1.1. *The square is selfish*

Proof. Let Q be a square, and let K be $\mathcal{F}_Q$-convex. We show that K is also a square.

Let ac be a diameter of K. By the definition of $\mathcal{F}_Q$-convexity, a and c must belong to a square included in K, which can only be the square $abcd$ with ac as a diagonal. See Figure 1.

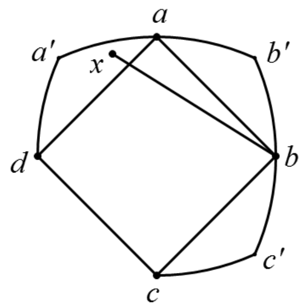

FIGURE 1

The discs of radius diam K and centres at a, b, c, d intersect in a "curved square" $\widetilde{a'b'c'd'}$ with a in the circular arc $\widetilde{a'b'}$, $b \in \widetilde{b'c'}$, $c \in \widetilde{c'd'}$, $d \in \widetilde{d'a'}$. Clearly, $K \subset \widetilde{a'b'c'd'}$.

Assume the existence of a point $x \in K$ in $\widetilde{a'ad}$. The points x, b belong to some square P included in $\widetilde{a'b'c'd'}$. This is only possible if b is a vertex of $P = a^*bc^*d^*$. We claim that an edge, say ba^*, of P lies between bx and ba, possibly along one of these. Indeed, if $a^* \in \widetilde{abb'} \setminus ab$, then its diagonal bd^* lies entirely in the triangle abd, as a^*ab and d^*db are similar, and so $\angle bdd^* = \angle baa^* < \pi/4$; but then $x \notin P$. Due to the claim, bc^* gets out of $\widetilde{a'b'c'd'}$, except for the case $x \in ad$, when $a^* = a$ and $P = abcd$.

Hence, $K = abcd$ and the proof is finished. $\square$

We presented the above proof as an introductory example. In many cases the proof of selfishness is necessarily more involved. In fact, Theorem 1.1 will be strengthened in the next section (Theorem 2.4).

2. Selfish convex bodies and non-selfish convex bodies

A continuous real function defined on an interval $[a, b]$ is *unimodal* if it is non-decreasing on a subinterval $[a, c]$ and non-increasing on $[c, b]$. A C^2-arc A in $\mathbb{R}^2$ will be called here *unimodal*, if its curvature radius at $x \in A$ is a unimodal function of arc-length (from an endpoint of A to x).

Theorem 2.1. *Suppose $K \subset \mathbb{R}^2$ is a long convex body. If at least one of the arcs in bd K between the two points of $K \cap S_K$ is unimodal, then K is not selfish. In particular, the convex bodies bounded by ellipses different from circles are non-selfish.*

Proof. We show that the disc B is $\mathcal{F}_K$-convex, for this K.

Let K' be similar to K and such that $K' \cap S = \{(0, -1), (0, 1)\}$. Let K' move, such that its lower part rolls inside B, the contact point $p(\theta)$ moving to the right. Here, θ denotes the angle of rotation of K'. Of course, in this way the upper part of K' gets out of B, as diam$K' = 2$. Apply a homothety of centre p which transforms K' into a convex body (of maximal size) K'' inside B. Thus, K'' has a second contact point $q(\theta)$ with S in its upper part. Due to the C^2 condition, p and q depend continuously on θ. The movement can be continued, until $p(\theta)$ and $q(\theta)$ become the point corresponding to the point of maximal radius of curvature on the right arc in bdK' joining $(0, -1)$ to $(0, 1)$. On the way, $p(\theta)$ and $q(\theta)$ realize all distances between 0 and 2.

