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Selfishness of convex bodies and discrete point sets

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Dedicated to the memory of Professor Michel Deza

ABSTRACT

Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$-*convex* if for any pair of distinct points $x, y \in M$, there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$.

A family $\mathcal{F}$ of compact sets is called *complete* if $\mathcal{F}$ contains all compact $\mathcal{F}$-convex sets. Generalizing the definition in Yuan and Zamfirescu (2016), a compact set K will be called *selfish*, if the family $\mathcal{F}_K$ of all sets similar to K contains all compact $\mathcal{F}_K$-convex sets.

In this paper, we investigate the selfishness of rectangles, isosceles triangles, regular n-gons, and some finite sets.

1. Introduction

At the 1974 meeting about convexity in Oberwolfach, the second author proposed the investigation of this very general kind of convexity: let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$-*convex* if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$.

It is clear that usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are just some examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

In 1980, Blind, Valette and Zamfirescu [2] first investigated rectangular convexity, which was also studied by Böröczky, Jr. [3], in 1990. In 2014 Zamfirescu [20] studied the right convexity. Yuan and Zamfirescu [17,16] investigated the right triple convexity, which is the discrete version of the former one. Later, Yuan, Zamfirescu and Zhang [19] studied the isosceles triple convexity.

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Bruckner [8] and also Magazanik, Perles [13] investigated L_n sets, which are $\mathcal{F}$-convex sets when $\mathcal{F}$ is the family of all polygonal paths in the plane with at most n edges. Magazanik, Perles [14] and Breen [4,5,7,6] dealt with staircase connectedness, which is also a kind of $\mathcal{F}$-convexity, $\mathcal{F}$ being the family of all staircases. Hyperconvex sets with respect to a convex body K defined by Mayer in 1935 [15] can also be regarded as $\mathcal{F}$-convex sets when $\mathcal{F} = \{\bigcap \mathcal{T}' : \mathcal{T}' \subset \mathcal{T}\}$, where $\mathcal{T}$ is the family of all translates of K. The case when K is a Euclidean ball, has become quite well researched recently [1,12]. Furthermore, for a subset of the vertex set of a graph, the g-convexity investigated by Farber and Jamison [11], the T-convexity studied by Changat and Mathew [9], and M-convexity researched by Duchet [10] can also be regarded as examples of $\mathcal{F}$-convexity for suitable families $\mathcal{F}$.

A family $\mathcal{F}$ of compact sets is called *complete* if $\mathcal{F}$ contains all compact $\mathcal{F}$-convex sets. A compact set K is called *selfish*, if the family $\mathcal{F}_K$ of all sets similar to K is complete. In [18], Yuan and Zamfirescu introduced and investigated the selfishness of convex bodies. For triangles, they showed that all non-acute ones are non-selfish; for acute ones, they proved that every equilateral triangle is selfish, and that there exist acute triangles which are not selfish. For quadrilaterals they proved that every rhombus is selfish, that neither the family of all rhombi, nor the family of all rectangles is complete, and that there exist quadrilaterals which are not selfish. In 3-dimensional space, they proved that no circular cylinder is selfish. They also proposed two problems:

Problem 1. Is every rectangle selfish?

Problem 2. Is every isosceles acute triangle selfish?

In this paper, we first continue the research line in [18] and prove that rectangles, acute isosceles triangles and all regular polygons are selfish, thus answering the two open problems mentioned above in the affirmative. Then we obtain some results on the selfishness of finite sets.

Now we present some notation.

Let $x, y \in \mathbb{R}^d$ be two distinct points. We denote by $\|x\|$ the Euclidean norm of x, by xy the straight line determined by x, y, and by $\overline{xy}$ the line-segment with endpoints x and y. Let L_{xy} denote the line perpendicular to xy, passing through the point x.

For a set S, let $\operatorname{diam} S = \sup\{\|x - y\| : x, y \in S\}$. A 2-point set $\{x, y\} \subset M$ with $\|x - y\| = \operatorname{diam} M$ is called a *diametral pair* of M, while $\overline{xy}$ is a *diameter* of M.

For any compact set $C \subset \mathbb{R}^d$, let S_C be the smallest hypersphere containing C in its convex hull, bd C and relint C be the boundary and relative interior of C. For compact sets C_1, C_2, $C_1 \sim C_2$ means that C_1 and C_2 are similar.

For $\alpha \in \mathbb{R}$, $\lceil \alpha \rceil$ is the smallest integer not less than α.

2. Selfishness of convex bodies

First we discuss the selfishness of rectangles.

Theorem 2.1. *Every rectangle is selfish in the plane.*

Proof. Let R_h denote a rectangle whose length-to-width ratio is h ($h \geqslant 1$). Let K be a compact $\mathcal{F}_{R_h}$-convex set in the plane. We want to prove that $K \in \mathcal{F}_{R_h}$.

Suppose that the line-segment $\overline{ac}$ is a diameter of K. By the definition of $\mathcal{F}_{R_h}$-convexity, at least one of the two rectangles in $\mathcal{F}_{R_h}$ with $\overline{ac}$ as a diagonal is included in K. Assume w.l.o.g. that the rectangle $abcd$ is contained in K, where $\|a - b\| = h\|a - d\|$. The four disks with radius equal to diam K, centered at a, b, c, d, respectively, intersect in a "curved rhombus" $\widetilde{a'b'c'd'}$, as shown in Fig. 1. Clearly, $K \subset \widetilde{a'b'c'd'}$.

Let m, n be the midpoints of the line-segments $\overline{bc}$, $\overline{ab}$, respectively. By symmetry, we only need to verify that there is no point of K lying in the union of the "curved triangle" $\widetilde{mbb'}$ minus $\overline{mb}$ and the "curved triangle" $\widetilde{nba'}$ minus $\overline{bn}$, see Fig. 1.

Assume, on the contrary, that $K \cap \widetilde{mbb'} \setminus \overline{mb} \neq \emptyset$. For any $u \in$ relint $\overline{bc}$, the line au meets bd K at a and some point u'. Then $K \subset \widetilde{a'b'c'd'}$ and $K \cap \widetilde{mbb'} \setminus \overline{mb} \neq \emptyset$ imply $u' \in \widetilde{bcb'} \setminus \overline{bc}$. According to the definition of $\mathcal{F}_{R_h}$-convexity, there is a rectangle $R_{au'} \in \mathcal{F}_{R_h}$ such that $a, u' \in R_{au'}$ and $R_{au'} \subset K$. Noticing that $a, u' \in$ bd K and a is an extreme point of K, we have $a, u' \in$ bd $R_{au'}$ and a is a vertex of

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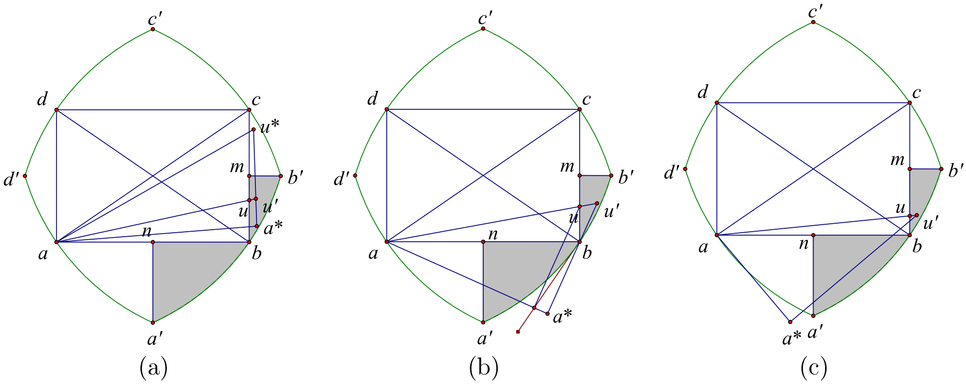

Fig. 1. $K \cap \widetilde{mbb'} \setminus \overline{mb} \neq \emptyset$.

$R_{au'}$. However, $\angle abu' > \pi/2$ implies $\|a-u'\| > \|a-b\|$, hence a, u' cannot lie on the same edge of $R_{au'}$, otherwise the diagonal of $R_{au'}$ is longer than the diameter of K.

If u' lies in the relative interior of an edge $\overline{a^*u^*}$ of $R_{au'}$, as shown in Fig. 1(a), then a^*u^* is the only supporting line of K through u'. Since $b, c \in K$, they are lying on the same side of a^*u^*. Then a^*, u^* belong to the “curved triangle” $\widetilde{bb'c}$. Furthermore, $u' \notin \overline{bc}$ implies that $\{a^*, u^*\} \neq \{b, c\}$. Therefore, $\angle a^*au^* < \angle bac$, which contradicts $R_{au'} \in \mathcal{F}_{R_h}$.

