

Right quadruple convexity*

Dandan Li , Liping Yuan †

*College of Mathematics and Information Science, Hebei Normal University,
050024 Shijiazhuang, People's Republic of China*

Tudor Zamfirescu

*Fachbereich Mathematik, TU Dortmund, 44221 Dortmund, Germany
Institute of Mathematics "Simion Stoilow", Roumanian Academy, Bucharest, Roumania*

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Abstract

A set of four points $w, x, y, z \in \mathbb{R}^d$ (always $d \geq 2$) form a rectangular quadruple if their convex hull is a non-degenerate rectangle. The set M is called rq -convex if for every pair of its points we can find another pair in M , such that the four points form a rectangular quadruple. In this paper we start the investigation of rq -convexity in Euclidean spaces.

Keywords: rq -convex sets, parallelotopes, finite sets, Platonic solids.

Math. Subj. Class.: 53C45, 53C22

1 Introduction

Let \mathcal{F} be a family of sets in \mathbb{R}^d . A set $M \subset \mathbb{R}^d$ is called \mathcal{F} -convex if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$.

The third author proposed at the 1974 meeting on Convexity in Oberwolfach the investigation of this very general kind of convexity. Usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are just some examples of \mathcal{F} -convexity (for suitably chosen families \mathcal{F}).

Blind, Valette and the third author [1], and also Böröczky Jr [2], investigated rectangular convexity. Magazanik and Perles dealt with staircase connectedness [5]. The third author

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†Corresponding author.

E-mail address: ddli68@163.com (Dandan Li), lpyuan@hebtu.edu.cn (Liping Yuan), tuzamfirescu@gmail.com (Tudor Zamfirescu)

studied right convexity [8]; then the second and the third author generalized the latter type of convexity and investigated the right triple convexity (see [7] and [6]). All these concepts are particular cases of \mathcal{F} -convexity. The rectangular convexity is obtained if \mathcal{F} is the family of all non-degenerate rectangles in \mathbb{R}^d .

In this paper we present a discretization of rectangular convexity, the right quadruple convexity, which constitutes a generalization of rectangular convexity. As usual, for $M \subset \mathbb{R}^d$, $\text{bd}M$ denotes its boundary, $\text{int}M$ its interior, $\text{diam}M = \sup_{x,y \in M} \|x - y\|$ its diameter, and $\text{conv}M$ its convex hull. A set of four points $w, x, y, z \in \mathbb{R}^d$ (always $d \geq 2$) form a *rectangular quadruple* if $\text{conv}\{w, x, y, z\}$ is a non-degenerate rectangle. Let \mathcal{R} be the family of all rectangular quadruples. Here, we shall choose \mathcal{F} to be this family \mathcal{R} .

Let $M \subset \mathbb{R}^d$. A pair of points $x, y \in M$ is said to enjoy the *rq-property in M* if there exists another pair of points $z, w \in M$, such that $\{w, x, y, z\}$ is a rectangular quadruple. The set M is called *rq-convex*, if every pair of its points enjoys the *rq-property in M*. This property is the *right quadruple convexity*.

Let $A \subset \mathbb{R}^d$. We call A^* an *rq-convex completion* of A , if A^* is *rq-convex*, $A^* \supset A$ and $\text{card}(A^* \setminus A)$ is minimal (but possibly infinite). Let $\gamma(A) = \text{card}(A^* \setminus A)$, which is called the *rq-convex completion number* of A , in case A is finite. For finite n , let $\gamma(n) = \sup\{\gamma(A) : \text{card}A = n\}$.

For distinct $x, y \in \mathbb{R}^d$, let \overline{xy} be the line through x, y , xy the line-segment from x to y , H_{xy} the hyperplane through x orthogonal to \overline{xy} , and C_{xy} the hypersphere of diameter xy . For $S_1, S_2 \subset \mathbb{R}^d$, let $d(S_1, S_2) = \inf\{d(x, y) \mid x \in S_1, y \in S_2\}$ denote the *distance* between S_1 and S_2 . The d -dimensional unit ball (centred at $\mathbf{0}$) is denoted by B_d ($d \geq 2$). Let us remark that every open set in \mathbb{R}^d is *rq-convex*.

2 Not simply connected *rq-convex* sets

In \mathbb{R}^2 , all compact rectangularly convex sets are conjectured to be extremely circular and symmetric. A planar convex set is *extremely circular* if its set of extreme points lies on a circle. Analogously, it is reasonable to conjecture that all compact *rq-convex* sets have an extremely circular and symmetric convex hull. Consequently, when investigating a compact connected *rq-convex* set M , we may reasonably start by assuming that $\text{conv}M$ is extremely circular and symmetric. We shall now take $\text{bdconv}M$ to be a circle. If M is simply connected we get the disc. So, assume $(\text{conv}M) \setminus M \neq \emptyset$.