Notice that we did not impose any differentiability condition on bd$K$ at its diametral pair of points. Indeed, if K' is not differentiable at $(0, -1)$, it will first just turn around $(0, -1)$, and $p(\theta)$ will remain at $(0, -1)$ as θ travels through some interval, while $q(\theta)$ will move from the very beginning. □

The condition card$(K \cap S_K) = 2$ alone does not guarantee non-selfishness. For a rhombus R, for example, the circular disc is not $\mathcal{F}_R$-convex, and indeed R is selfish! See Theorem 2.4. But, while each rhombus is selfish, the family $\mathcal{R}$ of all of them is not complete! Again, the disc is $\mathcal{R}$-convex.

Not every polygon is selfish either. For instance, if an edge ab of the polygon P is a diameter of S_P, then the disc is $\mathcal{F}_P$-convex, as one easily verifies. So, among the triangles, all non-acute ones are non-selfish. Are all acute ones selfish?

Theorem 2.2. *The equilateral triangle is selfish.*

Proof. Let T be an equilateral triangle, and let K be $\mathcal{F}_T$-convex.

Let ab be a diameter of K. The points a and b belong to an equilateral triangle included in K, which can happen only if ab is a side of that triangle. Let c be its third vertex, and consider the Reuleaux triangle $\widetilde{abc}$. Of course, $K \subset \widetilde{abc}$.

Assume, like in the preceding proof, that there is a point $x \in K \setminus abc$, for example in the component D of $\widetilde{abc} \setminus abc$ with ac on its boundary. There exists an equilateral triangle T' containing x and b, and included in K. Then T' must obviously have a vertex at b and an edge by between bx and ba, possibly along one of these. Then the other edge of T' starting at b is not included in $\widetilde{abc}$, unless $x \in ac$ and $y = a$.

Consequently, $K = abc$. □

One may be tempted to believe that every acute triangle is selfish. However, this is wrong.

Theorem 2.3. *There exist non-selfish acute triangles.*

Proof. Let $abcde$ be a regular pentagon. Consider $x \in cd$ and $y \in bc$ such that $xy \| bd$. Since $\|x-d\| < \|x-y\|$ at $x = d$, but $\|x-d\| > \|x-y\|$ at $x = c$, for some position $x = x_0$, $y = y_0$, we have $\|x_0 - d\| = \|x_0 - y_0\|$. Now take $z_0 \in ae$ with $\|a - z_0\| = \|c - x_0\|$, and put $\alpha = \angle cdz_0$. The triangle x_0dz_0 is acute. Indeed, its largest angles are $\angle dx_0z_0 = \angle dca = 2\pi/5$ and α. For x at the midpoint of cd, still $\|x-d\| < \|x-y\|$, so $\|x_0 - d\| > \|x_0 - c\|$, hence $\|z_0 - e\| > \|z_0 - a\|$. Denoting by m' the midpoint of the smaller arc $\widetilde{ae}$ of the circumscribed circle of $abcde$ and by m the midpoint of ae itself, we have $\angle z_0dc < \angle mdc < \angle m'dc = \pi/2$. See Figure 2.

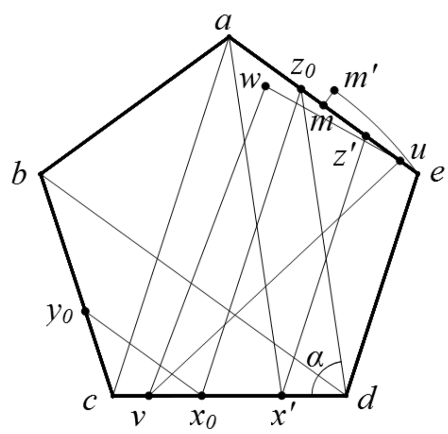

Figure 2

We shall prove that the triangle x_0dz_0 is not selfish, by showing that $abcde$ is $\mathcal{F}_{x_0dz_0}$-convex.

It suffices to show that any pair of points $u, v \in \mathrm{bd}\,abcde$ belong to some triangle from $\mathcal{F}_{x_0dz_0}$ included in $abcde$.

Choose $x' \in cd$ and $z' \in ae$ to satisfy $ax' \| dz_0$ and $x'z' \| de$.

