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Double normals of most convex bodies

Alain Rivière [a], Joël Rouyer [b], Costin Vîlcu [b,*], Tudor Zamfirescu [c,b,d]

[a] *Laboratoire Amiénois de Mathématiques Fondamentales et Appliquées, CNRS, UMR 7352, Faculté de Sciences d'Amiens, 80 039 Amiens Cedex 1, France*
[b] *Simion Stoilow Institute of Mathematics of the Roumanian Academy, Bucharest, Roumania*
[c] *Fakultät für Mathematik, Technische Universität Dortmund, 44221 Dortmund, Germany*
[d] *College of Mathematics and Information Science, Hebei Normal University, 050024 Shijiazhuang, PR China*

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ABSTRACT

We consider a typical (in the sense of Baire categories) convex body K in $\mathbb{R}^{d+1}$. The set of feet of its double normals is a Cantor set, having lower box-counting dimension 0 and packing dimension d. The set of lengths of those double normals is also a Cantor set of lower box-counting dimension 0. Its packing dimension is equal to $\frac{1}{2}$ if $d = 1$, is at least $\frac{3}{4}$ if $d = 2$, and equals 1 if $d \geq 3$. We also consider the lower and upper curvatures at feet of double normals of K, with a special interest for local maxima of the length function (they are countable and dense in the set of double normals). In particular, we improve a previous result about the metric diameter.

* Corresponding author.
E-mail addresses: Alain.Riviere@u-picardie.fr (A. Rivière), Joel.Rouyer@ymail.com (J. Rouyer), Costin.Vilcu@imar.ro (C. Vîlcu), tudor.zamfirescu@mathematik.tu-dortmund.de (T. Zamfirescu).

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1. Introduction and results

Let $\mathbb{E}$ be the Euclidean space of dimension $d+1$, with $d \geq 1$, and let $\mathcal{K}$ be the set of all convex bodies (i.e., compact convex sets with non-empty interior) in $\mathbb{E}$. For $K \in \mathcal{K}$, a *chord* is a line-segment xy joining boundary points x and y of K. A chord xy is called a *normal* of K if it is orthogonal to some supporting hyperplane at the point x called *foot*. An *affine diameter* is a chord with parallel supporting hyperplanes at its endpoints, while a *double normal* is an affine diameter orthogonal to those supporting hyperplanes. Thus, a double normal is a normal with two feet. In this paper, $\mathcal{N}(K)$ stands for the set of (oriented) double normals of K, $\ell(c)$ denotes the length of an oriented chord c, and $\mathcal{L}(K) = \{\ell(b) | b \in \mathcal{N}(K)\}$.

It is well known that every normal to a convex body K is a double normal if and only if K has constant width. On the other hand, the shortest and the longest affine diameter are double normals, but are there others?

Answering a question proposed by V. Klee [15], N. H. Kuiper proved in 1964 that every convex body in $\mathbb{E}$ has at least $d+1$ non-oriented double normals [19]. Moreover, for any $\mathcal{C}^{2-}$-function $f : \mathbb{P}^d \to \mathbb{R}$ ($\mathbb{P}^d$ is the projective space seen as the set of line directions of $\mathbb{E}$) there exists a symmetric convex body K in $\mathbb{E}$ with centre 0, for which the set of directions of double normals coincides with the critical set $z \in \mathbb{P}^d | (df)_z = 0$ of f. Conversely, for any convex body K in $\mathbb{E}$ there exists a centrally symmetric convex body K' with $\mathcal{C}^{2-}$-boundary and a $\mathcal{C}^{2-}$-function $f : \mathbb{P}^d \to \mathbb{R}$ whose critical set coincides with the set of double normal directions of K, and of K'. Here $\mathcal{C}^{2-}$ stands for a class of regularity between $\mathcal{C}^1$ and $\mathcal{C}^2$. More important for our paper, he also proved the following result.

Theorem A. ([19]) *If $d \leq 2$, $\mathcal{L}(K)$ has measure 0, while for $d \geq 3$ there exists a $\mathcal{C}^{2-}$ centrally symmetric strictly convex body $K^\star$ in $\mathbb{E}$ and a (non-rectifiable) arc $\gamma : [0,1] \to \mathcal{N}(K^\star)$ such that $\mathcal{L}(K^\star) = \{\ell(\gamma(t)) | t \in [0,1]\}$ is a non-degenerate interval.*

Two years later, A. S. Besicovitch and T. Zamfirescu [4] proved the existence of a planar convex body K with an interior point x such that $\mathcal{L}(K)$ and the set of ratios in which x divides affine diameters through it are uncountable. Their construction provides convex curves whose set of double normals is homeomorphic to any chosen compact subset of $\mathbb{R}$.

Recently, J. P. Moreno and A. Seeger devoted Sections 4 and 5 in [22] to the study of double normals. They prove, among other results, that $\mathcal{L}(K)$ is finite for any full-dimensional polytope K in $\mathbb{E}$ (compare to our Lemma 8).

Kuiper's results are closely related to billiards. Indeed, on a convex billiard table, 2-periodic trajectories correspond to double normals. A classical result of G. Birkhoff [5] states that in any planar convex billiard table K there always exist trajectories of period n, for any integer $n \geq 2$.

The set $\mathcal{B}$ of strictly convex planar sets, having a $\mathcal{C}^r$ boundary (for some $r \geq 2$) with positive curvature everywhere, endowed with a suitable metric, is a Baire space.

M. J. Dias Carneiro, S. Oliffson Kamphorst and S. Pinto de Carvalho [9] proved that for most billiard tables $K \in \mathcal{B}$, for every integer $n \geq 2$, there are at most finitely many n-periodic trajectories; in particular, $\mathcal{N}(K)$ and thus $\mathcal{L}(K)$ are finite. For results in similar directions, see [6], [8], [9], [16], [17], [18], [23], [27].

The problem of counting double normals extends beyond convexity, to the framework of Riemannian manifolds, see for instance [13], [24], [26].

In this paper we study double normals from the point of view of Baire categories. Our results strongly contrast the abovementioned ones on the finiteness of the sets of double normals.

The next fundamental fact, independently discovered by V. Klee [14] and P. Gruber [11], is essential for our topic.

Theorem B. ([11], [14]) *The boundary of most $K \in \mathcal{K}$ is of differentiability class $\mathcal{C}^1 \setminus \mathcal{C}^2$ and strictly convex.*

Our work is also related to the articles [3], [30], [31], [32], which focus on intersections of infinitely many affine diameters or normals for typical convex bodies. Let us mention here that, for $d \geq 2$, double normals of a typical convex body are pairwise disjoint [25]. For other Baire category results about convex bodies, see e.g. the survey [34].

We prove in this paper the following results.

For most $K \in \mathcal{K}$, the set of feet of double normals is a Cantor set (i.e., a set homeomorphic to the standard Cantor set) having lower box-counting dimension 0 and packing dimension d (Theorem 1 in Section 3, and Theorems 2–3 in Section 4). Recall that the lower box-counting dimension is greater than or equal to the Hausdorff dimension and the upper box-counting dimension is greater than or equal to the packing dimension, so these results provide the typical Hausdorff and upper box-counting dimension as well. Note that Theorems 1–2 are a little stronger, for they are stated for the sets of double normals rather than the sets of their feet (see Remark 3).

Let ℓ_K be the map which associates to an oriented chord of K its length. Obviously, ℓ_K is Lipschitz continuous with respect to any standard metric of $\mathbb{E}^2$ (we shall choose one after Lemma 1). Double normals are related to the critical points of ℓ_K, see Lemmas 3 and 4.

The set of non-oriented double normals of K is denoted by $\widetilde{\mathcal{N}}(K)$, and $\widetilde{\ell}_K$ stands for the corresponding length map.

For most $K \in \mathcal{K}$, $\widetilde{\ell}_K$ is injective. It will follow that $\mathcal{L}(K)$ is a Cantor set and has lower box-counting dimension 0. In particular, its Lebesgue measure vanishes, though the function ℓ_K does not satisfy the hypotheses of regularity of Sard's theorem. For most $K \in \mathcal{K}$, the packing dimension of $\mathcal{L}(K)$ is equal to $\frac{1}{2}$ if $d = 1$, is at least $\frac{3}{4}$ if $d = 2$, and equals 1 if $d \geq 3$ (Theorems 4–5 and Corollary 3 in Section 5).

Again for most $K \in \mathcal{K}$, the set of maximizing chords (local maxima of the length function) is countable and dense in $\mathcal{N}(K)$ (Propositions 2–3 in Section 6).

The last author considered in [28], [29], [33] the lower and upper curvatures γ_i^τ and γ_s^τ and proved, among other results, the following.

Theorem C. *For most $K \in \mathcal{K}$, at each point $x \in \partial K$, $\gamma_i^\tau(x) = 0$ or $\gamma_s^\tau(x) = \infty$ for any tangent direction τ at x, and both equalities hold at most points.*

The curvature of a convex body is deeply related to double normals, see [2], [35] and Remark 6.