Theorem 2.1. *If $\text{conv}M$ is a disc and $(\text{conv}M) \setminus M$ lies in a circular disc of radius r at distance at least $(\sqrt{3} - 1)r$ from $\text{bdconv}M$, then M is *rq-convex*.*

This theorem gives a useful sufficient condition for the *rq-convexity* of a set M which is not simply connected, regardless the shape of $(\text{conv}M) \setminus M$. Notice that it allows both M and its complement to have arbitrarily many components.

Proof. Let Q be a square circumscribed to the disc D of radius r including $(\text{conv}M) \setminus M$. We may suppose that the origin $\mathbf{0}$ is the centre of D , so $D = rB_2$.

We have $Q \subset \text{conv}M$. Indeed, Q is obviously included in the disc concentric with D of radius $\sqrt{2}r$, which in turn must be included in $\text{conv}M$, since the distance from D to $\text{bdconv}M$ is at least $(\sqrt{3} - 1)r > (\sqrt{2} - 1)r$.

Let $x, y \in M$. We verify the *rq-convexity* of M at these two points.

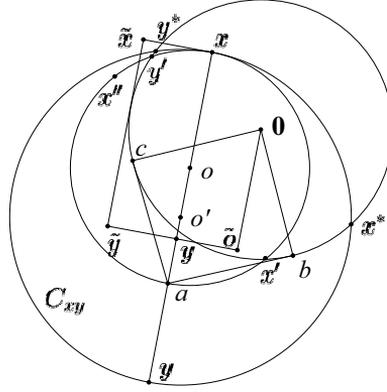


Figure 1: $x \in D, y \notin D$.

Case 1: $x, y \in D$. Choose Q to have a side s parallel to xy . Then x, y and their orthogonal projections on s are vertices of a rectangle. Moreover, these latter vertices lie in $(\text{conv}M) \setminus D$.

Case 2: $x \in D, y \notin D$. Let Γ_2, Γ_3 be the two circles concentric with $\text{bd}D$, of radii $r\sqrt{2}, r\sqrt{3}$. Let a be the point of $\overline{xy} \cap \Gamma_2$ such that $x \notin ay$. In case $y \in ax$, consider the rectangle $xy\tilde{y}\tilde{x}$, such that \overline{xy} separates $\mathbf{0}$ from \tilde{x}, \tilde{y} (if $\mathbf{0} \notin \overline{xy}$) and $\tilde{x}\tilde{y}$ is tangent to $\text{bd}D$. See Figure 1. Let \tilde{o} be the orthogonal projection of $\mathbf{0}$ onto $y\tilde{y}$. We have

$$\|\tilde{y}\|^2 = \|\tilde{y} - \tilde{o}\|^2 + \|\tilde{o}\|^2 = r^2 + \|\tilde{o}\|^2 \leq r^2 + \|y\|^2 \leq r^2 + \|a\|^2 = 3r^2.$$

Hence, \tilde{y} , and of course \tilde{x} too, lie in M , and the rq -property is satisfied in x, y .

We reconsider the case $y = a$. Choose $b, c \in \text{bd}D$ such that $ab\mathbf{0}c$ be a square and $b, \mathbf{0}$ be not separated by \overline{xy} . Let $\{x', y'\} = C_{ax} \cap \text{bd}D$, where x' is closer than y' from b . Let x'' be the point of C_{ax} diametrically opposite to x' . For any position of x ,

$$\angle \mathbf{0}ax' \leq \angle \mathbf{0}ab.$$

Hence, $\angle \mathbf{0}ax'' \geq \angle \mathbf{0}ac$, whence $x'' \notin D$. (This confirms the rq -property in x, a .) It follows that $\angle x'oy' < \pi$, where o is the centre of C_{ax} and the angle is measured towards $\mathbf{0}$.

In case $a \in xy$, consider the circle C_{xy} with centre o' , which cuts $\text{bd}D$ in x^*, y^* (the former being closer than the latter from b). We have

$$\angle xo'x^* < \angle xo'x' < \angle xox'$$

and

$$\angle xo'y^* < \angle xo'y' < \angle xoy',$$

whence

$$\angle x^*o'y^* < \angle x'oy' < \pi,$$

where both angles are taken towards $\mathbf{0}$. Consequently, the points $x^+, y^+ \in C_{xy}$ diametrically opposite to x^*, y^* (respectively) lie outside D . They also lie in different half-circles

determined by x, y on C_{xy} . Of these two half-circles, at least one is contained in the disc $\text{conv}M$.

So, either $\{x, y, x^*, x^+\} \subset M$ or $\{x, y, y^*, y^+\} \subset M$, and the rq -property is again satisfied at x, y .

Case 3: $x, y \notin D$. Besides the trivial cases $x, y \in \text{int}M$ and $x, y \in \text{bdconv}M$, we only have the simple situation $x \in \text{int}M, y \in \text{bdconv}M$. In that situation, the circle C_{xy} has necessarily two opposite arcs in M starting at x , respectively y . This proves the rq -property at x, y . □

Conjecture 2.2. *Each simply connected rq -convex set in \mathbb{R}^2 is convex.*

3 Unbounded rq -convex sets

An infinite family \mathcal{K} of closed convex sets is said to be *uniformly bounded below* if, for some $\lambda > 0$, each of the sets contains a translate of the disc λB_2 .