If $u \in az_0$, $v \in cx'$ and $\|a-u\| \le \|c-v\|$, then the triangle $uv'u'$ is the desired triangle, u', v' being the points of cd with $uu' \| z_0d$ and $uv' \| ac$.

The position $u = z_0$ is special, because in this case not only $z_0x_0d \in \mathcal{F}_{x_0dz_0}$, but also $z_0x_0y_0 \in \mathcal{F}_{x_0dz_0}$. Indeed, z_0x_0d and $z_0x_0y_0$ are congruent, as they share the side x_0z_0, $\|x_0 - d\| = \|x_0 - y_0\|$, and $\angle dx_0z_0 = \angle dca = \angle edb = \angle z_0x_0y_0$.

Now, for any $u \in z_0e$ and $v \in x_0d$ with $\|a-u\| \ge \|c-v\|$, there exists a triangle with uv as a side, similar to $z_0x_0y_0$ (and with same orientation), and included in $abcde$.

For $u \in z'e$ and $v \in x_0c$, take w such that $\angle vuw = 2\pi/5$ and $\angle uwv = \alpha$. See Figure 2b. Notice that the triangles vuw and acx' are similar, and $\|u - v\| \le \|u - c\| \le \|e - c\| = \|a - c\|$, whence $\|u - w\| \le \|c - x'\|$.

Then $w \in abcde$, because $\|u - w\| \le \|c - x'\| = \|a - z'\| \le \|u - a\|$ and $\angle vuw = \angle cea \le \angle vua$.

All essentially different positions of the pair u, v in $cd \cup ea$ have thus been considered, and this ends the proof, because all other positions are either analogous or unproblematic. □

Remark. The proof above shows that, not only the triangle x_0dz_0, which is similar to acz', is non-selfish, but also each triangle acs, for $s \in mz'$.

Regarding families $\mathcal{F}$ of sets, an interesting case is that of all rectangles. This $\mathcal{F}$-convexity was studied in [1] and [2]. The family $\mathcal{F}$ is not complete, but a characterization of $\mathcal{F}$-convexity in the compact case is still missing.

Theorem 2.4. *Every rhombus in* $\mathbb{R}^2$ *is selfish.*

Proof. Let $R \in \mathbb{R}^2$ be a rhombus, with an angle $\gamma \ge \pi/2$.

Let $K \in \mathcal{K}$ be $\mathcal{F}_R$-convex. For $x \in \mathrm{bd}K$, choose $y, z \in \mathrm{bd}K$ such that $\angle yxz = \gamma$. If $\|x - y\| < \|x - z\|$, rotate the angle about x, until $\|x - y\|$ and $\|x - z\|$ become equal. This rotation is not possible if $xz \subset \mathrm{bd}K$. In that case, z will be suitably moved in xz.

Of all such triples $\{x, y, z\} \subset \mathrm{bd}K$, choose the largest one (i.e. of largest diameter), and let this be already done.

Claim. $xy \cup xz \subset \mathrm{bd}K$.

Assume the contrary. Take $y', z' \in \mathrm{bd}K$ such that $y'z' \| yz$ and $\overline{y'z'}$ separates x from yz. The is a rhombus $R' \in \mathcal{F}_R$ containing y', z' and included in K. Due to the assumption, $y'z'$ cannot be a diameter of R'. Therefore, $R' = y'z'z''y''$, where $\angle y'z'z'' = \gamma$ or $\angle z'y'y'' = \gamma$. See Figure 3a.