We continue this investigation by considering the lower and upper curvatures at feet of double normals. We prove that at any foot x of a maximizing chord c of a typical convex body and in any tangent direction τ, $\gamma_s^\tau(x) = \infty$ and $\gamma_i^\tau(x) \geq \ell(c)^{-1}$, with equality if c is a metric diameter (a chord of globally maximal length); this improves [36, Th. 11]. Moreover, at both feet of a typical double normal, $\gamma_s^\tau(x) = \infty$ in any direction τ. Finally, given a fixed line-segment $c = xy$, for most convex bodies admitting c as double normal, $\gamma_s^\tau(x) = \infty$ and $\gamma_i^\tau(x) = 0$ in any direction τ (Theorems 6–9 in Section 7).

Statements similar to our theorems, but involving only centrally-symmetric convex bodies in $\mathbb{E}$, can also be proven. In this case, due to a variant of Theorem B for these bodies, see also [19, Theorem 2], all double normals intersect at the symmetry centre. The formal statements and the proofs are left to the interested reader. This paper also leaves open several questions, see Remarks 4, 5 and 7.

2. Preliminaries

The space $\mathcal{K}$, endowed with the Pompeiu–Hausdorff metric d_{PH}, is a Baire space. This allows us to state that *most* convex bodies, or *typical* convex bodies enjoy a given property, meaning that the set of those bodies that do not enjoy it is meagre, i.e. of first Baire category. (Recall that a subset of a topological space is said to be of *first Baire category*, if it is included in a countable union of closed sets of empty interior. Otherwise, it is called of second category.) Of course, it is also equivalent to state that the set of bodies that do enjoy the considered property is *residual*, meaning that it contains a dense countable intersection of open sets (a dense G_δ-set). A *Baire space* is a topological space in which every open set is of second category. We shall need the following (almost obvious) lemma.

Lemma 1. ([1]) *If Z is a space of second Baire category (in itself), Y is residual in Z, and X is residual in Y, then X is residual in Z.*

In this article, we shall apply the lemma when Z is a Baire space.

By *oriented* chord (respectively metric diameter, double normal) we mean an ordered pair of points corresponding to the endpoints of the non-oriented chord. It follows that ℓ

is nothing but the Euclidean metric on $\mathbb{E}$, and not, strictly speaking, a length function. Moreover $\mathcal{N}(K)$ and the set $\mathcal{C}(K) \stackrel{\text{def}}{=} \partial K \times \partial K$ of (possibly degenerate) oriented chords are subsets of $\mathbb{E}^2 = \mathbb{E} \times \mathbb{E}$ and inherit its metric. The distance we choose on $\mathbb{E}^2$ is given by

$$d\left((x, y), (x', y')\right) = \max\left(\|x - x'\|, \|y - y'\|\right);$$

thus, the ball centred at $c \in \mathbb{E}^2$ of radius r coincides with the Cartesian product of the balls of radii r centred at the entries of c.

An oriented chord which is a local maximum (a strict local maximum) of ℓ_K is said to be *maximizing* (respectively *strictly maximizing*). We define $\mathcal{M}(K)$ (resp. $\mathcal{M}^S(K)$) as the set of maximizing chords (respectively strictly maximizing chords).

From now on, unless otherwise specified, the words *double normal* will refer to an oriented double normal. The set of feet of double normals is denoted by $\mathcal{F}(K)$. The set of oriented affine diameters of K is denoted by $\mathcal{D}(K)$.

Some more general notation follows. As usual, $\mathbb{N}$ stands for the set of positive integers. We denote by $\mathbb{N}_n$ the set of positive integers smaller than or equal to n and by $\mathbb{N}_n^0$ the set of non-negative integers smaller than n. Given an n-tuple $x = (x_1, \ldots, x_n) \in \mathbb{E}^n$ and a subset I of $\mathbb{N}_n$, x_I denotes the set $\{x_i | i \in I\}$.

For any subset A of $\mathbb{E}$, ∂A stands for the boundary of A, $\operatorname{conv}(A)$ for the convex hull of A (i.e., the intersection of all convex sets containing A), $\langle A \rangle$ for the affine space spanned by A and $\overrightarrow{A}$ for the direction of $\langle A \rangle$, that is, the linear space of differences of vectors in $\langle A \rangle$.

For distinct x, $y \in \mathbb{E}$, xy stands for the line-segment joining x to y and $\overline{xy}$ for the whole line. The open ball, closed ball and sphere centred at x of radius r are denoted by $\mathbb{B}(x, r)$, $\bar{\mathbb{B}}(x, r)$ and $\mathbb{S}(x, r)$ respectively. We shall also use this notation when x belongs to $\mathbb{E}^2$.

Given a metric space X, for $A \subset X$, $\mathring{A}$ stands for the interior of A in X. The set of non-empty compact subsets of X is denoted by $\mathfrak{H}(X)$. It is endowed with the Pompeiu–Hausdorff distance induced by the distance on X. Since line-segments are compact subsets of $\mathbb{E}$, this metric also induces a distance on $\widetilde{\mathcal{N}}(K)$ for any $K \in \mathcal{K}$, with respect to which, the canonical map $\phi_K : \mathcal{N}(K) \to \widetilde{\mathcal{N}}(K)$ is 1-Lipschitz.

The next Lemma is obvious and left to the reader.

Lemma 2. *Let $K_n \in \mathcal{K}$ tend to $K \in \mathcal{K}$.*

(1) *Let $(x_n, y_n) \in \mathcal{N}(K_n)$ converge to $(x, y) \in \mathbb{E}^2$. Then (x, y) is a double normal of K.*
(2) *Let $C_n \subset \mathcal{N}(K_n)$ converge in $\mathfrak{H}(\mathbb{E}^2)$ to some limit C. Then $C \subset \mathcal{N}(K)$.*

Applying Lemma 2 with $K_n = K$, we get that $\mathcal{N}(K)$ is compact. Hence, $\mathcal{N}$ can be seen as a map from $\mathcal{K}$ to $\mathfrak{H}(\mathbb{E}^2)$. Note that Lemma 2 easily implies the upper semi-continuity of this map, in the sense of [20, p. 173].

Double normals are related to the critical points of ℓ_K. More precisely, we have the following two lemmas.

Lemma 3. *If $b = (x, y)$ is a local maximum of ℓ_K, then b is a double normal.*

Proof. Assume that b is not a double normal. Then the hyperplane H normal to $\overline{xy}$ through one extremity of b, say x, is not a supporting hyperplane. It follows that there exists $x_n \in \partial K$ tending to x and separated from y by H. Thus, $\|y - x_n\| > \|x - y\|$ and (x, y) is not a local maximum of ℓ_K. □

The next lemma is Proposition 1 in [18]; see also Proposition 2 in [8].

Lemma 4. *If ∂K is $\mathcal{C}^1$ then $b \in \mathcal{C}(K)$ is a double normal if and only if $\ell(b) > 0$ and $(d\ell_K)_b = 0$.*

The following lemma is central to this paper.

Lemma 5. *Let $b \in \mathcal{M}^S(K)$. Then, for any $\varepsilon > 0$, there exists a neighbourhood $\mathcal{U}$ of K in $\mathcal{K}$ such that for any $K' \in \mathcal{U}$ there exists a maximizing chord $b' \in \mathcal{M}(K')$ satisfying $d(b, b') < \varepsilon$.*

Proof. Since b is a strict local maximum of ℓ_K, there exists $r \in \,]0, \min(\varepsilon, \ell(b))[$ such that

$$\ell(b) > \max_{c \in \mathbb{S}(b,r) \cap \mathcal{C}(K)} \ell(c).$$

Hence, there is a neighbourhood $\mathcal{U}$ of K such that for any $K' \in \mathcal{U}$ there exists $c' \in \mathcal{C}(K') \cap \mathbb{B}(b, r)$ verifying

$$\ell(c') > \max_{c \in \mathbb{S}(b,r) \cap \mathcal{C}(K')} \ell(c).$$

It follows that the global maximum b' of $\ell_{K'}$ on $\bar{\mathbb{B}}(b, r) \cap \mathcal{C}(K')$ is not achieved on the boundary of the ball, and thus, it is a maximizing chord. □

We will often use implicitly the following criterion in order to prove that a chord is strictly maximizing.

Lemma 6. *Let $(x, y) \in \mathcal{N}(K)$, $K \in \mathcal{K}$. If there exists $\eta > 0$ and $\alpha < \pi/2$ such that for any $(x', y') \in \mathbb{B}((x, y), \eta) \cap \mathcal{C}(K)$ the angles $\measuredangle yxx'$ and $\measuredangle xyy'$ are smaller than α, then $(x, y) \in \mathcal{M}^S(K)$.*

The proof is elementary and left to the reader.

Corollary 1. *For any polytope $K \in \mathcal{K}$, $\mathcal{M}^S(K) = \mathcal{M}(K)$.*

The next lemma will be invoked in the proof of Theorem 5. It seems to be interesting by itself.