Theorem 3.1. *Let \mathcal{K} be a family of pairwise disjoint closed convex sets in \mathbb{R}^d . If \mathcal{K} is finite or uniformly bounded below, then the closure of the complement of $\bigcup \mathcal{K}$ is rq -convex.*

Proof. We may assume that all sets in \mathcal{K} possess interior points, because the case of empty interior is irrelevant. Let M be the closure of $\mathbb{R}^d \setminus \bigcup \mathcal{K}$, and choose $x, y \in M$. Clearly, the only interesting case is when $x, y \in \text{bd}M$.

The condition of uniform boundedness below for infinite \mathcal{K} guarantees that $x \in \text{bd}M$ only if x is a boundary point of some member of \mathcal{K} .

Let M' be the intersection of M with an arbitrary 2-dimensional plane $\Pi \ni x, y$. For some $K_x, K_y \in \mathcal{K}, x \in \text{bd}K_x, y \in \text{bd}K_y$. Consider the supporting hyperplane H_x of K_x at x , the line $H'_x = H_x \cap \Pi$, and analogously H_y and H'_y . If H'_x, H'_y are not orthogonal to xy , there are six different situations in the neighbourhood of x and y , depicted in Figure 2 (subfigures (a)-(f)). (In the figure only the generic case is depicted, when $K_x \cap \Pi$ and $K_y \cap \Pi$ are not degenerate; but the proof works in all cases.)

In the situations of Figure 2 (subfigures (a),(c) and (e)), the circle C_{xy} has two opposite arcs inside M , so M has the rq -property at x, y . In the cases of Figure 2 (subfigures (b),(d) and (f)), a thin rectangle with xy as a side has its short sides in M , so the rq -property is again verified.

The only remaining case is that of at least one of the lines H'_x, H'_y , say the first, being orthogonal to xy . In this case, there is a short line-segment $x(x+v) \subset M$ in any direction v orthogonal to xy . Now, if $y+v \in M$, we found the right quadruple $\{x, y, y+v, x+v\}$. If $y+v \notin M$, i.e. $y+v \in \text{int}K_y$, then $y-v \notin K_y$. Thus, $\{x, y, y-v, x-v\}$ is a suitable rectangular quadruple. □

We can drop the convexity condition if the considered sets are bounded.

Theorem 3.2. *The complement of any bounded set in \mathbb{R}^d is rq -convex.*

The easy proof is left to the reader.

A *plane tiling* \mathcal{T} is a countable family $\{T_1, T_2, \dots\}$ of closed sets with non-empty interiors, which cover the plane without gaps or overlaps. Every closed set $T_i \in \mathcal{T}$ is called a *tile of \mathcal{T}* . We consider the special case in which each tile is a polygon. If the corners and sides of a polygon coincide with the vertices and edges of the tiling, we call the tiling

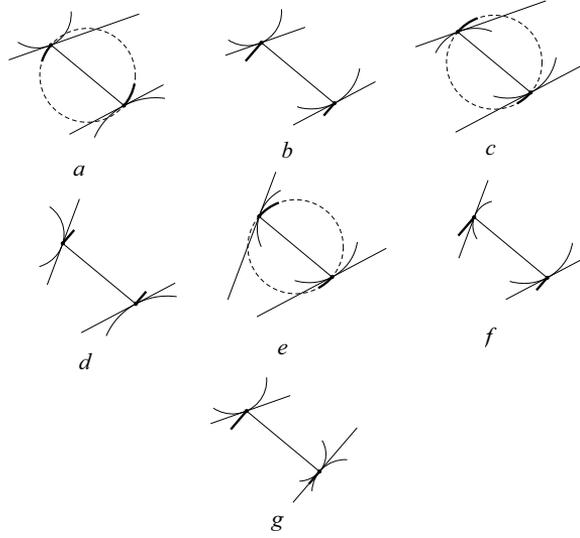


Figure 2: Illustration for the proof of Theorem 3.1.

edge-to-edge. A so-called *type* describes the neighbourhood of any vertex of the tiling. If, for example, in some cyclic order around a vertex there are a triangle, then another triangle, then a square, next a third triangle, and last another square, then its type is $(3^2.4.3.4)$. We consider plane edge-to-edge tilings in which all tiles are regular polygons, and all vertices are of the same type. Thus, the vertex-type defines our tiling up to similarity.

There exist precisely eleven such tilings [3]. These are (3^6) , $(3^4.6)$, $(3^3.4^2)$, $(3^2.4.3.4)$, $(3.4.6.4)$, $(3.6.3.6)$, (3.12^2) , (4^4) , $(4.6.12)$, (4.8^2) , and (6^3) . They are called *Archimedean tilings*.

Theorem 3.3. *The Archimedean tilings (4^4) , (3^6) , (6^3) , $(3.6.3.6)$, $(3^4.6)$, $(3.3.4.3.4)$, $(4.8.8)$ have rq -convex vertex sets.*

Theorem 3.4. *The vertex sets of the Archimedean tilings $(3.3.3.4.4)$, $(3.4.6.4)$, $(4.6.12)$, $(3.12.12)$ are not rq -convex.*

The proofs of Theorems 3.3 and 3.4 are also left to the reader.