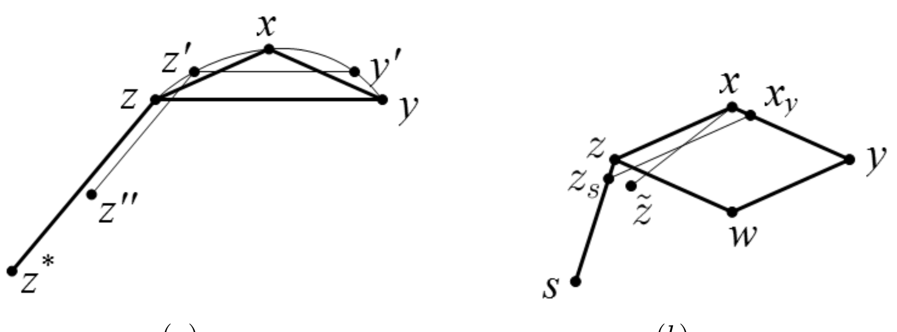

FIGURE 3

Letting $y' \to y$ (and $z' \to z$), we conclude that either $zz^* \subset K$, or $yy^* \subset K$, or both, where $\{y^*, z^*\}$ and x are separated by $\overline{yz}$, $\angle yzz^* = \angle zyy^* = \gamma$, and $\|z - z^*\| = \|y - y^*\| = \|y - z\|$.

Assume w.l.o.g. that $zz^* \subset K$. If $z^* \notin \mathrm{bd}K$, the same rotation technique (about z) provides points $z_1, z_2 \in \mathrm{bd}K$ with $\angle z_1zz_2 = \gamma$ and $\|z_1 - z\| = \|z_2 - z\| >$

$\|z^* - z\| = \|y - z\|$. But $\|y - z\| > \|x - z\|$, whence $\|z_1 - z\| > \|x - z\|$, contradicting the maximality of $\{x, y, z\}$. The same happens for $z^* \in \mathrm{bd}K$. The Claim is proved.

Assume now that K is not the rhombus R_0 of vertices y, x, z, and w. Then there is a point $\tilde{z} \in \mathrm{bd}K \setminus (xz \cup wz)$ close to z, or an analogous one close to y. Assume w.l.o.g. the existence of $\tilde{z}$. Notice that $R_0 \subset K$, hence $\tilde{z} \notin R_0$. See Figure 3b.

The points $x, \tilde{z}$ belong to a rhombus from $\mathcal{F}_R$ included in K. The only possibility for this rhombus is to have a vertex at x (and its angle $\pi - \gamma$ there). Letting $\tilde{z} \to z$, we obtain the existence of a point $s \in K$ with $\|x - z\| = \|z - s\|$ and $\angle xzs = \gamma$. If $s \notin \mathrm{bd}K$, our rotation technique would prove the existence of a triple larger than $\{x, y, z\}$. So, $s \in \mathrm{bd}K$.

But now consider points $x_y \in xy$ and $z_s \in zs$ close to x, respectively z, such that $x_y z_s \| xz$. A rhombus from $\mathcal{F}_R$ included in K and containing x_y, z_s cannot have $x_y z_s$ as a diagonal, and must therefore have a third vertex $w' \in \overline{zs} \cup \overline{xy}$. But then $\{x_y, z_s, w'\}$ would be larger than $\{x, y, z\}$.

Hence, $K = R_0$. □

Concerning the selfishness of long polytopes, it is clear that a necessary condition is that their diameter is not contained in a face. Is this also sufficient? We have seen, indeed, that every rhombus is selfish.

Theorem 2.5. *There exists a long polygon with its diameter not among the sides, which is not selfish.*

Proof. Let $abcd$ be a quadrilateral with $S_{abcd} = \mathrm{bd}B = S$, $\angle acb = \angle bac = \pi/8$, and $\angle acd = \angle dac = \pi/16$.

Take $b', d' \in S$ such that $\angle a\mathbf{0}b' = \angle b'\mathbf{0}c = \pi/2$, $\angle a\mathbf{0}d' = 3\pi/4$, and $\angle d'\mathbf{0}c = \pi/4$.

Turn $abcd$ about c such that it remains similar to itself and $a \in S$. When a moves towards b', b remains inside B, and when it takes the position b', b arrives on S at the midpoint m of $\widetilde{b'c}$.