Lemma 7. *For all $K \in \mathcal{K}$, the map ℓ_K is 2-Hölder continuous. More precisely, for any b_0, $b_1 \in \mathcal{N}(K)$, $|\ell_K(b_0) - \ell_K(b_1)| \leq H_K d(b_0, b_1)^2$, where $H_K = 2/\min \mathcal{L}(K)$.*

Proof. Assume that $\ell(b_0) \leq \ell(b_1)$ and set $\varepsilon \overset{\text{def}}{=} d(b_0, b_1)$. Let x, x' be the feet of b_0 and $x+u$, $x'+u'$ be the feet of b_1, where $\max(\|u\|, \|u'\|) = \varepsilon$. Since b_1 is included in the strip of $\mathbb{E}$ between the hyperplanes normal to b_0 through x and x', we have $\langle x' - x, u\rangle \geq 0$ and $\langle x' - x, u'\rangle \leq 0$. It follows that

$$\begin{aligned}
\ell(b_1)^2 &= \|x' - x + u' - u\|^2 \\
&= \ell(b_0)^2 + \|u\|^2 + \|u'\|^2 - 2\langle x' - x, u\rangle + 2\langle x' - x, u'\rangle - 2\langle u, u'\rangle \\
&\leq \ell(b_0)^2 + 4\varepsilon^2,
\end{aligned}$$

whence

$$\ell(b_1) - \ell(b_0) \leq \frac{4}{\ell(b_1) + \ell(b_0)}\varepsilon^2 \leq H_K \varepsilon^2. \quad \square$$

Remark 1. 2-Hölder maps defined on a space connected by Lipschitz continuous arcs are constant.

Remark 2. It is a classical result that the restriction of a map of class $\mathcal{C}^2$ to a compact set of critical points is always 2-Hölder, but in our case ℓ_K is not so regular.

3. A Cantor set

In this section, we prove the following theorem.

Theorem 1. *For most $K \in \mathcal{K}$, $\mathcal{N}(K)$ is a Cantor set.*

Proof. Recall that a famous theorem of Brouwer assures that a compact metric space is a Cantor set if and only if it is non-empty, totally disconnected, and perfect. The compactness is clear from Lemma 2. The non-emptiness follows from the fact that any metric diameter (i.e., longest chord) is, by Lemma 3, a double normal. Thus, it remains to prove the last two properties, to which Lemmas 10 and 11 below are devoted. $\square$

Remark 3. When $K \in \mathcal{K}$ is of differentiability class $\mathcal{C}^1$ (the typical case, by Theorem B), the projection $\mathcal{N}(K) \to \mathcal{F}(K)$ that maps a double normal to its first foot is a bijection. Since $\mathcal{N}(K)$ is compact, it is a homeomorphism. Similarly, any small enough compact subset of $\mathcal{N}(K)$ is homeomorphic to its image by the canonical map $\mathcal{N}(K) \to \widetilde{\mathcal{N}}(K)$. It follows that Theorem 1 holds for $\mathcal{F}(K)$ and $\widetilde{\mathcal{N}}(K)$, too.

A finite set $X \subset \mathbb{E}$ is said to be *standard* if for any two disjoint subsets X_1, X_2 with cardinality at most $d+1$, we have

$$\dim\left(\overrightarrow{X_1} \cap \overrightarrow{X_2}\right) = \max\left(0, \dim\left(\overrightarrow{X_1}\right) + \dim\left(\overrightarrow{X_2}\right) - d - 1\right).$$

A polytope is said to be *standard* if for any two faces F, G that do not have a common vertex, we have

$$\dim\left(\overrightarrow{F} \cap \overrightarrow{G}\right) = \max\left(0, \dim\left(\overrightarrow{F}\right) + \dim\left(\overrightarrow{G}\right) - d - 1\right).$$

Clearly, a polytope with a standard set of vertices is standard.

Lemma 8. *If $K \in \mathcal{K}$ is a standard polytope then $\mathcal{N}(K)$ is finite.*

Proof. Let $(x, y) \in \mathcal{N}(K)$ and F_x, F_y be the minimal-dimensional faces containing x and y respectively. Clearly F_x and F_y are included in two parallel supporting hyperplanes H_x and H_y, whence they cannot have a common vertex. On the one hand, K is standard, whence

$$\dim\left(\overrightarrow{F_x} \cap \overrightarrow{F_y}\right) = \max\left(0, \dim\left(\overrightarrow{F_x}\right) + \dim\left(\overrightarrow{F_y}\right) - d - 1\right).$$

On the other hand, $\overrightarrow{F_x}$ and $\overrightarrow{F_y}$ are subspaces of $\overrightarrow{H_x} = \overrightarrow{H_y}$, whence

$$\dim\left(\overrightarrow{F_x} \cap \overrightarrow{F_y}\right) \geq \max\left(0, \dim\left(\overrightarrow{F_x}\right) + \dim\left(\overrightarrow{F_y}\right) - d\right).$$

It follows that $\dim\left(\overrightarrow{F_x}\right) + \dim\left(\overrightarrow{F_y}\right) \leq d$ and $\dim\left(\overrightarrow{F_x} \cap \overrightarrow{F_y}\right) = 0$. Hence (x, y) is the only double normal whose extremities lie in minimal faces F_x and F_y. We proved that the cardinal of $\mathcal{N}(K)$ is not greater then the number of ordered pairs of facets of K. $\square$

Lemma 9. *The set of n-tuples $x \in \mathbb{E}^n$ such that $x_{\mathbb{N}_n}$ is standard contains an open and dense set in $\mathbb{E}^n$.*

Proof. First notice that the set $U \subset \mathbb{E}^n$ of all n-tuples x such that for any $I \subset \mathbb{N}_n$, $\dim \overrightarrow{x_I} = \min(\#I - 1, d + 1)$ (points in *generic* position) is open and dense. We have to prove that, for any non-empty disjoint subsets I, J with cardinality at most $d+1$, the set

$$U_{I,J} \stackrel{\text{def}}{=} \{x \in U |\, \dim(\overrightarrow{x_I} \cap \overrightarrow{x_J}) = \max(0, \#I + \#J - 3 - d)\}$$

is open and dense. Put $k \overset{\text{def}}{=} \max(0, \#I + \#J - d - 3)$. Note that $\dim \overrightarrow{x_I} \cap \overrightarrow{x_J}$ is always greater than or equal to k, and that $\operatorname{rank}(M_{I,J}) = \#I + \#J - 2 - \dim(\overrightarrow{x_I} \cap \overrightarrow{x_J})$, where $M_{I,J}$ is a $(d+1) \times (\#I + \#J - 2)$ matrix, whose columns are vectors $x_i - x_{\min I}$ ($i \in I$, $i \neq \min I$) and $y_j - y_{\min J}$ ($j \in J$, $j \neq \min J$). So $x \notin U_{I,J}$ if and only if $\operatorname{rank}(M_{I,J}) < \#I + \#J - 2 - k$, that is, if all minors of $M_{I,J}$ of order greater than or equal to $\#I + \#J - 2 - k$ vanish. Such minors are polynomials on $\mathbb{E}^n$, whence $U_{I,J}$ is open, and dense if and only if it is not empty. The latter fact being obvious, the proof is finished. □

Lemma 10. *For most $K \in \mathcal{K}$, $\mathcal{N}(K)$ is totally disconnected.*

Proof. We have

$$\begin{aligned}
\mathcal{A} &\overset{\text{def}}{=} \left\{K \in \mathcal{K} \middle| \exists C \in \mathfrak{H}(\mathbb{E}^2),\, C \subset \mathcal{N}(K),\, C \text{ connected, } \operatorname{diam}(C) > 0\right\} \\
&= \bigcup_{n \in \mathbb{N}} \left\{K \in \mathcal{K} \middle| \exists C \in \mathfrak{H}(\mathbb{E}^2),\, C \subset \mathcal{N}(K),\, C \text{ connected, } \operatorname{diam}(C) \geq \frac{1}{n}\right\} \\
&\overset{\text{def}}{=} \bigcup_{n \in \mathbb{N}} \mathcal{A}_n.
\end{aligned}$$

We claim that $\mathcal{A}_n$ is closed. Choose a sequence $\{K_p\}_{p=1}^{\infty}$ in $\mathcal{A}_n$ converging to $K \in \mathcal{K}$. By definition of $\mathcal{A}_n$, there exist compact connected sets $C_p \subset \mathcal{N}(K_p)$ whose diameter is at least $\frac{1}{n}$. Let Q be a compact neighbourhood of $K \times K$ in $\mathbb{E}^2$. For p large enough, C_p belongs to $\mathfrak{H}(Q)$, which is compact. Hence, one can extract from $\{C_p\}_{p=1}^{\infty}$ a converging subsequence; let C be its limit. Clearly, $\operatorname{diam}(C) \geq \frac{1}{n}$, and by Lemma 2, $C \subset \mathcal{N}(K)$. It is well known (and easy to check) that a Pompeiu–Hausdorff limit of connected compact sets is connected. Hence, C is connected, K belongs to $\mathcal{A}_n$ and thus $\mathcal{A}_n$ is closed.