Now suppose that u' is a vertex of $R_{au'}$. Let $R_{au'} = aa^*u'u^*$, where a^* lies below au'. If $\angle au'a^* = \arctan h$, as shown in Fig. 1(b), then clearly a^* is on the right side of L_{bd}, which means $a^* \notin \widetilde{a'b'c'd'}$. If $\angle u'aa^* = \arctan h$, as shown in Fig. 1(c), then

$$\|a-a^*\| = \frac{\|a-u'\|}{\sqrt{1+h^2}} > \frac{\|a-b\|}{\sqrt{1+h^2}}.$$

Now let u move towards b such that

$$\|u-b\| \leqslant \frac{h^2}{2(1+h^2)}\|b-c\|.$$

Clearly, $\angle caa^* = \angle cab + \angle u'aa^* - \angle uab = \pi/2 - \angle uab < \pi/2$, so $aa^* \cap \text{conv}\,\widetilde{a'd'}$ is a line-segment with length

$$\begin{aligned} 2\|a-c\|\cos\angle caa^* &= 2\|a-c\|\sin\angle uab = 2\|a-c\|\frac{\|u-b\|}{\|a-u\|} \\ &< 2\|a-c\|\frac{\|u-b\|}{\|a-b\|} \leqslant 2\sqrt{1+h^2}\|b-c\|\frac{h^2}{2(1+h^2)}\|b-c\|\frac{1}{h\|b-c\|} \\ &= \frac{h\|b-c\|}{\sqrt{1+h^2}} = \frac{\|a-b\|}{\sqrt{1+h^2}} < \|a-a^*\|, \end{aligned}$$

which implies that $a^* \notin \widetilde{a'b'c'd'}$. Both contradict $R_{au'} \subseteq K$.

Consequently, no point of K is in $(\widetilde{add'} \setminus \overline{ad}) \cup (\widetilde{cbb'} \setminus \overline{bc})$, whence K is contained in the “curved polygon” $\widetilde{aa'bcc'd}$, shown in Fig. 2.

Assume now $K \cap \widetilde{nba'} \setminus \overline{nb} \neq \emptyset$. For every $e \in \text{relint}\,\overline{ad}$ and $v \in \text{relint}\,\overline{ab}$, ev meets $\text{bd}\,K$ at e and some point $v' \in \widetilde{aba'} \setminus \overline{ab}$. Since K is $\mathcal{F}_{R_h}$-convex, there is a rectangle $R_{ev'} \in \mathcal{F}_{R_h}$ such that $e, v' \in R_{ev'} \subset K$ and $e, v' \in \text{bd}\,R_{ev'}$. By $e \neq a$ we know that $\|e-v'\| > \|a-b\|$ if $v \to b$, hence e, v' cannot lie on the same side of $R_{ev'}$. Because ad is the only supporting line of K through e, and $\angle dav' > \angle dab = \pi/2$, e must be a vertex of $R_{ev'}$. Otherwise there is an edge $\overline{e^*v^*}$ of $R_{ev'}$ satisfying $e^*, v^* \in \overline{ad}$ and $e \in \text{relint}\,\overline{e^*v^*}$. Assume $e^* \in \overline{ae}$, as shown in Fig. 2(a). Then $\angle ee^*v' > \angle dav' > \pi/2$, which contradicts $\angle ee^*v' \leqslant \pi/2$.

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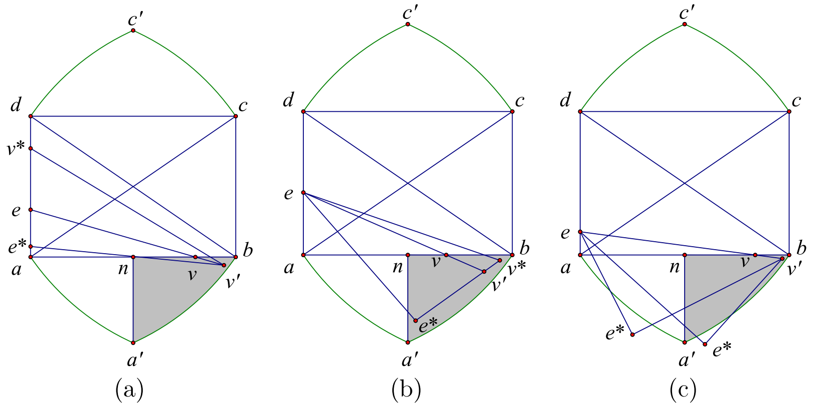

Fig. 2. $K \cap ((\widetilde{add'} \setminus \overline{ad}) \cup (\widetilde{cbb'} \setminus \overline{bc})) = \emptyset$ and $K \cap \widetilde{nba'} \setminus \overline{nb} \neq \emptyset$.

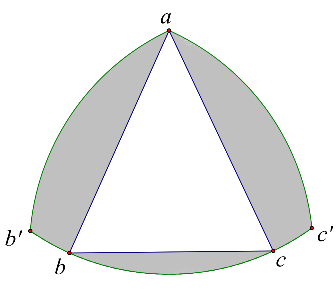

Fig. 3. Curved triangle $\widetilde{ab'c'}$.

If there is an edge $\overline{e^*v^*}$ of $R_{ev'}$ such that $v' \in \operatorname{relint} \overline{e^*v^*}$, then e^*v^* is the only supporting line of K through v'. Thus e^*, v^* belong to the "curved triangle" $\widetilde{aa'b}$. Assume w.l.o.g. e^* lies on the left of v', as shown in Fig. 2(b). When $e \to a$ and $v \to b$, we have $v' \to b$, and then $v' \to v^*$. If $\angle ee^*v' = \pi/2$, then we may have $\angle ev'e^* > \angle ev^*e^* = \arctan h$, or $\angle e^*ev' \to \angle e^*ev^* = \arctan h$. Both will lead to $e^* \notin \widetilde{aa'b}$, a contradiction. If $\angle ev^*v' = \pi/2$, then $\angle ev'e^* > \pi/2$, and therefore $e^* \notin \widetilde{aa'b}$, also a contradiction.

If v' is a vertex of $R_{ev'}$, then let $R_{ev'} = ee^*v'v^*$, where e^* lies under ev', as shown in Fig. 2(c). Thus we get $\angle ev'e^* = \arctan h$, or $\angle v'ee^* = \arctan h$. When $e \to a$ and $v \to b$, we have $e^* \notin \widetilde{aa'bcc'd}$, again a contradiction.

The proof is complete. □

Now we discuss the selfishness of isosceles acute triangles.

Lemma 2.2. *Every isosceles triangle with apex angle less than $\pi/3$ is selfish.*

Proof. Let I_α be an isosceles triangle with apex angle α $(0 < \alpha < \pi/3)$. Suppose K is a compact $\mathcal{F}_{I_\alpha}$-convex set in the plane.

Let $\overline{ab}$ be a diameter of K. Then there is an isosceles triangle $\triangle abc$ with leg $\overline{ab}$ and apex angle $\angle bac = \alpha$ contained in K. Clearly, K is contained in $\widetilde{ab'c'}$ (see Fig. 3), which is the intersection of the three disks with radius $\operatorname{diam} K$, centered at a, b, c, respectively. Now we prove that K must be the triangle $\triangle abc$ of vertices a, b, c.

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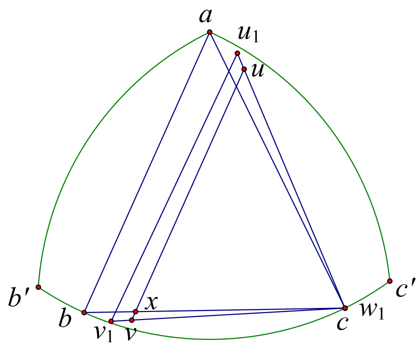

Fig. 4. Neither u nor v is a vertex of T.