4 rq -convex skeleta of parallelotopes

As already remarked in [1], for $d \geq 3$, there is not even any conjectured characterization of rectangularly convex sets in \mathbb{R}^d . Among the sets mentioned in [1] as rectangularly convex we find the cylinder $K \times [0, 1]$ with a $(d - 1)$ -dimensional compact convex set K as basis. In particular, any *right parallelotope*, i.e. the cartesian product of d pairwise orthogonal line-segments, is rectangularly convex and, a fortiori, rq -convex.

Theorem 4.1. *The 1-skeleton of any right parallelotope is rq -convex.*

Proof. Let $P = I_1 \times I_2 \times \dots \times I_d$ be our parallelotope, where $I_i = [0, a_i]$ ($i = 1, \dots, d$). We verify the rq -property at the points x, y belonging to the 1-skeleton of P .

Case 1: x, y belong to parallel edges of P . We have without loss of generality

$$\begin{aligned}x &= (x_1, 0, \dots, 0), \\y &= (y_1, a_1, \dots, a_i, 0, \dots, 0).\end{aligned}$$

Then we choose as third and fourth point

$$\begin{aligned}u &= (y_1, 0, \dots, 0), \\v &= (x_1, a_1, \dots, a_i, 0, \dots, 0).\end{aligned}$$

Indeed, $xuvy$ is a rectangle.

Case 2: x, y belong to two edges of P having a common endpoint. Say without loss of generality that

$$\begin{aligned}x &= (x_1, 0, \dots, 0), \\y &= (0, y_2, 0, \dots, 0).\end{aligned}$$

Then take

$$\begin{aligned}u &= (x_1, 0, a_3, 0, \dots, 0), \\v &= (0, y_2, a_3, 0, \dots, 0),\end{aligned}$$

completing the vertex set of a rectangle $xuvy$.

Case 3: x, y belong to two non-parallel disjoint edges of P . If

$$\begin{aligned}x &= (x_1, 0, \dots, 0), \\y &= (0, \dots, 0, y_i, a_{i+1}, \dots, a_d),\end{aligned}$$

then we choose

$$\begin{aligned}u &= (x_1, 0, \dots, 0, a_{i+1}, \dots, a_d), \\v &= (0, \dots, 0, y_i, 0, \dots, 0),\end{aligned}$$

and again we get the vertices of a rectangle.

Case 4: x, y belong to the same edge of P . This is immediate. □

Contrary to the case of an arbitrary cylinder, the following is true.

Theorem 4.2. *The boundary of any right parallelotope is rq -convex.*

Proof. Take x, y on the boundary of the parallelotope P . We show that they have the rq -property.

If x, y belong to the same facet F , choose their orthogonal projections onto the opposite facet F' ; the four points are vertices of a rectangle.

If $x \in F, y \in F'$, choose the projection x' of x onto F' and the projection y' of y onto F ; we get the rectangle $xx'y'y'$.

If x, y belong to two adjacent facets F, F^* , respectively, take the orthogonal projections x^* and y^* of x and y (respectively) onto $F \cap F^*$. We complete the rectangles $xx^*y^*\tilde{y}$ and $yy^*x^*\tilde{x}$. Then, clearly, $\{x, \tilde{x}, y, \tilde{y}\} \subset \text{bd}P$ is a rectangular quadruple. □

Theorem 4.3. *Not every convex cylinder has an rq -convex boundary.*

Proof. Take a cylinder $Z = E \times [0, 1] \subset \mathbb{R}^3$, where $E \subset \mathbb{R}^2$ is convex and $\text{bd}E$ is a long ellipse. Choose x on the long axis of $\text{bd}E \times \{1\}$, close to one of its endpoints $\{e\} = \{e_p\} \times \{1\}$, and let $\{e'\} = \{e'_p\} \times \{1\}$ be the other endpoint. Let $\{y_\varepsilon\} = \{e'_p\} \times \{\varepsilon\}$, where $\varepsilon \geq 0$. See Figure 3.

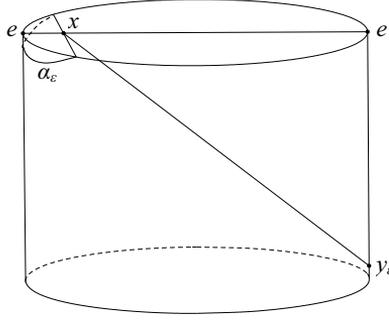


Figure 3: A convex cylinder without rq -convex boundary.

The plane H_{xy_ε} cuts $(\text{bd}E) \times [0, 1]$ along an arc α_ε of an ellipse. Let $f(\varepsilon) = d(\{x\}, \alpha_\varepsilon)$. The function f is increasing, and $f(0) > 0$.