By virtue of Lemma 9, standard polytopes are dense in $\mathcal{K}$, and by Lemma 8, they cannot belong to $\mathcal{A}_n$. Hence $\mathcal{A}_n$ has empty interior, and thus $\mathcal{A}$ is meagre. □

Lemma 11. *For most $K \in \mathcal{K}$, $\mathcal{N}(K)$ is perfect.*

Proof. Choose any countable dense set Z in $\mathbb{E}^2$. The assumption that $\mathcal{N}(K)$ is not perfect implies that there exist $b \in \mathcal{N}(K)$, $r > 0$, $u \in Z$ such that

$$\mathcal{N}(K) \cap \bar{\mathbb{B}}(u, r) = \{b\}.$$

We have

$$\mathcal{A} \overset{\text{def}}{=} \{K \in \mathcal{K} | \mathcal{N}(K) \text{ not perfect}\} \subset \bigcup_{(n,u) \in \mathbb{N} \times Z} \mathcal{A}_{n,u}$$

with

$$\mathcal{A}_{n,u} = \left\{ K \in \mathcal{K} \middle| \exists b \in \mathcal{N}(K) \text{ s.t. } \mathcal{N}(K) \cap \bar{\mathbb{B}}\left(u, \frac{1}{n}\right) = \{b\} \right\}$$
$$= \left\{ K \in \mathcal{K} \middle| \# \left(\mathcal{N}(K) \cap \bar{\mathbb{B}}\left(u, \frac{1}{n}\right)\right) = 1 \right\}.$$

We have to prove that the closure of $\mathcal{A}_{n,u}$ has empty interior, that is, for any $K_0 \in \mathcal{K}$ and any $\varepsilon > 0$ there exists $K_3 \in \mathcal{K}$ such that $d_{PH}(K_0, K_3) < \varepsilon$ and such that a whole neighbourhood of K_3 does not intersect $\mathcal{A}_{n,u}$.

First, we can find a polytope K_1 such that $d_{PH}(K_0, K_1) < \varepsilon$. If $\mathcal{N}(K_1) \cap \bar{\mathbb{B}}\left(u, \frac{1}{n}\right)$ is empty then the set will remain empty for any K in a whole neighbourhood of K_1, because otherwise the limit of a converging subsequence of double normals of K tending to K_1 would belong to $\bar{\mathbb{B}}\left(u, \frac{1}{n}\right)$. Hence we can set $K_3 = K_1$ and the proof is finished.

If $\mathcal{N}(K_1) \cap \bar{\mathbb{B}}\left(u, \frac{1}{n}\right)$ is not empty then we can move and dilate slightly K_1 such that the modified polytope K_2 satisfies $d_{PH}(K_0, K_2) < \varepsilon$ and $\mathcal{N}(K_2) \cap \mathbb{B}\left(u, \frac{1}{n}\right) \neq \emptyset$. Let b_2 belong to $\mathcal{N}(K_2) \cap \mathbb{B}\left(u, \frac{1}{n}\right)$. Consider a rectangle $R = x_3x_3'y_3y_3'$ whose centre is the midpoint of b_2, such that x_3y_3' is parallel to b_2, longer than $\ell(b_2)$. If it is not too long nor too wide, then (x_3, y_3) and (x_3', y_3') belong to $\mathbb{B}\left(u, \frac{1}{n}\right)$ and the distance from $K_3 \overset{\text{def}}{=} \operatorname{conv}(K_2 \cup R)$ to K_0 is less than ε. Still reducing the width x_3x_3' if necessary, we may assume that the hyperplanes normal to the diagonals of R through their extremities do not intersect K_2, whence those hyperplanes are supporting K_3, and (x_3, y_3) and (x_3', y_3') are double normals of K_3. Also, one can easily check that any segment between x_3 (respectively y_3) and any point of K_3 makes an angle less than $\pi/2$ with x_3y_3, whence $(x_3, y_3) \in \mathcal{M}^S(K_3)$. Of course, the same holds for (x_3', y_3'). Now, by Lemmas 5 and 6, there is a whole neighbourhood U of K_3 such that any $K \in U$ admits at least two double normals in $\mathbb{B}\left(u, \frac{1}{n}\right)$, hence U does not intersect $\mathcal{A}_{n,u}$. □

4. Dimensions

In this section, we prove that for most convex bodies K the lower box-counting dimension of $\mathcal{F}(K)$ is 0 and its packing dimension is d. Let us recall their definitions.

If A is a metric space and δ is a positive number, a subset $F \subset A$ is called a *δ-set* if any two distinct points of F have a distance at least δ. Let's denote by $P_\delta(A)$ the supremum of the cardinals of all δ-sets of A. The *lower* and *upper box-counting dimension* of A are defined as

$$\underline{\dim}_B A = \liminf_{\delta \to 0} \frac{\ln P_\delta(A)}{-\ln \delta}$$
$$\overline{\dim}_B A = \limsup_{\delta \to 0} \frac{\ln P_\delta(A)}{-\ln \delta}.$$

It is well-known that the lower box-counting dimension is greater than or equal to the Hausdorff dimension [10, (3.17)].

The fact that a compact countable set may have arbitrarily large box-counting dimension leads to the definition of the so-called *packing dimension*:

$$\dim_P A = \inf_{\{A_i\}_{i=1}^{\infty}} \sup_{i\in\mathbb{N}} \overline{\dim}_B A_i,$$

where the infimum is taken over all the coverings $\{A_i\}_{i=1}^{\infty}$ of A. It is clear that this dimension is lower than or equal to the upper box-counting dimension, and vanishes for any countable set. There also exists a similar dimension derived from the lower box-counting dimension, but we shall not use it in this paper. Note that, classically, the packing dimension is defined in a completely different way, involving outer measures. See [10, 3.3 and 3.4] or [21, Section 5.9] for the original definition and the equivalence between those definitions.

It is easy to see that for any subset A of $\mathbb{E}$, $P_\delta\left(\overline{A}\right) = P_\delta\left(A\right)$ and thus $\overline{\dim}_B \overline{A} = \overline{\dim}_B A$. This fact, together with Baire's theorem leads to the following lemma.

Lemma 12. *Let s be a positive number. If A is a complete metric space in which any open set has upper box-counting dimension at least s, then $\dim_P A \geq s$.*

It follows that $\overline{\dim}_B A = \dim_P A$ whenever A is complete and enjoys some kind of homogeneity, as can be expected for the set of double normals of a typical convex body.

Theorem 2. *For most $K \in \mathcal{K}$, the lower box-counting dimension of $\mathcal{N}(K)$ is 0.*

Using a general result of Gruber [12, p. 20], the proof of the theorem almost completely reduces to the upper semi-continuity of the maps $K \mapsto \mathcal{N}(K)$ (Lemma 2) and $A \mapsto P_\delta(A)$ ([12, p. 20]). However, in order to make the paper more self-contained, we choose to give a more geometrical, direct proof.

Proof. Define

$$\mathcal{A} \overset{\text{def}}{=} \left\{ K \in \mathcal{K} \middle| \liminf \frac{\ln P_\delta\left(\mathcal{N}(K)\right)}{-\ln \delta} > 0 \right\} = \bigcup_n \mathcal{A}_n,$$

where

$$\mathcal{A}_n = \left\{ K \in \mathcal{K} \middle| \forall \delta \leq \frac{1}{n} \ : \ \frac{\ln P_\delta\left(\mathcal{N}(K)\right)}{-\ln \delta} \geq \frac{1}{n} \right\}.$$

We first prove that $\mathcal{A}_n$ is closed. Let $K_p \in \mathcal{A}_n$ tend to $K \in \mathcal{K}$. Let us fix $\delta \leq 1/n$; we want to prove that

$$P_\delta\left(\mathcal{N}(K)\right) \geq \delta^{-1/n}.$$

Since $K_p \in \mathcal{A}_n$ we have $P_\delta(\mathcal{N}(K_p)) \geq \delta^{-1/n}$. So there are $N \stackrel{\text{def}}{=} \lceil \delta^{-1/n} \rceil$ double normals $b_p^1, \ldots, b_p^N$ in K_p forming a δ-set. By extraction, one can assume the convergence of each sequence $\{b_p^i\}_p$ $(i \in \mathbb{N}_N)$ to some limit $b^i \in \mathcal{N}(K)$ (by Lemma 2). Obviously $\{b^i | i \in \mathbb{N}_N\}$ is a δ-set of double normals. So $P_\delta(\mathcal{N}(K)) \geq \delta^{-1/n}$, and $K \in \mathcal{A}_n$.

Clearly, if $\mathcal{N}(K')$ is finite for some $K' \in \mathcal{K}$ then K' does not belong to $\mathcal{A}_n$. Hence, by Lemmas 7 and 8, $\mathcal{A}_n$ has empty interior and $\mathcal{A}$ is meagre. □

Lemma 13. *For any $K \in \mathcal{K}$, any $(x, y) \in \mathcal{N}(K)$ and any $\varepsilon > 0$ there exist $K' \in \mathcal{K}$, $o \in \mathbb{E}$, $R > 0$ such that $d_{PH}(K, K') < \varepsilon$ and $\mathbb{S}(o, R) \cap \partial K'$ contains two spherical caps symmetrical to each other with respect to o, one of them included in $\mathbb{B}(x, \varepsilon)$.*

Proof. Let o be the midpoint of xy and Δ the open subset of $\mathbb{E}$ bounded by the two hyperplanes through x and y, normal to $x - y$.