Continuing the rotation of $abcd$ with a on S, b gets out of B; let b'' be its new position. The side $b''c$ cuts S at some point b^*. A suitable homothety of centre c brings the (rotated) $abcd$ to a position $a^*b^*cd^* \subset B$.

As b'' can become any point in $\widetilde{b'c}$, we conclude that any two points of S at distance between 0 and $\|m - c\|$ belong to some polygon from $\mathcal{F}_{abcd}$ included in B.

Finally, let $abcd$ turn in opposite direction, i.e. with a moving towards d'. When a arrives there, d reaches the midpoint m' of $\widetilde{d'c}$. Thus, any two points of S at distance between $\|m' - c\|$ and 2 belong to some polygon from $\mathcal{F}_{abcd}$ included in B.

As $\|m - c\| = \|m' - c\|$, the $\mathcal{F}$-convexity of B is verified, and so our polygon is not selfish. □

As at least some rectangles (by Theorem 1.1 the square) are selfish in $\mathbb{R}^2$, we might be inclined to think that also (some) circular cylinders in $\mathbb{R}^3$ are selfish...

Theorem 2.6. *No circular cylinder in $\mathbb{R}^3$ is selfish.*

Proof. Let Z be a circular cylinder. Denote by $r(Z)$ the ratio between its height and the diameter of its base. We show that the unit ball B is $\mathcal{F}_Z$-convex.

Let $x, y \in S$. A plane Π through $-x, y$ cuts S along a circle C_Π. Then $Z_\Pi = \mathrm{conv}(C_\Pi \cup (-C_\Pi))$ is an inscribed cylinder of B. As Π varies, $r(Z_\Pi)$ takes all values between 0 and $\tan \angle x(-x)y$. The latter ratio is obtained for Π orthogonal to xy. If Π_0 is this position of Π, consider in Π_0 a circle C tangent to C_{Π_0} at y, included in B. Take the cylinder $Z_C = \mathrm{conv}(C \cup (C + x - y))$. As the radius of C varies between 0 and $\|x + y\|/2$, $r(Z_C)$ takes all values from $\tan \angle x(-x)y$ upwards. So, one of these cylinders, a Z_Π or a Z_C, must be similar to Z, contains x, y, and is included in B. □

3. Families of convex bodies and selfishness

The convex bodies of constant width are selfish and the family of all of them complete.

The family of all convex bodies is of course complete. But the family of all unions (or of all connected unions) of pairs of convex bodies is not.

Clearly, if the family $\mathcal{F}$ is contained in the family $\mathcal{G}$, and a set M is $\mathcal{F}$-convex, then M is $\mathcal{G}$-convex, too.

Theorem 3.1. *If the families of sets $\{\mathcal{F}_\iota\}_{\iota \in I}$ are complete, then $\bigcap_{\iota \in I} \mathcal{F}_\iota$ is also complete.*

Proof. Assume there exists an $\bigcap_{\iota \in I} \mathcal{F}_\iota$-convex set $M \notin \bigcap_{\iota \in I} \mathcal{F}_\iota$. Then M does not belong to $\mathcal{F}_\iota$, for some $\iota \in I$. But M is $\mathcal{F}_\iota$-convex, by the remark preceding the statement. Thus, $\mathcal{F}_\iota$ is not complete, and a contradiction is obtained. □

So the intersection of complete families is complete. Is this so also for the union?

Theorem 3.2. *There exist selfish convex bodies K, K', such that $\mathcal{F}_K \cup \mathcal{F}_{K'}$ is not complete.*

Proof. We shall choose K, K' to be a square and a thin rhombus (having an angle at most $\pi/4$), respectively. Then the disc is $\mathcal{F}_K \cup \mathcal{F}_{K'}$-convex.