The plane $H_{y_\varepsilon x}$ cuts $\text{bd}Z$ along a closed curve (reduced to a single point if $\varepsilon = 0$), of diameter $g(\varepsilon)$. This function g is also increasing, and $g(0) = 0$.

Therefore, for $\varepsilon > 0$ small enough,

$$g(\varepsilon) < f(0) < f(\varepsilon).$$

Choose $y = y_\varepsilon$. The above inequalities show that there is no rectangle $xyy'x'$ with $x', y' \in \text{bd}Z$.

Consider now the sphere C_{xy} . The set $C_{xy} \cap Z$ has four components: a component Z_1 containing x , another one Z_2 containing y , a third Z_3 containing the point e^* diametrically opposite to e' in C_{xy} , and a fourth, $\{e'\}$. It is easily seen that the only pairs of diametrically opposite points in $Z_1 \cup Z_2 \cup Z_3 \cup \{e'\}$ are (x, y) and (e', e^*) . But $e^* \in \text{int}Z$, so $\text{bd}Z$ is not rq -convex. \square

5 rq -convexity of finite sets

In these last two sections, we shall use the following notation. For $x, y \in \mathbb{R}^d$, we set $W_{xy} = H_{xy} \cup H_{yx} \cup C_{xy}$. Let \mathcal{A} be the family of all finite point sets in \mathbb{R}^2 .

Theorem 5.1. *For any set $A \in \mathcal{A}$ with $\text{card}A = n \geq 3$, we have $\gamma(A) \leq n^2 - 2n$.*

Proof. If A is included in a line L , consider a line L' parallel to L and the orthogonal projection A' of A onto L' . Then obviously $A \cup A'$ is rq -convex and $\text{card}A' = n \leq n^2 - 2n$, since $n \geq 3$.

If A is not included in any line, let $A = \{a_1, a_2, \dots, a_n\}$, and assume that $a_1 a_2$ is a side of the polygon $\text{conv}A$. Obviously there are at most $n - 2$ lines L_1, \dots, L_{n-2} passing through the remaining points of A and parallel to $\overline{a_1 a_2}$. Also, there are at most n lines L'_1, \dots, L'_n passing through the points of A orthogonally onto $L_0 = \overline{a_1 a_2}$.

The set

$$A' = \bigcup_{0 \leq i \leq n-2; 1 \leq j \leq n} (L_i \cap L'_j)$$

is obviously rq -convex and has at most $n(n - 1)$ points, whence $\gamma(A) \leq n^2 - 2n$. \square

Thus, for any $n \geq 3$, $\gamma(n) \leq n^2 - 2n$. In particular, $\gamma(3) = 3$.

Theorem 5.2. *There are precisely two kinds of 6-point rq -convex sets in \mathcal{A} , shown in Figure 4.*

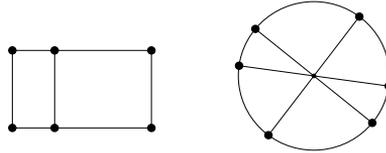


Figure 4: 6-point rq -convex sets.

Proof. Let $F = \{a, b, c, d, e, f\}$ be a 6-point rq -convex set. We assume without loss of generality that $\{a, b, c, d\} \in \mathcal{R}$, where $\angle abc = \frac{\pi}{2}$. By the definition of rq -convexity, e, f must meet one of the following seven conditions.

- $C_1.$ $\{\{e, f, a, b\}, \{e, f, c, d\}\} \subset \mathcal{R}$;
- $C_2.$ $\{\{e, f, a, c\}, \{e, f, b, d\}\} \subset \mathcal{R}$;
- $C_3.$ $\{\{e, f, a, d\}, \{e, f, b, c\}\} \subset \mathcal{R}$;
- $C_4.$ $\{\{e, f, a, b\}, \{e, f, a, c\}, \{e, f, a, d\}\} \subset \mathcal{R}$;
- $C_5.$ $\{\{e, f, b, a\}, \{e, f, b, c\}, \{e, f, b, d\}\} \subset \mathcal{R}$;
- $C_6.$ $\{\{e, f, c, a\}, \{e, f, c, b\}, \{e, f, c, d\}\} \subset \mathcal{R}$;
- $C_7.$ $\{\{e, f, d, a\}, \{e, f, d, b\}, \{e, f, d, c\}\} \subset \mathcal{R}$.

Clearly, C_1 and C_3 generate the same kind of set F , and so do C_4, C_5, C_6 and C_7 .

Case 1: e, f satisfy C_1 . By the definition of rq -convexity, $e, f \in W_{ab} \cap W_{cd}$, and so $e, f \in (H_{ab} \cup H_{ba}) \setminus \{a, b, c, d\}$ and ef, ab are parallel. Without loss of generality, we may suppose $e \in H_{ab}, f \in H_{ba}$, which leads to the three solutions depicted in Figure 5, all of them providing a 6-point set of the first type.

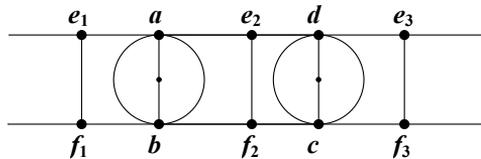


Figure 5: e, f satisfy C_1 .