We choose $R > \|x - y\|/2$ small enough to ensure that

$$K' \stackrel{\text{def}}{=} \operatorname{conv}(K \cup (\bar{\mathbb{B}}(o, R) \setminus \Delta))$$

satisfies $d_{PH}(K_0, K_1) < \varepsilon$. It remains to prove that a whole neighbourhood of the poles $p^+ \stackrel{\text{def}}{=} o + \frac{R}{\|o-x\|}(x - o)$ and $p^- \stackrel{\text{def}}{=} 2o - p$ in $\partial K'$ is included in $\mathbb{S}(o, R)$. Let $B^\pm$ be the connected component of $\bar{\mathbb{B}}(o, R) \setminus \Delta$ that contains $p^\pm$. Assume that there exist $p_n \in \mathbb{S}(o, R)$, tending to p^+, and interior to some line-segments $a_n b_n$ with $a_n \in B^+$ and $b_n \in \operatorname{conv}(K \cup B^-)$. Passing if necessary to a subsequence, we may assume that b_n converges to $b \in \operatorname{conv}(K \cup B^-)$. The hyperplane H_n through p_n and normal to $(x - y)$ separates a_n and b_n, and the connected component of $B^+ \setminus H_n$ containing p^+ tends to $\{p^+\}$, whence $a_n \to p^+$. Since $\|p_n - a_n\| \to 0$, $\measuredangle b_n a_n o \to \pi/2$. It follows that b should belong to the hyperplane through p^+ normal to $x - y$, and we get a contradiction.

Of course, the same proof holds for p^-. □

Lemma 14. *Let K be a convex body in $\mathbb{E}$ and $b_1, \ldots, b_n \in \mathcal{N}(K)$ be n double normals. Assume that each foot of b_i $(i \in \mathbb{N}_n)$ admits a neighbourhood in ∂K which does not contain any line-segment. Then there exists a sequence $K_p \in \mathcal{K}$ tending to K when p tends to ∞, such that $b_1, \ldots, b_n$ belong to $\mathcal{M}^S(K_p)$ for any p.*

Proof. Let u_i, v_i be the feet of b_i $(i = 1, \ldots, n)$, and consider the convex cone of revolution $C_{i,p}^+$ (respectively $C_{i,p}^-$) with apex u_i (respectively v_i), axis $\overline{u_i v_i}$, angle $\frac{\pi}{2} - \frac{1}{p}$ between the axis and the generatrices, and containing v_i (respectively u_i). Since K is locally strictly convex near u_i and v_i, the intersection K_p of K and all these cones tends to K when p tends to ∞ and clearly $b_i \in \mathcal{M}^S(K_p)$. □

Theorem 3. *For most $K \in \mathcal{K}$, we have $\dim_P \mathcal{F}(K) = d$.*

Proof. Let $\mathcal{U}$ be a countable base of open sets of $\mathbb{E}$. For $N \geq 1$ and $V \in \mathcal{U}$, we define

$$\Omega_{V,N} = \left\{ K \in \mathcal{K} \left| \begin{array}{l} \mathcal{F}(K) \cap V = \emptyset \\ \text{or} \\ \exists \delta \in \left]0, \frac{1}{N}\right[\text{ s.t. } \frac{\ln P_\delta(\mathcal{F}(K)\cap V)}{-\ln \delta} > d - \frac{1}{N} \end{array} \right. \right\}.$$

If for a given $V \in \mathcal{U}$, K lies in the intersection of all these $\Omega_{V,N}$, it satisfies $\overline{\dim}_P \mathcal{F}(K) \cap V = d$ whenever $\mathcal{F}(K) \cap V \neq \emptyset$, and it follows that $\dim_P \mathcal{F}(K) = d$ by Lemma 12. Thus we just have to check the density in $\mathcal{K}$ of $\mathring{\Omega}_{V,N}$.

Let $V \in \mathcal{U}$, $N \geq 1$, $K_0 \in \mathcal{K}$ and $\varepsilon > 0$; we look for some $K_3 \in \mathring{\Omega}_{V,N}$ such that $d_{PH}(K_0, K_3) < \varepsilon$. If K_0 belongs to $\mathring{\Omega}_{V,N}$ then the proof is over, otherwise there exists $K_1 \in \mathcal{K}$ such that $d_{PH}(K_0, K_1) < \varepsilon$ and $\mathcal{F}(K_1) \cap V$ contains at least one element x_1; let y_1 be the other foot of a double normal issued from x_1.

By Lemma 13, there exist $K_2 \in \mathcal{K}$, $R > 0$, $o \in K_2$ such that $d_{PH}(K_0, K_2) < \varepsilon$ and ∂K_2 contains two open subsets $U^{\pm}$ of $S \overset{\text{def}}{=} \mathbb{S}(o, R)$, image one to the other by the symmetry $\sigma : p \mapsto 2o - p$, and such that $U^+ \subset V$. Choose $x \in U^+$ and let $r > 0$ be small enough to ensure that $C^+ \overset{\text{def}}{=} \bar{\mathbb{B}}(x, r) \cap S \subset U^+$.

Since $\dim C^+ = d$, we can choose $0 < \delta < 1/N$ such that

$$\frac{\ln P_\delta(C^+)}{\ln 2 - \ln \delta} > d - \frac{1}{N}$$

and a δ-set $F \subset C^+$ with cardinality $P_\delta(C^+)$. Clearly $(x, \sigma(x)) \in \mathcal{N}(K_1)$ for $x \in F$. By Lemma 14, there exists $K_3 \in \mathcal{K}$ such that $d_{PH}(K_0, K_3) < \varepsilon$ and $(x, \sigma(x)) \in \mathcal{M}^S(K_3)$ for any $x \in F$.

By virtue of Lemma 5, there is a neighbourhood V of K_3 in $\mathcal{K}$ such that for any $K \in V$, and any $x \in F$, there exists a double normal $(\tilde{x}, \tilde{y}) \in \mathcal{N}(K)$ verifying $\widetilde{x} \in \mathbb{B}(x, \delta/4) \cap V$. From this we get that $P_{\delta/2}(\mathcal{F}(K) \cap V) \geq P_\delta(C^+)$ and thus $K \in \Omega_{V,N}$. Hence $K_3 \in \mathring{\Omega}_{V,N}$ and the proof is complete. □

Remark 4. The reader may ask why this theorem is stated for $\mathcal{F}$ instead of $\mathcal{N}$. As a matter of fact, obviously,

$$\dim_P \mathcal{F}(K) \leq \dim_P \mathcal{N}(K) \leq \dim_P \mathcal{D}(K).$$

Since this set is canonically one to one mapped (for a $\mathcal{C}^1$ strictly convex body K) on the unit sphere of $\mathbb{E}$, one may think that $\overline{\dim}_P \mathcal{D}(K) = d$, in which case Theorem 3 would hold for $\mathcal{N}$ as well. However, this bijection is not (known to be) regular enough to get any conclusion on the dimension of $\mathcal{D}(K)$.

For a smooth strictly convex body, there is a diametral map $\Delta_K : \partial K \to \partial K$ which associates to a point x the only point x' such that $(x, x') \in \mathcal{D}(K)$. Hence $\mathcal{D}(K) \subset \partial K \times \partial K$ is the graph of this map. However, this map is not necessarily Lipschitz continuous, or regular enough to carry any dimensional information. Indeed, K. Adiprasito and T. Zamfirescu proved that it behaves rather badly in the typical case, for it maps a set of full measure on a set of measure zero [1].

Nevertheless, if $d = 1$, an elementary argument of monotony of Δ_K shows that $\dim \mathcal{D}(K) = 1$ for any reasonable notion of dimension. See Lemma 16.

5. Critical values

This section focuses on the set of lengths of double normals. As seen earlier, double normals can be seen as critical points of the length function, so their lengths are critical values.

We first show that for typical $x \in \mathbb{E}^n$, positive distances $d(\langle x_I\rangle, \langle x_J\rangle)$ are pairwise distinct $(I,\, J \subset \mathbb{N}_n)$.