Indeed, take the unit disc B. It suffices to prove that for any two points $x, y \in S$, there is a square or a rhombus similar to K' containing x, y and included in B. For $\angle x\mathbf{0}y \leq \pi/2$, we find a suitable square. For $\angle x\mathbf{0}y > \pi/2$, a rhombus similar to K' will be the useful one. □

Theorem 3.3. *Let $\mathcal{F}$ be a family of sets and $\mathcal{G}$ a family of $\mathcal{F}$-convex sets. If some set is $\mathcal{G}$-convex, then it is also $\mathcal{F}$-convex.*

Proof. Let M be the set in the statement, and $x, y \in M$. There exists $G \in \mathcal{G}$ such that $x, y \in G$ and $G \subset M$. Also, there is an $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset G \subset M$. Thus, M is $\mathcal{F}$-convex. □

For most convex bodies K in $\mathbb{R}^d$ (in the sense of Baire categories), $\mathrm{card}(K \cap S_K) = d + 1$ [8]. Moreover, for all convex bodies in $\mathbb{R}^d$, except those in a nowhere dense subset, $\mathrm{card}(K \cap S_K) \geq d + 1$.

The latter assertion was explicitly stated in [7], and proved in [8] for $d = 2$; the extension to higher dimensions presents no difficulty. Hence, the set of long convex bodies is small in $\mathcal{K}$. However, since the set $\mathcal{K}'$ of all convex bodies K, for which

$K \cap S_K$ contains a diametral pair of S_K, is closed in $\mathcal{K}$, it is itself a Baire space. Most of these convex bodies are long. Inspired by Theorem 2.1, we want to look for the chances of a convex body in $\mathcal{K}'$ to be selfish.

Theorem 3.4. *For all pairs of convex bodies $(K, K') \in \mathcal{K} \times \mathcal{K}$ except those of a nowhere dense family of pairs, K is not $\mathcal{F}_{K'}$-convex.*

Proof. Let $\{a, b\}$ be a diametral pair of $K \in \mathcal{K}$. Choose, if possible, $K_1 \in \mathcal{F}_{K'}$ such that $a, b \in K_1$ and $K_1 \subset K$. This represents a necessary condition for K to be $\mathcal{F}_{K'}$-convex. It follows that $\{a, b\}$ is diametral in K_1, too. The set of pairs (K, K'), for which the preceding choice is possible, is obviously closed in $\mathcal{K} \times \mathcal{K}$. Thus, to prove the theorem it will suffice to find, arbitrarily close to (K, K'), a pair for which the choice is not possible.

Indeed, let $\mathcal{N}$ be a neighbourhood of (K, K_1) in $\mathcal{K} \times \mathcal{K}$. Choose $(K^*, K_1^*) \in \mathcal{N}$ such that ab is a diameter for both K^* and K_1^*, K^* is a polytope with a vertex at a, and K_1^* is smooth. In this manner, $K_1^* \setminus K^* \neq \emptyset$, and K_1^* is similar to a set K'^* arbitrarily close to K'. Hence, the above choice is impossible for the pair (K^*, K'^*). □

A fortiori, for all convex bodies $K \in \mathcal{K}$ except those of a nowhere dense family, and for most $K' \in \mathcal{K}$, K is not $\mathcal{F}_{K'}$-convex. But it is well known that most convex bodies are smooth (see [4], [3]). Is the theorem true perhaps for all smooth K'? Indeed, an inspection of the preceding proof confirms that only the smoothness of K' was used, so the following is also true.

Theorem 3.5. *For every smooth $K' \in \mathcal{K}$, and for all convex bodies $K \in \mathcal{K}$ except those of a nowhere dense family, K is not $\mathcal{F}_{K'}$-convex.*

Considering $\mathcal{K}'$ instead of $\mathcal{K}$ does not change the situation, and theorems analogous to Theorems 3.4 and 3.5 are true with $\mathcal{K}'$ instead of $\mathcal{K}$. So, the chances of a long convex body in $\mathcal{K}'$, and even of a convex body in $\mathcal{K}$, to be selfish are not bad!

Let $\mathcal{F}(K)$ be the set of all compact $\mathcal{F}_K$-convex sets.