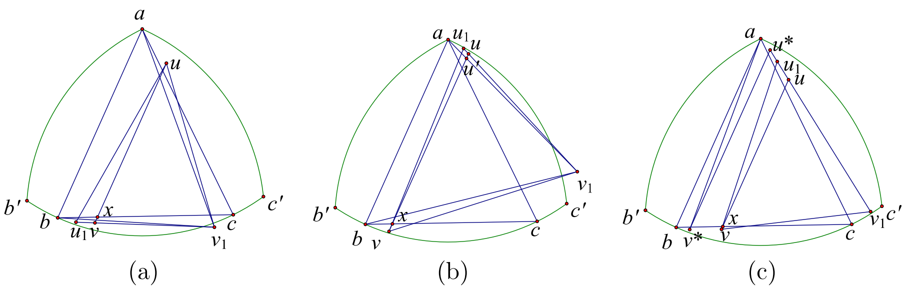

Fig. 5. Only one of u and v is a vertex of T.

Suppose on the contrary that $K \setminus \triangle abc \neq \emptyset$, which means that at least one of the sets $\widetilde{ab'b} \setminus \overline{ab}$, $\widetilde{ac'c} \setminus \overline{ac}$ and $(\mathrm{conv}\,\widetilde{bc}) \setminus \overline{bc}$ has points in K. Assume w.l.o.g. $((\widetilde{ac'c} \setminus \overline{ac}) \cup (\mathrm{conv}\,\widetilde{bc} \setminus \overline{bc})) \cap K \neq \emptyset$.

For any $x \in \mathrm{relint}\,\overline{bc}$, the line through x parallel to ab intersects $\mathrm{bd}\,K$ at $u \in \widetilde{ac'c}$ and $v \in \mathrm{conv}\,\widetilde{bc}$. Then at least one of the points u, v lies outside of the triangle $\triangle abc$.

Since K is $\mathcal{F}_{I_\alpha}$-convex, there is an isosceles triangle T with apex angle α such that $u, v \in T \subset K$ and $u, v \in \mathrm{bd}\,T$.

First we claim that we can choose suitably x, such that both u, v are vertices of T.

Assume that neither u nor v is a vertex of T. Since $u, v \in \mathrm{bd}\,K$, the points u, v cannot lie in the relative interior of the same edge of T. Now we may assume that $u \in \mathrm{relint}\,\overline{u_1w_1}$, $v \in \mathrm{relint}\,\overline{v_1w_1}$, where u_1, w_1, v_1 are the vertices of T, as shown in Fig. 4. Clearly, u_1w_1 is the unique supporting line of K through u, and v_1w_1 the unique supporting line of K through v. Therefore a, b, c are on the same side of u_1w_1 and v_1w_1. Thus $u_1, w_1 \in \widetilde{ac'c}$, $v_1, w_1 \in \widetilde{bc}$, which implies $w_1 = c$. Since at least one of u, v is outside of $\triangle abc$, $\angle u_1w_1v_1 = \angle ucv > \angle acb = (\pi - \alpha)/2 > \alpha$, contradicting $T \in \mathcal{F}_{I_\alpha}$.

Assume now that u is a vertex of T, and there is an edge $\overline{u_1v_1}$ of T such that $v \in \mathrm{relint}\,\overline{u_1v_1}$. Clearly, u_1, v_1 are in $\mathrm{conv}\,\widetilde{bc}$. Assume w.l.o.g. v_1 is on the right of v, as shown in Fig. 5(a). In the triangle $\triangle abv_1$, we have $\angle abv_1 \geqslant \angle abc = (\pi - \alpha)/2 \geqslant \angle uv_1u_1 > \angle av_1b$. Therefore $\|a - v_1\| > \|a - b\|$, which means v_1 is outside $\widetilde{ab'c'}$, a contradiction.

Now, assume that v is a vertex of T, and there is an edge $\overline{u_1v_1}$ of T such that $u \in \mathrm{relint}\,\overline{u_1v_1}$. Clearly, u_1, v_1 are in $\widetilde{acc'}$. Assume w.l.o.g. u_1 is above u. At this moment we have $u \notin \overline{ac}$, otherwise $\{u_1, u, v_1\} \subset \overline{ac}$, but $\angle vu_1v_1 < \angle vuv_1 = \angle bac = \alpha$ cannot be an interior angle of T. If $\angle vu_1v_1 = (\pi - \alpha)/2$, as shown in Fig. 5(b), then in the triangle $\triangle abv_1$, we have $\angle bav_1 = \angle vu'v_1 \geqslant \angle vuv_1 > \angle vu_1v_1 = (\pi - \alpha)/2 \geqslant \angle u_1v_1v > \angle av_1b$. Therefore $\|b - v_1\| > \|a - b\|$, which means v_1 is outside $\widetilde{ab'c'}$, also a contradiction. If $\angle vu_1v_1 = \alpha$, as shown in Fig. 5(c), then let u^* be the end point of the line-segment $u_1v_1 \cap K$ which

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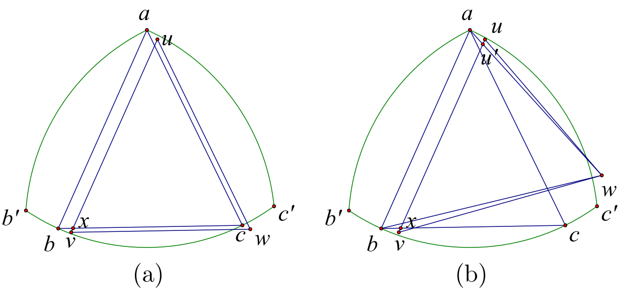

Fig. 6. w and c are on the same side of uv.

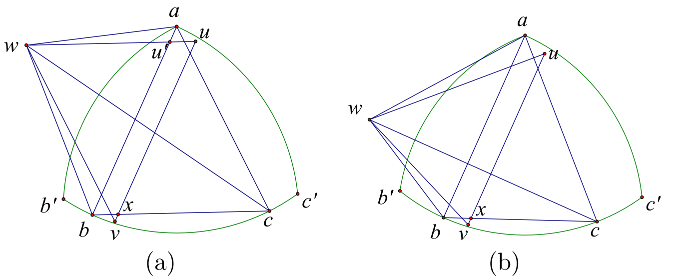

Fig. 7. w and c are on different sides of uv.

lies near u_1. If $u^* \neq a$, draw through u^* a line parallel to $\overline{ab}$ intersecting the boundary of K at v^*, and let $u = u^*$, $v = v^*$. If $u^* = a$, since u is not in $\triangle abc$, we can choose a suitable x such that $\angle uav > \alpha$. Therefore the claim is proved.

Suppose $T = \triangle uvw$. When $x \to b$, we have $\|u - v\| \to \|a - b\|$. Then $\overline{uv}$ must be a leg of T, which implies that $\angle uwv \neq \alpha$.

Case 1. w and c are on the same side of uv.

If $\angle vuw = \alpha$, as shown in Fig. 6(a), then c is in the interior of $\triangle uvw$ ($u, v \notin \triangle abc$), or in the relative interior of an edge of T (one of u, v is in $\triangle abc$, the other is not). The discussion above implies $w \notin \widetilde{ab'c'}$.

If $\angle uvw = \alpha$, as shown in 6(b), then we consider $\|b - w\|$. Since K is a compact convex set, $a, b, w \in K$ implies $\overline{aw}, \overline{bw} \subset K$. By $u, v \in \text{bd}\, K$, both intersections $\overline{aw} \cap \overline{uv}$, $\overline{bw} \cap \overline{uv}$ are not empty. Let $\overline{aw} \cap \overline{uv} = \{u'\}$. So in $\triangle abw$, $\angle baw = \angle vu'w \geqslant \angle vuw = \angle uwv > \angle bwa$. Therefore $\|b - w\| > \|b - a\|$, which means that w is outside of $\widetilde{ab'c'}$.

Case 2. w and c are on different sides of uv.

If $\angle uvw = \alpha$, as shown in Fig. 7(a), then we have $\angle caw = \angle cab + \angle baw = \angle cab + \angle bu'w - \angle uwa = \angle cab + \angle vuw - \angle uwa$. If $x \to b$, then $u \to a$ and $\angle uwa \to 0$. Therefore $\angle caw \to \angle cab + \angle vuw = \alpha + (\pi - \alpha)/2 > \pi/2$, which will lead to w being outside $\widetilde{ab'c'}$.

Above all, the third vertex w of T must satisfy the following conditions: w and c are on different sides of uv and $\angle vuw = \alpha$, as shown in Fig. 7(b). Rotate b clockwise about a by an angle α, and denote the new position by b_1. If $x \to b$, we have $w \to b_1$. As K is compact, $b_1 \in K$. Similarly, in the triangle $\triangle ab_1b$ we rotate b clockwise about a by angle 2α and get the point b_2, which is still in K. Repeat the above processes, we can get $b_3, b_4, \ldots \in K$. As $\alpha > 0$, there must be one b_n such that $\angle b_nac > \pi/3$. Therefore $\|b_n - c\| > \|a - c\|$, and b_n is outside $\widetilde{ab'c'}$, a contradiction.