Lemma 15. *There is an open and dense set $U \subset \mathbb{E}^n$ such that for any $x \in U$ and for any four pairwise disjoint non-empty sets of indices $I,\, J,\, I',\, J' \subset \mathbb{N}_n$ of cardinality at most d, the distance between $\langle x_I\rangle$ and $\langle x_J\rangle$ and the distance between $\langle x_{I'}\rangle$ and $\langle x_{J'}\rangle$ are either distinct or both equal to* 0*.*

Proof. There is an open and dense set $V_0 \subset \mathbb{E}^n$ such that for any non-empty set of indices $I \subset \mathbb{N}_n$, $\dim \langle x_I\rangle = \#I - 1$. If $\#I + \#J > d + 3$ then $d(\langle x_I\rangle, \langle x_J\rangle) = 0$ for any $x \in V_0$. So, from now on, we assume implicitly that $\#I + \#J \leq d + 3$ and $\#I' + \#J' \leq d + 3$. Now, by Lemma 9, there is an open and dense set $V_1 \subset V_0$ such that for any disjoint sets of indices $I,\, J \subset \mathbb{N}_n$, $\overrightarrow{x_I} \cap \overrightarrow{x_J} = \{0\}$. Moreover, there exists a real valued rational function P_{IJ} on $\mathbb{E}^n$ whose restriction to V_1 satisfies $d(\langle x_I\rangle, \langle x_J\rangle)^2 = P_{IJ}(x)$. We have to prove that, given four pairwise disjoint sets of indices $I,\, J,\, I',\, J'$, the open set $U_{II'JJ'} = \{x \in V_1 | P_{IJ}(x) \neq P_{I'J'}(x)\}$ is dense in V_1. Since it is defined by polynomial inequalities, it is sufficient to prove that it is not empty. This latter fact is obvious because the sets of indices are disjoint. □

Theorem 4. *For most $K \in \mathcal{K}$, $\widetilde{\ell}_K : \widetilde{\mathcal{N}}(K) \to \mathbb{R}$ is injective.*

Proof. A set K has a non-injective function $\widetilde{\ell}_K$ if and only if there exist an integer n and two non-oriented double normals b_1, b_2 such that $d_{PH}(b_1, b_2) \geq \frac{1}{n}$ and $\ell(b_1) = \ell(b_2)$. For fixed n, the set $\mathcal{A}_n$ of such bodies is obviously closed in $\mathcal{K}$. Since a double normal realizes the distance between the affine spaces spanned by two disjoint faces (disjoint, because they lie in two parallel hyperplanes), by Lemma 15, there is a dense set of polytopes that does not intersect $\mathcal{A}_n$ and the proof is finished. □

Corollary 2. *For most $K \in \mathcal{K}$, $\mathcal{M}(K) = \mathcal{M}^S(K)$.*

Corollary 3. *For most $K \in \mathcal{K}$, $\mathcal{L}(K)$ is homeomorphic to the Cantor set and has lower box-counting dimension* 0*.*

Proof. By Theorem 1 and Remark 3, $\widetilde{\mathcal{N}}(K)$ is a Cantor set. Since, by Theorem 4, $\widetilde{\ell}:\widetilde{\mathcal{N}}(K)\to\mathbb{R}$ is injective, $\mathcal{L}(K)=\widetilde{\ell}(\widetilde{\mathcal{N}}(K))=\ell(\mathcal{N}(K))$ is also a Cantor set. Moreover, ℓ is Lipschitz continuous, whence, by Theorem 2,

$$\underline{\dim}_P\mathcal{L}(K)\leq\underline{\dim}_P\mathcal{N}(K)=0.\quad\square$$

Concerning the upper dimension, we get the following result.

Theorem 5. *For most $K\in\mathcal{K}$,*

- *if $d=1$, $\dim_P\mathcal{L}(K)=\frac{1}{2}$,*
- *if $d=2$, $\dim_P\mathcal{L}(K)\geq\frac{3}{4}$,*
- *if $d\geq 3$, $\dim_P\mathcal{L}(K)=1$.*

Remark 5. We conjecture that, in the case $d=2$, $\dim_P\mathcal{L}(K)$ cannot exceed 3/4 for any $K\in\mathcal{K}$. Obviously the conjecture implies the equality in Theorem 5.

The rest of the section is devoted to the proof and will be divided in several lemmas; the final compilation is postponed to the end of the section.

Lemma 16. *If $d=1$ and $K\in\mathcal{K}$ is $\mathcal{C}^1$ and strictly convex then $\overline{\dim}_B\mathcal{D}(K)=1$.*

Proof. Let $\Delta_K:\partial K\to\partial K$ be the function which associates to each point x the other extremity of the affine diameter starting at x. Thus $\mathcal{D}(K)$ is the graph of Δ_K.

It is easy to see that two distinct affine diameters of K always intersect inside K.

Thus Δ_K is locally monotone, in the following sense: for any homeomorphisms $\phi:[0,1]\to U\subset\partial K$, $\psi:[0,1]\to V\subset\partial K$ such that $x\in U$ and $\Delta_K(x)\in V$, $\psi^{-1}\circ\Delta_K\circ\phi$ is monotone. It follows that the dimension of the graph of Δ_K cannot exceed 1. $\square$

Let $\mathcal{V}$ be a basis of open sets of $\mathbb{R}$. For $V\in\mathcal{V}$, $\kappa>0$ and $N\in\mathbb{N}$, define

$$U^{\kappa}_{V,N}\stackrel{\text{def}}{=}\left\{K\in\mathcal{K}|\exists\delta\in\left]0,\frac{1}{N}\right[\text{ s.t. }\frac{\ln P_\delta(\ell(\mathcal{M}^S(K))\cap V)}{-\ln(\delta/2)}>\kappa-\frac{1}{N}\right\},$$
$$W_V\stackrel{\text{def}}{=}\{K\in\mathcal{K}|\mathcal{L}(K)\cap V=\emptyset\}.$$

Lemma 17. *For $d=1$, for all $V\in\mathcal{V}$ and $N>0$, $U^{1/2}_{V,N}$ is dense in $\mathcal{K}\setminus W_V$.*

Proof. Fix $K_0\in\mathcal{K}\setminus W_V$, $(x_0,y_0)\in\mathcal{N}(K)$ such that $\|x_0-y_0\|\in V$, and $\varepsilon>0$. By Lemma 13, there exists $K_1\in\mathcal{K}$ such that $d_{PH}(K_0,K_1)<\varepsilon$ and ∂K_1 contains two circle arcs $C^\pm$ sharing the same centre o, symmetrical to each other with respect to o, and such that the line x_0y_0 intersects ∂K_1 in two points $x\in C^+$ and $y\in C^-$. We may also assume that $2R\stackrel{\text{def}}{=}\|x-y\|\in V$. By considering even smaller arcs, one can assume without loss of generality that x is the midpoint of C^+; let a, b be its extremities. Put $\Theta = \measuredangle xoa$. Making if necessary C^+ even smaller, we may also assume without loss of generality that

$$\cos\theta \leq 1 - \theta^2/3$$

for any $\theta \in [0, \Theta]$. The lines tangent to C^+ at a and b intersect at some point c collinear with o and x. Note that the union K_2 of the triangle abc and K_1 is convex. Making if necessary C^+ even smaller, we may assume without loss of generality that $d_{PH}(K_0, K_2) < \varepsilon$. Let $u(\theta)$ be a unit vector such that $\measuredangle(u(\theta), a - o) = \theta$ and $Ru(\theta) \in C^+$. Now choose a positive integer n and define for $i = 0, \ldots, n$

$$\begin{aligned} \delta &= R\Theta^2/4n^2, \\ R_i &= R + i\delta, \\ v_i &= o + R_i u(i\Theta/n). \end{aligned}$$

Note that, for $i > 0$,

$$R_i \cos\frac{i\Theta}{n} < R\left(1 - \frac{\Theta^2}{12n^2}\right) < R,$$

whence all the v_i belong to the triangle abc. Let K_3 be the convex hull of K_2, the points v_i and their symmetrical points v_i' with respect to o. Since $K_1 \subset K_3 \subset K_2$ we have $d_{PH}(K_0, K_3) < \varepsilon$.

We claim that any triangle ov_iv_j with $1 \leq i < j \leq n$ is acute. Since $R_j > R_i$, it is clear that $\measuredangle ov_jv_i < \pi/2$. Moreover $\measuredangle ov_iv_j$ is acute if and only if $\frac{R_i}{R_j} > \cos\frac{(j-i)\Theta}{n}$. Now

$$\frac{R_i}{R_j} = \frac{R_j - (j-i)\delta}{R_j} = 1 - \frac{(j-i)\Theta^2}{4n^2}\frac{R}{R_j} \geq 1 - \frac{\Theta^2(j-i)}{4n^2}.$$

On the other hand

$$\cos\frac{(j-i)\Theta}{n} \leq 1 - \frac{(j-i)^2\Theta^2}{3n^2} \leq 1 - \frac{(j-i)\Theta^2}{3n^2},$$

and the claim is proven. Moreover, $\measuredangle ov_nb < \pi/2$ because $r_n > R$. It follows that $v_i \in \partial K_3$ and $(v_i, v_i') \in \mathcal{M}^S(K_3)$, $1 \leq i \leq n$. Obviously the set $\{\ell(v_i) | i \in \mathbb{N}_n\}$ is a δ-set; for n large enough, it is included in V. It follows that $P_\delta(\ell(\mathcal{M}^S(K_3)) \cap V) \geq n$. Since $\lim \frac{\ln n}{-\ln\delta} = \frac{1}{2}$, for n large enough, $K_3 \in U^{1/2}_{V,N}$. $\square$

Lemma 18. *For $d \geq 3$, for all $V \in \mathcal{V}$, $U^1_{V,N}$ is dense in $\mathcal{K} \setminus W_V$.*

Proof. Choose $K_0 \in \mathcal{K} \setminus W_V$, $b \in \mathcal{N}(K_0)$ such that $\ell_{K_0}(b) \in V$, and $\varepsilon > 0$. We have to prove that there exists $K \in U^1_{V,N}$ such that $d_{PH}(K_0, K) < \varepsilon$.