There are convex bodies K with huge $\mathcal{F}(K)$. However, $\mathcal{F}(K)$ cannot become $\mathcal{K}$. Indeed, let $K \in \mathcal{K}$, and consider a diameter ab of K. Take $x \notin ab$ in the interior of K. Then the triangle abx is not $\mathcal{F}_K$-convex, which can be easily checked.

Theorem 3.6. *For each $K \in \mathcal{K}$ there exists $K' \notin \mathcal{F}(K)$ such that $K \in \mathcal{F}(K')$.*

Proof. Let $K \in \mathcal{K}$, and consider an interior point x of K. Let r be the radius of a ball Δ centred at x and included in K. Choose $z \in \mathrm{bd}K$ such that $\|z-x\| \geq \|y-x\|$ for all $y \in \mathrm{bd}K$, and put $q = \|z-x\|$. Let T be an isosceles triangle with two angles equal to $\alpha = \arctan \frac{r}{q}$.

We prove that $K \in \mathcal{F}(T)$.

Let u, v be distinct points of K. First assume $x \notin \overline{uv}$. Take the chord bc through x, parallel to uv, and let Π be the plane determined by them. The line in Π containing x and orthogonal to bc meets $\mathrm{bd}\Delta$ at a, a^*; choose the intersection point a to be separated in Π from uv by bc (see Figure 4).

Since $\angle abc \geq \alpha$ and $\angle acb \geq \alpha$, the triangle abc includes an isosceles triangle $a'bc$ with $\angle a'bc = \angle a'cb = \alpha$.

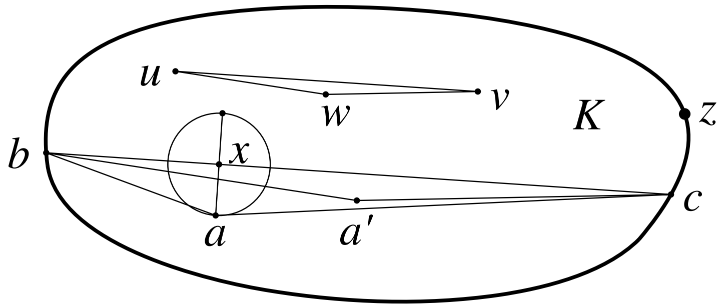

FIGURE 4

Now let wuv be homothetical to $a'bc$. Then wuv is similar to T, and included in the convex pentagon $a'cvub \subset K$. (This always holds, not only when bc is longer than uv.)

The case $x \in \overline{uv}$ is now obvious. Hence, $K \in \mathcal{F}(T)$.

It is immediately seen that $T \notin \mathcal{F}(K)$. □

4. Open problems

The investigation of selfishness was only initiated in this paper. There are many related questions, which wait to be answered. We select here just two.

We have seen that the square is selfish. This, however, doesn't straightforwardly generalize to all rectangles.

Problem 1. Is every rectangle selfish?

Similarly, the equilateral triangle is selfish, by Theorem 2.2, but not every acute triangle is selfish, by Theorem 2.3. None of the triangles mentioned in the Remark following Theorem 2.3 is isosceles.

Problem 2. Is every isosceles acute triangle selfish?

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Liping Yuan
College of Mathematics and Information Science, Hebei Normal University, 050024 Shijiazhuang, P.R. China and;
Hebei Key Laboratory of Computational Mathematics and Applications, 050024 Shijiazhuang, P. R. China
E-mail address: `lpyuan@mail.hebtu.edu.cn`

Tudor Zamfirescu
Fachbereich Mathematik, Universität Dortmund, 44221 Dortmund, Germany and;
Institute of Mathematics "Simion Stoilow", Roumanian Academy, Bucharest, Roumania and;
College of Mathematics and Information Science, Hebei Normal University, 050024 Shijiazhuang, P.R. China
E-mail address: `tudor.zamfirescu@mathematik.uni-dortmund.de`