Thus, $K = \triangle abc$, and the proof is complete. $\square$

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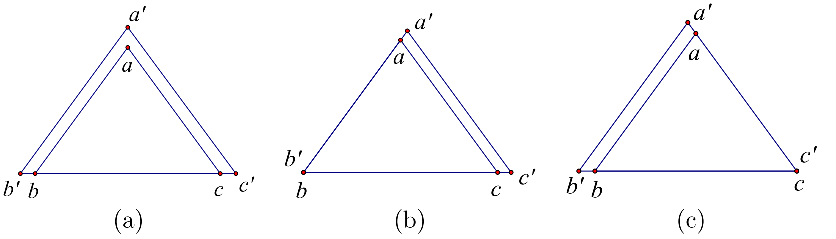

Fig. 8. $(bc \setminus \overline{bc}) \cap K \neq \emptyset$.

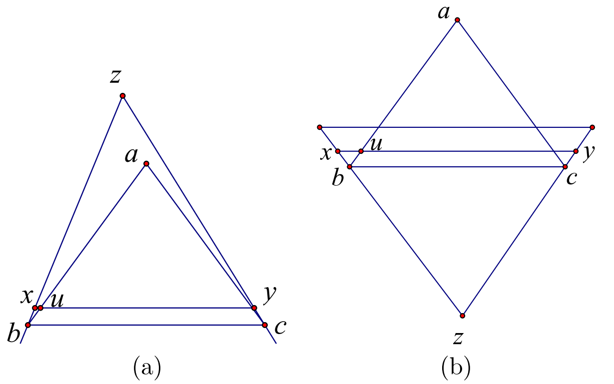

Fig. 9. Neither x nor y is a vertex of I_{xy}.

Lemma 2.3. *Every isosceles triangle with apex angle greater than $\pi/3$ and less than $\pi/2$ is selfish.*

Proof. Let I_α be an isosceles triangle with apex angle α ($\pi/3 < \alpha < \pi/2$). Suppose K is a compact $\mathcal{F}_{I_\alpha}$-convex set in the plane. We prove that K belongs to $\mathcal{F}_{I_\alpha}$.

For $a \in \text{bd}\, K$, choose $b, c \in \text{bd}\, K$ such that $\angle bac = \alpha$ and $\|a-b\| = \|a-c\|$. Among all such triples, let $\{a, b, c\} \subset \text{bd}\, K$ denote the largest one (with respect to their diameters).

We first prove that there is no point of K lying in $bc \setminus \overline{bc}$. Suppose on the contrary that $(bc \setminus \overline{bc}) \cap K \neq \emptyset$. As $b, c \in \text{bd}\, K$, there is no point of K that lies below bc, otherwise at least one of b, c would be an interior point of K. Let $K \cap bc = \overline{b'c'}$. Since K is $\mathcal{F}_{I_\alpha}$-convex, there exists an isosceles triangle in $\mathcal{F}_{I_\alpha}$ contained in K and containing b', c'. Clearly, b', c' must be vertices of the triangle. Hence there must be a triple $\{a', b', c'\} \subset K$ with $\angle b'a'c' = \alpha$ and $\|a'-b'\| = \|a'-c'\|$. $\{b, c\} \neq \{b', c'\}$ implies that a is an interior point of $\triangle a'b'c'$, as shown in Fig. 8(a), or the triple $\{a', b', c'\} \subset \text{bd}\, K$ is a larger one than $\{a, b, c\}$, as shown in Fig. 8(b)(c).

Now we claim that both $\overline{ab}$ and $\overline{ac}$ are contained in bd K. Suppose the contrary, and consider two cases.

Case 1. Neither $\overline{ab}$ nor $\overline{ac}$ is contained in bd K. For every point $u \in \text{relint}\, \overline{ab}$, through u draw a line L_u parallel to bc. Suppose $L_u \cap \text{bd}\, K = \{x, y\}$. Clearly, both x and y are outside $\triangle abc$. As K is an $\mathcal{F}_{I_\alpha}$-convex set, there must be an isosceles triangle I_{xy} in $\mathcal{F}_{I_\alpha}$ such that $x, y \in I_{xy} \subset K$. $x, y \in \text{bd}\, K$ implies $x, y \in \text{bd}\, I_{xy}$. Then we prove that both x and y must be vertices of I_{xy} when $u \to b$.

Assume that neither x nor y is a vertex of I_{xy}. Then x and y lie in the relative interiors of different edges of I_{xy}. Let z be the common vertex of the two edges. Then both xz and yz are supporting lines of K.

If $b \in zx$ and $c \in zy$, then a is an interior point of I_{xy} (as shown in Fig. 9(a)) or the vertex set of I_{xy} is a larger triple on bd K (see Fig. 9(b)), and both are contradictions.

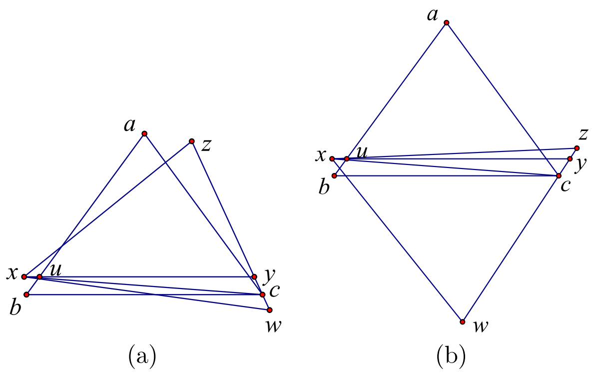

Fig. 10. x is a vertex and y lies in the relative interior of an edge $\overline{zw}$ of I_{xy}.

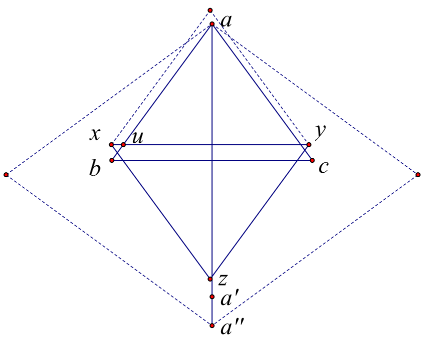

Fig. 11. Both x and y are vertices of I_{xy}.

If $b \notin zx$ and $c \in zy$, let $\overline{zw} = zx \cap \text{bd}\, K$. Then w lies in the open strip determined by xy and bc. Suppose the line through w parallel to bc meets bd K at v and w, and let $x = w$, $y = v$. Then x must be a vertex of I_{xy}.

If $b \notin zx$ and $c \notin zy$, we can choose u below this position such that both x and y are vertices of I_{xy}.

Suppose that x is a vertex and y lies in the relative interior of an edge $\overline{zw}$ of I_{xy}.

If $c \in zw$, suppose that w is below y. We can choose suitably u, such that $\|x - y\| > \|a - b\|$ and $\angle xcy > (\pi - \alpha)/2$. Since one of the side-lengths $\|x - w\|$ and $\|x - z\|$ must be greater than $\|x - y\|$ in the triangle $\triangle xzw$, $\angle wxz = (\pi - \alpha)/2$. If $\angle xzy = \alpha$, then $\angle xwy = (\pi - \alpha)/2$, see Fig. 10(a). We can assume $\|x - z\| < \|x - y\|$, otherwise $\{x, z, w\} \in \text{bd}\, K$ is larger than $\{a, b, c\}$. So $\angle bcz = \angle xyz < \alpha$. As $\angle xcy > (\pi - \alpha)/2$, $c \in \text{relint}\, \overline{yw}$. $\angle bcw = \pi - \angle bcz > \pi/2$ implies $\|b - w\| > \|b - c\|$. When $u \to b$, we have $x \to b$. Therefore $\|x - w\| > \|b - c\|$, and we get a larger triple on bd K. If $\angle xzy = (\pi - \alpha)/2$, then $\angle xwy = \alpha$, as shown in Fig. 10(b). We can assume $\|x - w\| < \|x - y\|$, otherwise $\{x, z, w\} \subset \text{bd}\, K$ is larger than $\{a, b, c\}$. So $\angle xyw < \angle xwy = \alpha$. Therefore $\angle bcz = \angle xyz = \pi - \angle xyw > \pi/2$, and $\|b - z\| > \|b - c\|$. When $u \to b$, we can get a larger triple on bd K.