Case 2: e, f satisfy C_2 . By the definition of rq -convexity, we have $e, f \in W_{ac} \cap W_{bd}$, so e, f are antipodal points of C_{ac} ; see Figure 6.

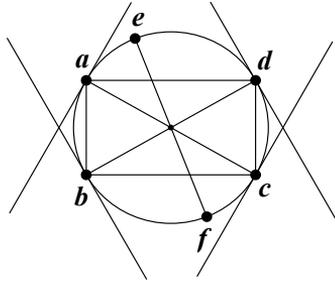


Figure 6: e, f satisfy C_2 .

Case 3: e, f satisfy C_4 . By the definition of rq -convexity, $(e, f) \in W_{ab} \cap W_{ac} \cap W_{ad}$. But $W_{ab} \cap W_{ac} \cap W_{ad} = \{a, b, d\}$, so we obtain no solution in this case. See Figure 7. \square

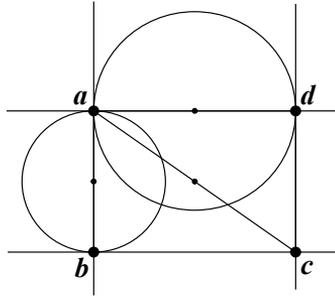


Figure 7: e, f satisfy C_3 .

Theorem 5.3. *There are precisely three kinds of 8-point rq -convex sets, shown in Figure 8.*

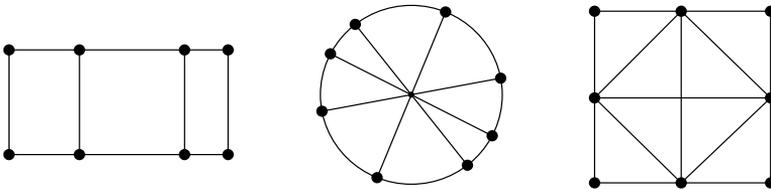


Figure 8: 8-point rq -convex sets.

The proof is ten pages long, so we decided not to include it into the paper. It is a case-by-case examination, treating separately those sets which contain a 6-point rq -convex subset and those which do not. It can be read in [4].

Theorem 5.4. *The smallest odd cardinality of an rq -convex set in \mathbb{R}^2 is 9.*

Proof. It is obvious that every rq -convex set contains a rectangular quadruple and quickly seen that no fifth point can be added to a rectangular quadruple keeping rq -convexity. Similarly, knowing what a 6-point rq -convex set looks like (Theorem 5.2), it is an easy exercise to establish that there is no 7-point rq -convex set containing a 6-point rq -convex set.

Next, we will consider the case that a 7-point rq -convex set does not contain any 6-point rq -convex subset. Let $F = \{a, b, c, d, e, f, g\}$ be such a set. Suppose a, c realise the diameter of F . Since F is rq -convex, there is another pair of antipodal points of C_{ac} in F , say $\{b, d\}$. Hence $\{a, b, c, d\} \in \mathcal{R}$, and put $\text{conv}\{a, b, c, d\} = R$.

For the set of points $\{x, y\} \subset \{e, f, g\}$, if there exist $z, w \in \{a, b, c, d\}$ such that $\{w, x, y, z\} \in \mathcal{R}$, then we say that $\{x, y\}$ is rq -good. Next we will prove that for any two points $x, y \in \{e, f, g\}$, $\{x, y\}$ is not rq -good. Suppose $\{e, f\}$ is rq -good. Then there exist $z, w \in \{a, b, c, d\}$, such that $\{e, f, z, w\} \in \mathcal{R}$.

Case 1: zw is a diagonal of R . Without loss of generality, we assume $zw = ac$, so $e, f \in W_{ac}$. Since $\{a, c\}$ realise the diameter of F , e, f are antipodal on C_{ac} . But so we obtain a 6-point rq -convex subset of F , which contradicts our assumption about F .

Case 2: zw is an edge of R . We assume without loss of generality that $zw = ab$. Then $e, f \in W_{ab}$. Clearly, e, f must be antipodal points of C_{ab} . Take a diameter a_0b_0 of C_{ab} orthogonal to \overline{ab} , such that \overline{ab} separates a_0 from cd . We may suppose that e belongs to the (smaller) arc of C_{ab} from a to a_0 ; see Figure 9. If $\|a - b\| \leq \|b - c\|$, then $\|c - e\| > \|c - a\|$,

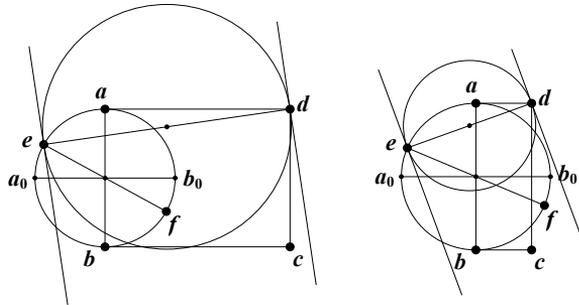


Figure 9: zw is an edge of R .

which is impossible. So, $\|a - b\| > \|b - c\|$.