By "combination" of K_0 and the convex bodies $K^\star$ provided by Theorem A, one can find K_1 such that $d_{PH}(K_0, K_1) < \varepsilon$ and $\mathcal{L}(K_1) \cap V$ contains an interval $[a, a + 2\Delta]$, with $0 < \Delta < 1$. Here, the convex bodies are combined using the same construction as in the proof of Lemma 13 at a neighbourhood of b, replacing the sphere by a rescaled and displaced copy of $K^\star$.

Put $\delta_0 \stackrel{\text{def}}{=} \frac{2\Delta}{M}$, where M is chosen large enough to ensure that $\delta_0 < \frac{1}{N}$ and

$$\frac{\ln \Delta}{\ln \Delta - \ln M} < \frac{1}{N}.$$

Let $b_i \in \mathcal{N}(K_1)$ $(i = 0, \dots, M)$ be a double normal of length $a + i\delta_0$. By Lemma 14, one can find $K_2 \in \mathcal{K}$ such that $d_{PH}(K_0, K_2) < \varepsilon$ and $b_i \in \mathcal{M}^S(K_2)$. Now,

$$P_{\delta_0}(\ell(\mathcal{M}^S(K_2)) \cap V) \geq M = \frac{\Delta}{\delta_0/2},$$

whence

$$\frac{\ln(P_{\delta_0}(\ell(\mathcal{M}^S(K_2)) \cap V))}{-\ln(\delta_0/2)} \geq 1 - \frac{\ln \Delta}{\ln \Delta - \ln M} > 1 - \frac{1}{N}$$

by the choice of M. Hence K_2 belongs to $U^1_{V,N}$ and the proof is complete. □

The next technical lemma is needed for the case $d = 2$.

Lemma 19. *Consider the classical parametrization of the unit sphere*

$$\phi : (\lambda, \theta) \mapsto (\cos\lambda \cos\theta, \cos\lambda \sin\theta, \sin\lambda),$$

choose $R > 0$, $A > 0$, $T \in \left]0, \frac{\pi}{4}\right[$ and define for any natural integer m and any $(i, j) \in \mathbb{N}^0_{m^2} \times \mathbb{N}^0_m$

$$\delta = \frac{RT^2}{16m^4},$$
$$r_{ij} = R + A - \left(jm^2 + i\right)\delta,$$
$$v_{ij} = r_{ij}\phi\left(\frac{iT}{m^2}, \frac{jT}{m}\right).$$

Then, for m large enough, for any $(i, j) \neq (i', j') \in \mathbb{N}^0_{m^2} \times \mathbb{N}^0_m$,

$$\langle v_{i'j'}, v_{i'j'} - v_{ij} \rangle > 0.$$

Proof. Put

$$D = \langle v_{i'j'}, v_{i'j'} - v_{ij} \rangle .$$

Since for $i + m^2 j \le i' + m^2 j'$, $\|v_{i'j'}\| \le \|v_{i,j}\|$, it is sufficient to check the sign of D for $i' + m^2 j' > i + m^2 j$. A straightforward computation shows that

$$D = r'(r' - Pr),$$

with

$$\begin{aligned} r &= r_{ij}, \\ r' &= r_{i'j'}, \\ P &= \cos \frac{Ti}{m^2} \cos \frac{Ti'}{m^2} \cos \frac{T(j'-j)}{m} + \sin \frac{Ti}{m^2} \sin \frac{Ti'}{m^2}. \end{aligned}$$

Thus $D > 0$ if and only if

$$P < \frac{r'}{r}.$$

We claim that these inequalities hold for m large enough, for any $(i,j) \neq (i',j') \in \mathbb{N}^0_{m^2} \times \mathbb{N}^0_m$. Assume, on the contrary, that there exist sequences m_p, i_p, j_p, i'_p and j'_p such that $m_p \to \infty$, i/m_p^2, i'_p/m_p^2, j_p/m_p, $j'_p/m_p \in [0,1]$ and the corresponding value of D is non-positive. Extracting if necessary subsequences, one may assume without loss of generality that the four ratios are converging in $[0,1]$; denote by α, α', β and β' the respective limits of Ti_p/m_p^2, Ti'_p/m_p^2, Tj_p/m_p and Tj'_p/m_p. Then P converges to

$$\cos\alpha \cos\alpha' \cos(\beta' - \beta) + \sin\alpha \sin\alpha' \le 1,$$

with equality if and only if $\alpha' = \alpha$ and $\beta' = \beta$. On the other hand, r'/r tends to 1. It follows that, if $\alpha \neq \alpha'$ or $\beta \neq \beta'$, a contradiction is found. From now on, we assume $\alpha' = \alpha$ and $\beta' = \beta$. We now discuss two cases.

Case 1. There exist arbitrarily large indices p such that $j_p = j'_p$. By extracting suitable subsequences, we may assume without loss of generality that $j_p = j'_p$ (and so $i'_p > i_p$) for all p. Then, since $\frac{T\left(i'_p - i_p\right)}{m_p^2} \to \alpha' - \alpha = 0$, for m large enough,

$$P = \cos \frac{T\left(i'_p - i_p\right)}{m_p^2} < 1 - \frac{T^2\left(i'_p - i_p\right)^2}{4m_p^4} \le 1 - \frac{T^2}{4m_p^4}\left(i'_p - i_p\right).$$

On the other hand,

$$\frac{r'}{r} = \frac{r - \left(i'_p - i_p\right)\delta}{r} > 1 - \frac{T^2}{16m_p^4}\left(i'_p - i_p\right)$$

and we get a contradiction.

Case 2. For p large enough, $\hat{\jmath}_p \overset{\text{def}}{=} j'_p - j_p > 0$. By extracting suitable subsequences, we may assume without loss of generality that this inequality holds for all p. Define α_p, $\hat{\alpha}_p$ and $\hat{\beta}_p$ by $i_p = \alpha_p m_p^2/T$, $i'_p = (\alpha_p + \hat{\alpha}_p)\, m_p^2/T$ and $\hat{\jmath}_p = m_p \hat{\beta}_p$; then $\lim_{p\to\infty} \hat{\alpha}_p = \lim_{p\to\infty} \hat{\beta}_p = 0$ and by straightforward computations

$$
\begin{aligned}
P &= \sin 2\alpha_p \sin \hat{\alpha}_p \sin^2 \frac{\hat{\beta}_p}{2} + Q \cos \hat{\alpha}_p \le \frac{\hat{\beta}_p^2}{4} |\hat{\alpha}_p| + Q, \\
Q &= \frac{1}{2} \left((\cos 2\alpha_p + 1) \cos \hat{\beta}_p - \cos 2\alpha_p + 1 \right).
\end{aligned}
$$

Since $\hat{\beta}_p \to 0$, for p large enough $\cos \hat{\beta}_p < 1 - \hat{\beta}_p^2/3$ whence

$$
Q < 1 - \frac{\hat{\beta}_p^2}{6}.
$$

For p large enough, $|\hat{\alpha}_p| < \frac{2}{21}$, whence

$$
P < 1 - \frac{\hat{\beta}_p^2}{7} = 1 - \frac{\hat{\jmath}_p^2 T^2}{7 m_p^2} \le 1 - \frac{\hat{\jmath}_p T^2}{7 m_p^2}.
$$

On the other hand

$$
\begin{aligned}
\frac{r'}{r} &= \frac{r - \left(\hat{\jmath}_p m_p^2 + i'_p - i_p \right) \delta}{r} \\
&\ge 1 - (\hat{\jmath}_p + 1) \frac{T^2}{16 m_p^2} \ge 1 - \frac{\hat{\jmath}_p T^2}{8 m_p^2},
\end{aligned}
$$

and we get another contradiction. This completes the proof. □

Lemma 20. *For $d = 2$, for all $V \in \mathcal{V}$, $U_N^{3/4}$ is dense in $\mathcal{K} \setminus W_V$.*