If $c \notin zw$, we can choose u closer to b, such that y is also a vertex of I_{xy}.

Suppose both x and y are vertices of I_{xy}. Hence there is a point $z \in K$ satisfying $\angle xzy = \alpha$, $\|z - x\| = \|z - y\|$. Since $a \in \text{bd}\, K$, z must be below xy. When $u \to b$, $z \to a'$, which is the reflected copy of the point a about bc, as shown in Fig. 11. Since K is compact, $a' \in K$. Suppose $aa' \cap \text{bd}\, K = \{a, a''\}$. So there exists an isosceles triangle $I_{aa''}$ in $\mathcal{F}_{I_\alpha}$ such that $a, a'' \in I_{aa''} \subset K$. As $\|a - a'\| > \|b - c\|$, a, a' must be vertices of $I_{aa''}$, otherwise another larger triple will appear on bd K. Hence there is a point p in K such that $\angle apa' = \alpha$ and $\|p - a\| = \|p - a'\|$, which contradicts the fact that $b, c \in \text{bd}\, K$.

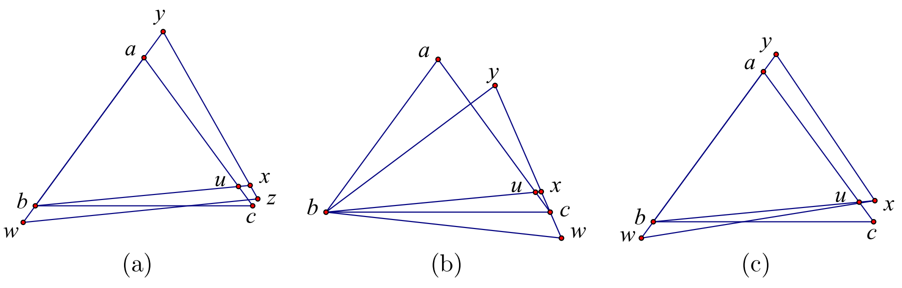

Fig. 12. At least one of b and x is not a vertex of I_{bx}.

Case 2. Exactly one of $\overline{ab}$ and $\overline{ac}$ is contained in bd K. Assume w.l.o.g. $\overline{ab} \subset$ bd K. For any $u \in$ relint $\overline{ac}$, suppose $bu \cap$ bd $K = \{b, x\}$. It is clear that x is outside the triangle $\triangle abc$. For points b, x, there is an isosceles triangle I_{bx} in $\mathcal{F}_{I_\alpha}$ such that $b, x \in I_{bx} \subset K$ and $b, x \in$ bd I_{bx}.

Suppose that neither b nor x is a vertex of I_{bx}. Then b, x lie in the relative interiors of different edges of I_{bx}. Let y be the intersection of the two edges. $\overline{ab} \subset$ bd K implies $y \in ab$. If $b \in$ relint $\overline{ay}$ or $y \in$ relint $\overline{ab}$, then $\angle byx < \angle abc = (\pi - \alpha)/2$ or $\angle byx > \angle bax > \alpha$, respectively. And therefore the angle byx cannot be an angle of I_{bx}. Hence $a \in$ relint $\overline{by}$. Suppose the other two vertices are w, z, with $b \in \overline{yw}$ and $x \in \overline{yz}$, as shown in Fig. 12(a). So $a, b \in \overline{yw}$, and yz is a supporting line of K. As $c \in$ bd K, c cannot lie above wz. Hence $\angle ywz \leqslant \angle ywc < \angle ybc = (\pi - \alpha)/2$, as $w \neq b$. Thus, $\angle ywz$ cannot be an interior angle of I_{bx}.

Suppose b is a vertex of I_{bx}, x is in the relative interior of an edge $\overline{yw}$ of I_{bx}. If $c \in yw$, then $\overline{bc} \subset I_{bx}$ as $\overline{ab} \subset$ bd K, see Fig. 12(b). The vertex set of I_{bx} is a larger triple on bd K.

If $c \notin yw$, we can choose u below this position such that x is also a vertex of I_{bx}.

Suppose x is a vertex of I_{bx} and b is in the relative interior of an edge $\overline{yw}$ of I_{bx}, with w below b. We have $yw = ab$ and $a \in \overline{yw}$, otherwise, $y \in$ relint $\overline{ab}$, $\angle wyx > \angle bac = \alpha$ cannot be an interior angle of I_{bx}. So $\|y - w\| > \|a - b\|$, see Fig. 12(c). And one of $\|x - w\|$ and $\|x - y\|$ must be greater than $\|b - x\|$ in the triangle $\triangle xyw$. Therefore $\{x, y, w\} \subset$ bd K is larger than $\{a, b, c\}$.

If both b, x are vertices of I_{bx}, then there is a point $y \in K$ with $\angle byx = \alpha$ and $\|y - b\| = \|y - x\|$. As $\angle abx < \angle abc = (\pi - \alpha)/2$ and $\overline{ab} \subset$ bd K, y must be below bx. When $u \to c$, then $y \to a'$, where a and a' are symmetric with respect to bc. Now, by a method similar to the one in case 1, we also get a contradiction.

The claim is proved. Therefore $ab \cap K \subset$ bd K, $ac \cap K \subset$ bd K. If both $(ab \setminus \overline{ab}) \cap K$ and $(ac \setminus \overline{ac}) \cap K$ are not empty, we can obtain a triple on bd K larger than $\{a, b, c\}$. So we assume w.l.o.g $(ab \setminus \overline{ab}) \cap K = \emptyset$. Then we prove $\overline{bc} \subset$ bd K. For $u \in$ relint $\overline{ab}$, $v \in$ relint $\overline{bc}$, suppose $uv \cap$ bd $K = \{u, x\}$. There exists an isosceles triangle I_{ux} in $\mathcal{F}_{I_\alpha}$ such that $u, x \in I_{ux} \subset K$ and $u, x \in$ bd I_{ux}. If $\overline{bc}$ is not included in the boundary of K, x is outside $\triangle abc$. When $u \to b$ and $v \to c$, then $x \to c$. So we can choose suitably u, v, such that $\angle abx < \angle aux < \alpha$.

If neither u nor x is a vertex of I_{ux}, they lie in the relative interiors of different edges. As $(ab \setminus \overline{ab}) \cap K = \emptyset$, the common vertex of the two edges must be b, see Fig. 13(a). But $(\pi - \alpha)/2 < \angle ubx < \alpha$, which means the angle ubx cannot be an angle of I_{ux}.

If u is a vertex of I_{ux}, and x is in the relative interior of an edge $\overline{yw}$ of I_{ux}, yw is a supporting line of K. Suppose y is above x. When $u \to b$ and $v \to c$, then $x, y \to c$. So we can assume $\angle uwy = \alpha$, as shown in Fig. 13(b). Therefore, when $u \to b$ and $v \to c$, $w \to a'$, a' being the mirror reflection point of a about bc. By using the same method as above, we can get a contradiction.

If x is a vertex of I_{ux}, and u is in the relative interior of an edge $\overline{yw}$ of I_{ux}, then $\overline{yw} \subset \overline{ab}$. Suppose y is above u, as shown in Fig. 13(c). As

$$\alpha > \angle aux > \angle ywx \geqslant \angle abx > \frac{\pi - \alpha}{2},$$

the angle ywx cannot be an angle of I_{ux}.

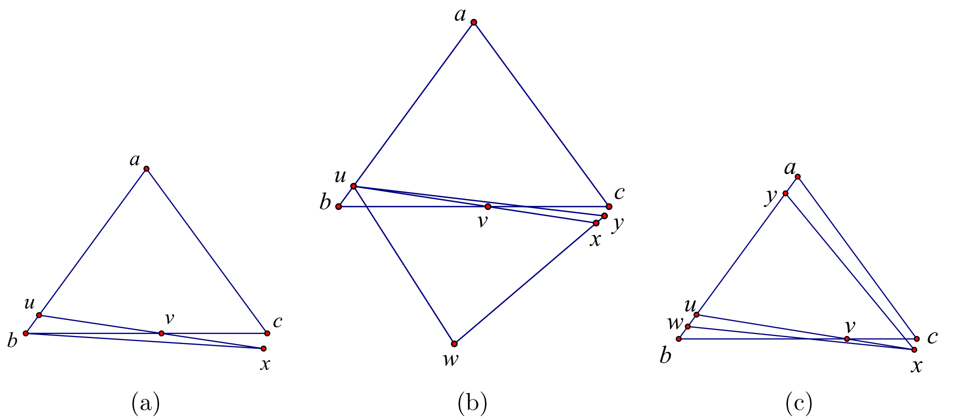

Fig. 13. At least one of u and x is not a vertex of I_{ux}.