As F is rq -convex, $\{e, d\}$ enjoys the rq -property and there exist $p, q \in \{a, b, c, f, g\}$ such that $p, q \in W_{ed}$. Clearly, $a, b, c \notin H_{de}$. Since $d \notin C_{ab}$, we have $f \notin H_{de}$. Further, we easily verify that $a, b, c, f \notin H_{ed} \cup H_{de}$. It follows that p, q are two antipodal points of C_{ed} . Since $\angle dae > \frac{\pi}{2}$, $\angle dce < \frac{\pi}{2}$, we get $a, c \notin C_{ed}$. Hence, $p = b \in C_{ed}$ or $p = f \in C_{ed}$, and $q = g$.

(i) $p = b \in C_{ed}$. We only can choose g such that b, g are antipodal points of C_{ed} . As $\angle ebf = \frac{\pi}{2}$, b, f, d are collinear. But then we get a 6-point rq -convex set $\{e, b, a, f, g, d\} \subset F$, contradicting our choice of F ; see Figure 10.

(ii) $p = f \in C_{ed}$. Now, f, g are antipodal points of C_{ed} . Hence, $efdg$ is a rectangle. The points g and b are separated by \overline{ad} , or $a \in dg$ if $e = a_0$. Also,

$$\|d - g\| = \|e - f\| = \|a - b\| > \|b - c\| = \|a - d\|.$$

It follows that $\|c - g\| > \|c - a\|$, and a contradiction is obtained. See Figure 11. Hence,

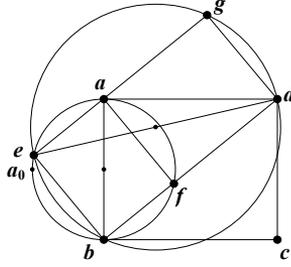


Figure 10: $p = b \in C_{ed}$.

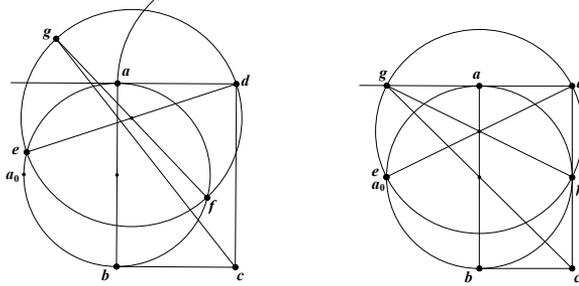


Figure 11: $p = f \in C_{ed}$.

$\{e, f\}$ is not rq -good.

Since F is rq -convex, we must have $\{e, f, g, a\}$, $\{e, f, g, b\}$, $\{e, f, g, c\}$, $\{e, f, g, d\} \in \mathcal{R}$, which is true only for $a = b = c = d$, which is impossible. Thus, there is no 7-point rq -convex set.

On the other hand, a 9-point rq -convex set is easily produced: just take the intersection $L \cap L'$, where L is the union of three horizontal lines and L' the union of three vertical lines. \square

Consider now the square lattice \mathbb{Z}^2 , and the usual norm

$$\|(x, y)\|_m = \max\{|x|, |y|\},$$

defining in \mathbb{Z}^2 the discs of radius $n \in \mathbb{Z}$

$$Q(n) = \{(x, y) : \|(x, y)\|_m \leq n\},$$

centred at the origin $\mathbf{0}$. The subset $Q(n) \setminus Q(n - 1)$ will be called *boundary of $Q(n)$* , and $Q(n - 1)$ its *interior*. Obviously, $Q(n)$ is rq -convex, for any $n \geq 1$, and so is its boundary too. Does $Q(n)$ remain rq -convex if one deletes parts of its interior (but not all of it)?

Theorem 5.5. *The set $Q(n) \setminus \{\mathbf{0}\}$ is rq -convex.*

Proof. For any pair of points $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $Q(n) \setminus \{\mathbf{0}\}$, consider the points (x_1, y_2) , $(y_1, x_2) \in \mathbb{Z}^2$. If none of them is $\mathbf{0}$, the rq -property is verified at (x, y) , as $(x_1, y_2), (y_1, x_2) \in Q(n)$.

Otherwise, assume without loss of generality that $x_1 = y_2 = 0$. We can also assume that both x_2, y_1 are positive, the other cases being symmetrical. Consider the points $x' = (-x_2, x_2 - y_1)$ and $y' = (y_1 - x_2, -y_1)$. Then $x', y' \in Q(n) \setminus \{0\}$ and x, y, y', x' are the vertices of a square. \square

Perhaps removing several layers of the boundary, thereby giving a set $Q(n) \setminus Q(m)$ for $m < n - 1$, will provide an rq -convex set?