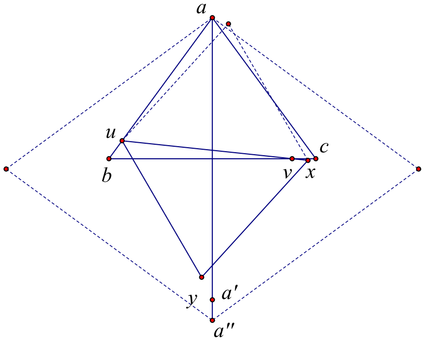

Fig. 14. Both u and x are vertices of I_{ux}.

If both u and x are vertices of I_{ux}, there is a point $y \in K$ with $\angle uyx = \alpha$ and $\|y-u\| = \|y-x\|$. The point y must be below ux when $v \to c$, otherwise it contradicts $\overline{ac} \subset \mathrm{bd}\, K$, see Fig. 14. Therefore, when $u \to b$ and $v \to c$, $y \to a'$, where a' is the mirror reflection point of a about bc. By the same method, we get a contradiction. □

It is proved in [18] that every equilateral triangle is selfish. Combining that with Lemmas 2.2 and 2.3, we obtain the following theorem, which answers the second open problem of [18] affirmatively.

Theorem 2.4. *Every acute isosceles triangle is selfish.*

Let T be an isosceles trapezoid. If an edge of T is a diameter of S_T, then the disk is clearly $\mathcal{F}_T$-convex. However, prohibiting all edges of T to be diameters of S_T does not guarantee the selfishness of T.

Theorem 2.5. *There exists a non-selfish isosceles trapezoid T no edge of which is a diameter of S_T.*

Proof. Let T be an isosceles trapezoid with a base angle of $\frac{2\pi}{5}$, and with three equally long sides and a longer fourth. It is clear that the smallest regular pentagon containing T is $\mathcal{F}_T$-convex, see Fig. 15. □

In [18] it is proved that both the equilateral triangle and the square are selfish. Furthermore, we have the following theorem.

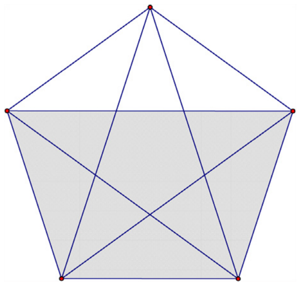

Fig. 15. A non-selfish isosceles trapezoid.

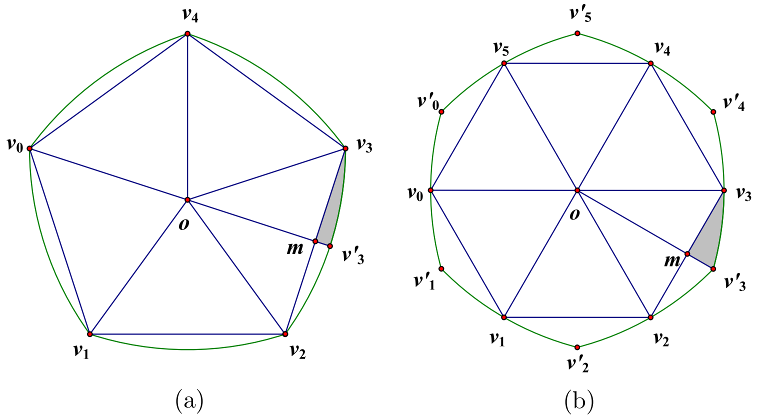

Fig. 16. $\widetilde{R_n}$.

Theorem 2.6. *Every regular convex polygon is selfish.*

Proof. Let R_n be a regular convex n-gon centered at o, and K a compact $\mathcal{F}_{R_n}$-convex set. We show that K is also a regular convex n-gon.

Let the line-segment $\overline{ab}$ be a diameter of K. By the definition of $\mathcal{F}_{R_n}$-convexity, there must be a regular convex n-gon with $\overline{ab}$ as a diameter and contained in K. Assume w.l.o.g. the regular n-gon is denoted by $\mathrm{conv}\{v_0, \ldots, v_{n-1}\}$, where $a = v_0$, $b = v_{\lceil \frac{n}{2} \rceil}$. The disks of radii $\|v_0 - v_{\lceil \frac{n}{2} \rceil}\|$ centered at $v_0, \ldots, v_{n-1}$ intersect in a "curved n-gon", $\widetilde{R_n}$. If n is odd, let $\widetilde{R_n} = v_0 \widetilde{\cdots v_{n-1}}$, see Fig. 16(a). If n is even, let $\widetilde{R_n} = v'_0 \cdots v'_{n-1}$, as shown in Fig. 16(b), where v_i is on the arc $\widetilde{v'_i v'_{i+1}}$ if $i = 0, \ldots, n-2$, and $v_{n-1} \in \widetilde{v'_{n-1} v'_0}$. Clearly, $K \subset \widetilde{R_n}$.

Denote by m the midpoint of $\overline{v_{\lceil \frac{n}{2} \rceil} v_{\lceil \frac{n}{2} \rceil - 1}}$. If n is odd, let $v'_{\lceil \frac{n}{2} \rceil}$ be the midpoint of the arc $v_{\lceil \frac{n}{2} \rceil} \widetilde{v_{\lceil \frac{n}{2} \rceil - 1}}$. Let $T = m v_{\lceil \frac{n}{2} \rceil} \widetilde{v'_{\lceil \frac{n}{2} \rceil}} \setminus \overline{m v_{\lceil \frac{n}{2} \rceil}}$, as shown in Fig. 16. Then we only need to prove $T \cap K = \emptyset$.

Suppose on the contrary that $T \cap K \neq \emptyset$. Hence $T \cap \mathrm{bd}\, K \neq \emptyset$. Let $u \in T \cap \mathrm{bd}\, K$. Since K is $\mathcal{F}_{R_n}$-convex, there must be a regular n-gon $R_{v_0 u}$ with $v_0, u \in R_{v_0 u} \subset K$. It is clear that $v_0, u \in \mathrm{bd}\, R_{v_0 u}$, and v_0 is a vertex of $R_{v_0 u}$.

If u is not a vertex of $R_{v_0 u}$, then there exists an edge of $R_{v_0 u}$, say $\overline{u_1 u_2}$, such that $u \in \mathrm{relint}\, \overline{u_1 u_2}$, as shown in Fig. 17. Hence $u_1 u_2$ is a supporting line of K through u, and $u_1, u_2 \in v_{\lceil \frac{n}{2} \rceil} \widetilde{v'_{\lceil \frac{n}{2} \rceil}} v_{\lceil \frac{n}{2} \rceil - 1}$. $u \notin \overline{v_{\lceil \frac{n}{2} \rceil} v_{\lceil \frac{n}{2} \rceil - 1}}$ implies $\{u_1, u_2\} \neq \{v_{\lceil \frac{n}{2} \rceil}, v_{\lceil \frac{n}{2} \rceil - 1}\}$. Therefore $\angle u_1 v_0 u_2 < \angle v_{\lceil \frac{n}{2} \rceil} v_0 v_{\lceil \frac{n}{2} \rceil - 1} = \frac{\pi}{n}$. But in a regular n-gon for any two non-adjacent points v_i, v_j, we have $\angle v_j v_i v_{j-1} = \frac{\pi}{n}$, a contradiction.

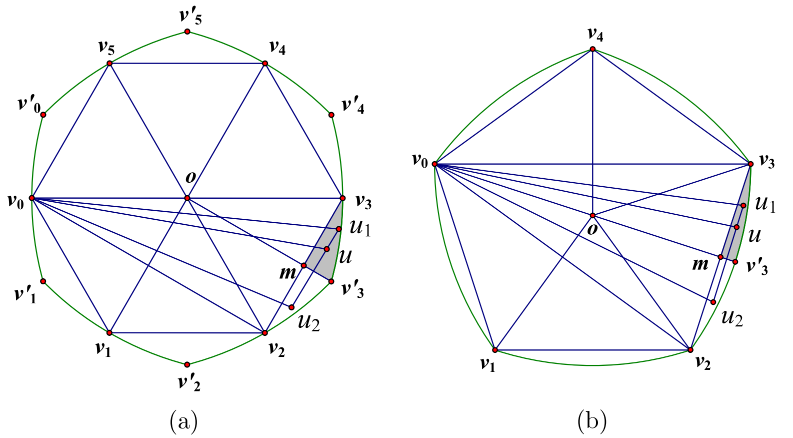

Fig. 17. u is not a vertex of R_{v_0u}.

If u is a vertex of R_{v_0u}, we claim that $\overline{v_0u}$ must be a diameter of R_{v_0u}. Indeed, since $\angle v_0mu > \pi/2$, we have $\|v_0 - u\| > \|v_0 - m\|$.

If n is odd, then

$$\|v_0 - m\| = \|v_0 - o\| + \|v_{\frac{n-1}{2}} - o\| \sin(\frac{n-2}{2n}\pi) > 2\|v_0 - o\| \sin(\frac{n-2}{2n}\pi),$$
$$\|v_0 - v_i\| = 2\|v_0 - o\| \sin(\frac{i}{n}\pi),\ i = 1, \dots, \frac{n-3}{2}.$$

Hence for any $i = 1, \dots, \frac{n-3}{2}$,

$$\|v_0 - v_i\| \leqslant 2\|v_0 - o\| \sin(\frac{n-3}{2n}\pi) < 2\|v_0 - o\| \sin(\frac{n-2}{2n}\pi) < \|v_0 - m\|.$$

If n is even, for any $i = 1, \dots, \frac{n-2}{2}$, $\|v_0 - v_i\| \leqslant \|v_0 - v_{\frac{n-2}{2}}\|$. But in the triangle $\triangle v_0 v_{\frac{n-2}{2}} m$, $\angle v_0 v_{\frac{n-2}{2}} m = \pi/2$, so $\|v_0 - v_{\frac{n-2}{2}}\| < \|v_0 - m\|$.

Therefore $\|v_0 - m\|$ is larger than all the distances between pairs of vertices in R_n, except the diametral pair. Recall that $\|v_0 - u\| > \|v_0 - m\|$, which forces $\overline{v_0u}$ to be a diameter of R_{v_0u}.

Let w_1, w_2 be the two adjacent vertices of v_0 in R_{v_0u}, and assume that w_1 lies above v_0. We will prove that at least one of w_1, w_2 is outside of "curved n-gon" $\widetilde{R_n}$, which contradicts to K being $\mathcal{F}_{R_n}$-convex.

If n is odd, then $\{\angle w_1v_0u, \angle w_2v_0u\} = \{\frac{n-1}{2n}\pi, \frac{n-3}{2n}\pi\}$.

If $\angle w_1v_0u = \frac{n-1}{2n}\pi$, then $\angle w_1v_0v_{\frac{n-1}{2}} = \angle w_1v_0u + \angle uv_0v_{\frac{n-1}{2}} \geqslant \frac{n-1}{2n}\pi + \frac{1}{2n}\pi = \pi/2$. So w_1 does not lie in $\widetilde{R_n}$.

If $\angle w_2v_0u = \frac{n-1}{2n}\pi$, then w_2 is on the left of line v_0v_1, as shown in Fig. 18(a). Because

$$\angle w_2v_0v_1 = \angle uv_0v_{\frac{n+1}{2}} = \frac{n-1}{2n}\pi - \angle uv_0v_1,\quad \frac{\|w_2 - v_0\|}{\|u - v_0\|} = \frac{\|v_0 - v_1\|}{\|v_0 - v_{\frac{n+1}{2}}\|},$$

we have $\triangle w_2v_0v_1 \sim \triangle uv_0v_{\frac{n+1}{2}}$. Hence $\angle v_0v_1w_2 = \angle v_0v_{\frac{n+1}{2}}u > \angle v_0v_{\frac{n+1}{2}}v_{\frac{n-1}{2}} = \frac{n-1}{2n}\pi$. Therefore $\angle v_{\frac{n+1}{2}}v_1w_2 = \angle v_{\frac{n+1}{2}}v_1v_0 + \angle v_0v_1w_2 > \frac{n-1}{2n}\pi + \frac{n-1}{2n}\pi = \frac{n-1}{n}\pi$. When $n \geqslant 3$, $\angle v_{\frac{n+1}{2}}v_1w_2 > \pi/2$, which implies $w_2 \notin \widetilde{R_n}$.

If n is even, then $\angle w_2v_0u = \frac{n-2}{2n}\pi$. Clearly w_2 is on the left side of v_0v_1, see Fig. 18(b). Noticing that

$$\angle w_2v_0v_1 = \angle uv_0v_{\frac{n}{2}} = \frac{n-2}{2n}\pi - \angle uv_0v_1,\quad \frac{\|w_2 - v_0\|}{\|u - v_0\|} = \frac{\|v_0 - v_1\|}{\|v_0 - v_{\frac{n}{2}}\|},$$

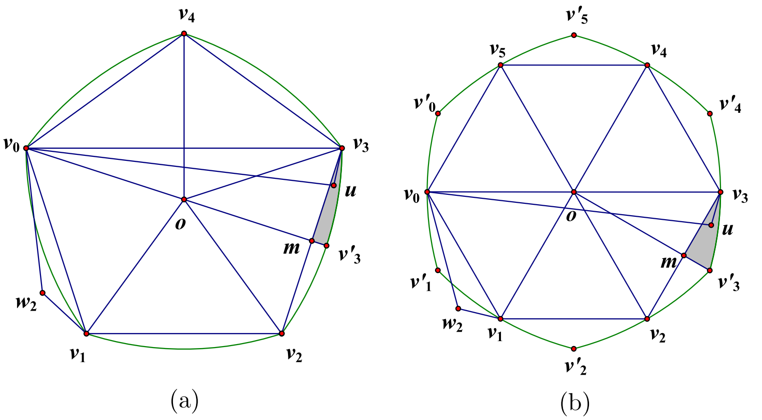

Fig. 18. u is a vertex of R_{v_0u}.

we have $\triangle w_2v_0v_1 \sim \triangle uv_0v_{\frac{n}{2}}$. So $\angle v_0v_1w_2 = \angle v_0v_{\frac{n}{2}}u > \angle v_0v_{\frac{n}{2}}v_{\frac{n-2}{2}} = \frac{n-2}{2n}$. Hence $\angle v_{\frac{n+2}{2}}v_1w_2 = \angle v_{\frac{n+2}{2}}v_1v_0 + \angle v_0v_1w_2 > \frac{n-2}{2n}\pi + \frac{n-2}{2n}\pi = \frac{n-2}{n}\pi$. When $n \geqslant 4$, $\angle v_{\frac{n+2}{2}}v_1w_2 \geqslant \pi/2$, and therefore $w_2 \notin \widetilde{R_n}$.

The proof is complete. □

3. Selfishness of finite sets

In this section we investigate the selfishness of finite sets. Our first result about them parallels Theorem 2.6.

Theorem 3.1. *The vertex set of a regular polygon is selfish.*

Proof. Let V_n be the vertex set of a planar regular convex n-gon, and P be a finite $\mathcal{F}_{V_n}$-convex set in the plane. We show that P is also the vertex set of a regular n-gon.

Let $\{a, b\}$ be a diametral pair of P. Since P is $\mathcal{F}_{V_n}$-convex, there must be an n-point set $\{v_0, \ldots, v_{n-1}\} \in \mathcal{F}_{V_n}$, such that $\{a, b\} \subset \{v_0, \ldots, v_{n-1}\} \subset P$ and $\{a, b\}$ is also a diametral pair of $\{v_0, \ldots, v_{n-1}\}$. Assume w.l.o.g. $a = v_0$, $b = v_{\lceil \frac{n}{2} \rceil}$. It is clear that $P \subset \widetilde{R_n}$, where $\widetilde{R_n}$ is the "curved n-gon" described in the proof of Theorem 2.6, as shown in Fig. 19.

By a method similar to the one used in the proof of Theorem 2.6, we get $(\widetilde{mv_{\lceil \frac{n}{2} \rceil}v'_{\lceil \frac{n}{2} \rceil}} \setminus \{v_{\lceil \frac{n}{2} \rceil}\}) \cap P = \emptyset$. So $P \subset (\mathrm{intconv}\{v_0, \ldots, v_{n-1}\} \cup \{v_0, \ldots, v_{n-1}\})$, see Fig. 20.