Theorem 5.6. *The set $Q(n) \setminus Q(n - 2)$ is not rq -convex, for any $n \geq 3$.*

Proof. Assume first that $n > 3$. Consider the points $x = (n, 2 - n)$ and $y = (n - 3, n - 1)$. The point $(n - 3, 2 - n)$ does not belong to $Q(n) \setminus Q(n - 2)$. The line H_{yx} meets $Q(n)$ again at $y' = (-n, n - 4)$. But the fourth vertex $x' = (3 - n, -n - 1)$ of the square $xyy'x'$ lies outside $Q(n)$, whence $Q(n) \setminus Q(n - 2)$ is not rq -convex.

Now, consider the set $Q(3) \setminus Q(1)$. In this case take the points $x = (3, -1)$, $y = (-1, 2)$. As $(-1, -1) \notin Q(3) \setminus Q(1)$ and $H_{yx} \cap Q(3) \setminus Q(1) = \emptyset$, the result is proven. \square

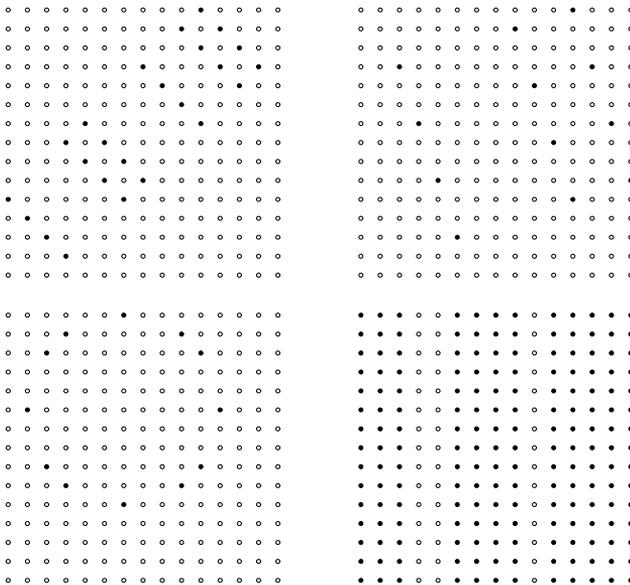


Figure 12: rq -convex proper subsets of $Q(n)$.

It seems that $Q(n) \setminus \{0\}$ is the only rq -convex proper subset of $Q(n)$ containing the boundary of $Q(n)$ but different from it. However, this is not proven. Other proper subsets of $Q(n)$ which are rq -convex abound. For some examples, see Figure 12, where the solid black dots form rq -convex proper subsets of $Q(n)$.

6 rq -convexity of the vertex sets of Platonic solids

Due to their symmetry, the vertex sets of the cube, regular octahedron, regular dodecahedron, and regular icosahedron are all rq -convex. Among the Platonic solids, only the

regular tetrahedron lacks this property. But what is the rq -convex completion number of the vertex set of the regular tetrahedron?

Theorem 6.1. *The rq -convex completion number of the vertex set of the regular tetrahedron is 3.*

Proof. Let $T = \{a, b, c, d\}$ denote the vertex set of a regular tetrahedron in \mathbb{R}^3 . Obviously, for any $x, y \in T$, we have $T \cap W_{xy} = \{x, y\}$. Also, it is easily seen that there is no 5-point rq -convex set containing T . Suppose there is a 6-point rq -convex set $\{a, b, c, d, x, y\}$. The only suitable pair of points $x, y \in W_{ab} \cap W_{cd}$ is obtained when $\{x, y\} = (H_{ab} \cap C_{cd}) \cup (H_{ba} \cap C_{cd})$. But then b, c do not enjoy the rq -property in $\{a, b, c, d, x, y\}$. Hence $\gamma(T) \geq 3$.

Next, we only need to prove $\gamma(T) \leq 3$. Let a_1, b_1, c_1 denote the midpoints of ad, bd, cd , respectively. See Figure 13. The line $L_a \ni a_1$ parallel to \overline{bc} and the anal-

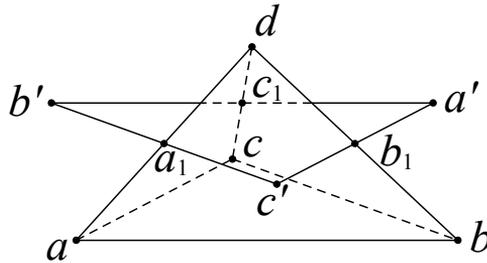


Figure 13: A 7-point rq -convex set containing the vertices of a regular tetrahedron.

ogous lines L_b and L_c are coplanar. Put

$$\{a'\} = L_b \cap L_c, \quad \{b'\} = L_c \cap L_a, \quad \{c'\} = L_a \cap L_b.$$

Obviously, $ab'dc', bc'da', ca'db'$ are squares, while $a'b'ab, b'c'bc, c'a'ca$ are rectangles. Thus, $\{a, b, c, d, a', b', c'\}$ is a 7-point rq -convex set, and $\gamma(T) = 3$. \square

Theorem 6.1 reveals the existence of 7-point rq -convex sets in \mathbb{R}^3 , in contrast with the inexistence of such sets in \mathbb{R}^2 . What happens in higher dimensions?

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