Assume that there is a $u \in P$, such that $u \in \triangle omv_{\lceil \frac{n}{2} \rceil} \setminus \overline{mv_{\lceil \frac{n}{2} \rceil}}$, as shown in Fig. 20. As P is $\mathcal{F}_{V_n}$-convex, there is an n-point set $V_{v_0u} \in \mathcal{F}_{V_n}$ satisfying $v_0, u \in V_{v_0u} \subset P$. Suppose the vertices in V_{v_0u} adjacent to v_0 are w_1, w_2, and assume that w_1 is above v_0. For $\angle w_1v_0w_2 = \frac{n-2}{n}\pi$, there must exist a positive integer k, such that $\angle w_1v_0u = \frac{k}{n}\pi$, $\angle w_2v_0u = \frac{n-2-k}{n}\pi$. If $w_2 \in (\mathrm{intconv}\{v_0, \ldots, v_{n-1}\} \cup \{v_0, \ldots, v_{n-1}\})$, then by $w_2 \neq v_1$, we have $\angle w_2v_0u < \angle v_1v_0u \leqslant \angle v_1v_0v_{\lceil \frac{n}{2} \rceil} = \frac{\lceil \frac{n}{2} \rceil - 1}{n}\pi$. So $n - 2 - k < \lceil \frac{n}{2} \rceil - 1$. For k is integer, $k \geqslant n - \lceil \frac{n}{2} \rceil$. Therefore $\angle w_1v_0u = \frac{k}{n}\pi \geqslant \frac{n - \lceil \frac{n}{2} \rceil}{n}\pi = \angle v_{n-1}v_0v_{\lceil \frac{n}{2} \rceil - 1}$. But u is above the line $v_0v_{\lceil \frac{n}{2} \rceil - 1}$, Hence $w_1 \notin (\mathrm{intconv}\{v_0, \ldots, v_{n-1}\} \cup \{v_0, \ldots, v_{n-1}\})$.

Consequently, $P = \{v_0, \ldots, v_{n-1}\}$ is the vertex set of a regular n-gon. □

From Theorem 3.1 we know that the vertex set of the equilateral triangle is selfish. The parallelism to the results of the previous section ends, however, when passing to other isosceles triangles.

Theorem 3.2. *The vertex set of no isosceles triangle, excepting the equilateral one, is selfish.*

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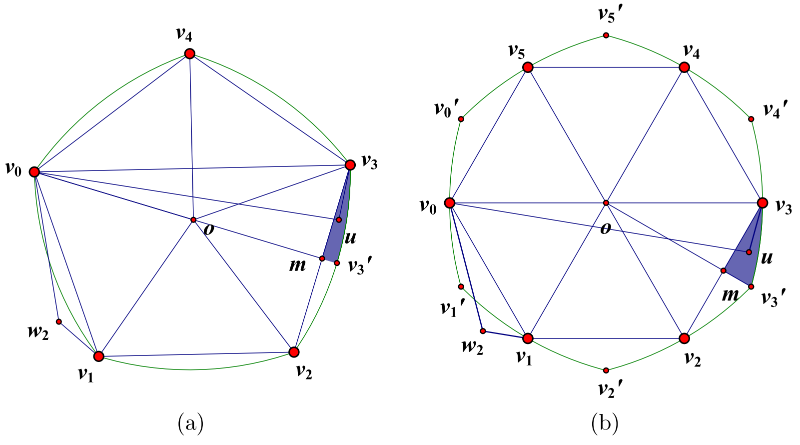

Fig. 19. $\widetilde{R_n}$.

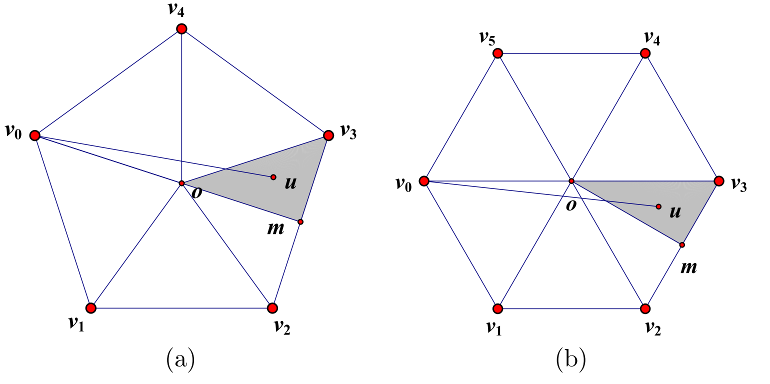

Fig. 20. $P \cap (\triangle omv_{\lceil \frac{n}{2} \rceil} \setminus \overline{mv_{\lceil \frac{n}{2} \rceil}}) \neq \emptyset$.

Proof. Let $\triangle abc$ be an isosceles triangle with $\|a-b\| = \|a-c\| \neq \|b-c\|$. Denote by e the intersection of the circle of radius $\|b-c\|$ centered at b and ca; denote by d the intersection of the circle of radius $\|b-c\|$ centered at c and ba, see Fig. 21. Since $\triangle abc$ is not equilateral, $d \neq a$ and $e \neq a$. It is easy to check that for any two points in $\{a, b, c, d, e\}$, there is a third point in the set, such that the three points form an isosceles triangle similar to $\triangle abc$. So $\{a, b, c\}$ is not selfish. □

Theorem 3.3. *There exist a parallelogram and an isosceles trapezoid, the vertex sets of which are non-selfish.*

Proof. Let $P = \{a, b, c, d\}$ be the vertex set of a parallelogram with $\|a-b\| = \sqrt{2}\|b-c\|$. Let $e = (a+b)/2$ and $f = (c+d)/2$. It is easily seen that $\{a, b, c, d, e, f\}$ is a 6-point $\mathcal{F}_P$-convex set, see Fig. 22(a).

Let now $V = \{a, b, c, d\}$ be the vertex set of an isosceles trapezoid, with $\angle abc = 3\pi/5$, and $\|a-b\| = \|b-c\| = \|c-d\|$. It is obvious that the vertex set of the regular pentagon is $\mathcal{F}_V$-convex, as one can verify in Fig. 22(b). □

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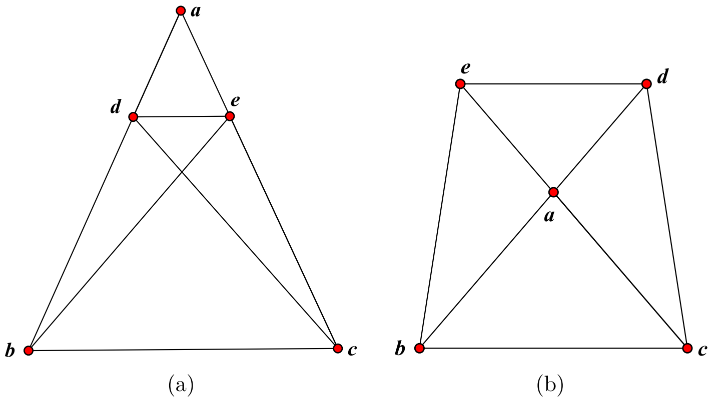

Fig. 21. $\triangle abc \sim \triangle bce \sim \triangle cbd \sim \triangle ade$.

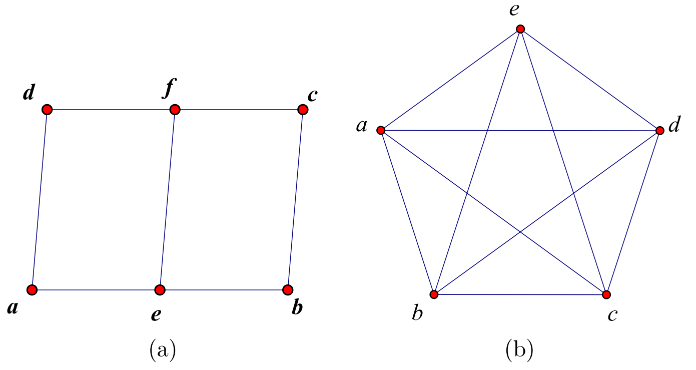

Fig. 22. Non-selfish parallelogram and isosceles trapezoid.

Remark. As a referee pointed out, it is natural to ask the following question.

For a convex polytope $P \in \mathbb{R}^d$, is there some relation between the selfishness of P and that of its vertex set?

Combining Sections 2 and 3, we have the following table.

Convex polygons	Selfishness of convex polygons	Selfishness of their vertex sets
Regular polygons	Yes	Yes
Non-equilateral isosceles acute triangles	Yes	No
Isosceles non-acute triangles	No	No
Rectangles with length ratio of long and short edges $\sqrt{2}$: 1	Yes	No

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