Proof. Choose $K_0 \in \mathcal{K}$, $(x_0, y_0) \in \mathcal{N}(K_0)$ such that $\|x_0 - y_0\| \in V$, and $\varepsilon > 0$; we have to prove that there exists $K \in U_{V,N}^{3/4}$ such that $d_{PH}(K_0, K) < \varepsilon$. By Lemma 13, one can find a convex body K_1 whose distance from K_0 is less than ε and whose boundary contains two spherical caps, symmetrical to each other with respect to some point o. Let R be the radius of that sphere; we may assume that $2R \in V$. One can also assume, without loss of generality, that $o = (0, 0, 0)$ and that those caps are centered at equatorial points $\pm e = (\pm R, 0, 0)$. Denote by C the cap centered at e, and, for $A > 0$, by $\hat{C}$ the convex hull of $C \cup \{(R + 2A, 0, 0)\}$. For A sufficiently small, $K_2 = K_1 \cup \hat{C} \cup \left(-\hat{C}\right)$ is convex and $d_{PH}(K_0, K_2) < \varepsilon$. Let ϕ be a classical parametrization of the unit sphere:

$$
\phi(\lambda, \theta) = (\cos\lambda \cos\theta, \cos\lambda \sin\theta, \sin\lambda).
$$

Let $T > 0$ be small enough to ensure that $(R + A)\,\phi\,([0, T] \times [0, T])$ is included in the interior of $\hat{C} \setminus K_1$. For any positive integer m, and any $(i, j) \in \mathbb{N}^0_{m^2} \times \mathbb{N}^0_m$, define

$$\delta = \frac{RT^2}{16m^4},$$
$$r_{ij} = R + A - \left(jm^2 + i\right)\delta,$$
$$v_{ij} = r_{ij}\phi\left(\frac{iT}{m^2}, \frac{jT}{m}\right).$$

For m large enough, all the v_{ij} lie in $\hat{C}$ and $V \overset{\text{def}}{=} \{v_{i,j} | i \in \mathbb{N}^0_{m^2}, j \in \mathbb{N}^0_m\}$ is included in the interior of $\hat{C} \setminus K_1$. Let K_3 be the closed convex hull of V and K_1. Since $K_1 \subset K_3 \subset K_2$, $d_{PH}\,(K_0, K_3) < \varepsilon$. By Lemma 19, for m large enough

$$\langle v_{ij}, v_{ij} - v_{i'j'}\rangle > 0.$$

Moreover, for $c \in C$

$$\langle v_{ij}, v_{ij} - c\rangle > 0$$

because $\|v_{ij}\| = r_{ij} > R = \|c\|$. Thus for any point $p \neq v_{ij}$ in

$$G \overset{\text{def}}{=} \hat{C} \cap K_3 = \operatorname{conv}(C \cup V)\,,$$

the angle $\measuredangle o v_{ij} p$ is less than $\pi/2$. It follows that $v_{ij} \in \partial K_3$ and that $(v_{ij}, -v_{ij})$ are maximizing chords of K_3.

For m large enough, all the lengths of those chords belong to V, whence

$$P_\delta(\ell(\mathcal{M}^S(K) \cap V) \geq m^3$$

and

$$\frac{\ln P_\delta(\ell(\mathcal{M}^S(K) \cap V)}{-\ln(\delta/2)} \geq \frac{3\ln m}{4\ln m - \ln\frac{RT^2}{32}} \underset{m\to\infty}{\rightarrow} \frac{3}{4},$$

whence K_3 belongs to $U^{3/4}_{V,N}$ if m is large enough. This ends the proof. □

Proof of Theorem 5. By Theorem B, Lemma 16 and Lemma 7, $\overline{\dim}_B\,(K) \leq 1/2$ for $d = 1$. Clearly this dimension is bounded from above by 1 in any case. So we just have to prove that $\dim_P\,(\mathcal{L}(K)) \geq d^* \overset{\text{def}}{=} \min\left(1, \frac{1+d}{4}\right)$.

For $N \geq 1$ and $V \in \mathcal{V}$, define

$$\Omega_{V,N} \overset{\text{def}}{=} \left\{ K \in \mathcal{K} \,\middle|\, \begin{array}{l} \exists \delta \in \left]0, \frac{1}{N}\right[\text{ s.t.} \frac{\ln P_\delta(\mathcal{L}(K) \cap V)}{-\ln\delta} > d^* - \frac{1}{N} \\ \text{or} \\ \mathcal{L}(K) \cap V = \emptyset \end{array} \right\}.$$

If, for a fixed $V \in \mathcal{V}$, K lies in infinitely many $\Omega_{V,N}$ then $\overline{\dim}_B(K \cap V) \geq d^*$ whenever $\mathcal{L}(K) \cap V$ is not empty. It follows by Lemma 12 that $\dim_P \mathcal{F}(K) = d^*$, for any K lying in the intersection of all $\Omega_{V,N}$ with $V \in \mathcal{V}$, $N > 1$. Thus, we just have to check the density of the interior of $\Omega_{V,N}$ in $\mathcal{K}$.

If $K_0 \in U^{d^*}_{V,N}$, then there exist $\delta < \frac{1}{N}$ and $M \overset{\text{def}}{=} 1 + \left\lceil \left(\frac{\delta}{2}\right)^{-d^*+1/N} \right\rceil$ double normals $b_1, \ldots, b_M$ whose lengths form a δ-set. Hence, by Lemma 5, for K close enough to K_0, there exist M double normals of K whose lengths form a $\delta/2$-set, thus $P_{\delta/2}(K) \geq M$ and $K \in \Omega_{V,N}$. It follows that $U^{d^*}_{V,N}$ is included in the interior of $\Omega_{V,N}$.

By Lemmas 17, 20 and 18, $U^{d^*}_{V,N}$ is dense in $\mathcal{K} \setminus W_V$, whence, $U^{d^*}_{V,N} \cup \mathring{W}_V$ is dense in $\mathcal{K}$ and included in $\Omega_{V,N}$. □

6. Critical points

As Gruber showed in [11], a typical convex body K is not $\mathcal{C}^2$. It follows that the usual classification of critical points of ℓ_K according to the Hessian does not work. However, one can distinguish local maxima, local minima, and other critical points. Since the curvature (and so the Hessian) is typically undefined, it is unclear whether those other critical points look like saddles.

The first proposition is almost obvious.

Proposition 1. *For a strictly convex body $K \in \mathcal{K}$, ℓ_K has no other local minima than the degenerate chords (x, x), $x \in \partial K$.*

Proof. Let $c = (x_1, x_2)$ be a non-degenerate chord of K and H_i $(i = 1, 2)$ be a supporting hyperplane through x_i. There are unit vectors $u_i \in \overrightarrow{H_i}$ $(i = 1, 2)$ such that the map

$$f : t \mapsto \|(x_1 + tu_1) - (x_2 + tu_2)\|$$

is non-increasing on $[0, \varepsilon]$ for some positive number ε. If ε is small enough, for any $t \in [0, \varepsilon]$, the line-segment joining $x_1 + tu_1$ and $x_2 + tu_2$ intersects K. By choosing the right orientation, this intersection determines a chord $c(t)$ which tends to c when t tends to 0. Now, due to the strict convexity, for any $t \in [0, \varepsilon]$,

$$\ell_K(c(t)) < f(t) \leq f(0) = \ell_K(c),$$

and c cannot be a local minimum. □

Local maxima are not very numerous either.

Proposition 2. *For most convex bodies $K \in \mathcal{K}$, the set $\mathcal{M}(K)$ is at most countable.*

Proof. By Theorem 4, for most $K \in \mathcal{K}$, $\widetilde{\ell}_K : \widetilde{\mathcal{N}}(K) \to \mathbb{R}$ is injective. Let $\mathcal{W}_K$ be a countable base of open sets of $\mathcal{C}(K) \setminus \{(x, x) | x \in \partial K\}$ such that $(x, y) \in V \in \mathcal{W}_K$ implies

$(y, x) \notin V$. Let $\mathcal{W}'_K \subset \mathcal{W}_K$ be the subset of those V such that $\ell_K|_V$ admits a maximum, which is necessarily unique by the injectivity of $\widetilde{\ell}_K$. Then the map $\mathcal{W}'_K \to \mathcal{M}(K)$ mapping V to this maximum is surjective and the proof is complete. □

However, we have the following proposition.

Proposition 3. *For most $K \in \mathcal{K}$, $\mathcal{M}^S(K)$ is dense in $\mathcal{N}(K)$.*

Proof. Let $\mathcal{K}^S$ be the set of all convex bodies K such that $\mathcal{M}^S(K) = \mathcal{M}(K)$. By Corollary 2, $\mathcal{K}^S$ is residual in $\mathcal{K}$, so by Lemma 1, it is sufficient to prove the conclusion for most $K \in \mathcal{K}^S$. Let $\mathcal{U}^2$ be a countable basis of open sets of $\mathbb{E}^2$. For $U \in \mathcal{U}^2$, define

$$\Phi_U \stackrel{\text{def}}{=} \{K \in \mathcal{K}^S | \mathcal{N}(K) \cap \overline{U} = \emptyset\}$$

$$\Psi_U \stackrel{\text{def}}{=} \{K \in \mathcal{K}^S | \mathcal{M}(K) \cap U \neq \emptyset\}.$$

Those sets are open in $\mathcal{K}^S$ by Lemma 2 and Lemma 5 respectively. If K belongs to the G_δ-set $\bigcap_{U \in \mathcal{U}^2} (\Phi_U \cup \Psi_U)$, then $\mathcal{M}(K)$ is dense in $\mathcal{N}(K)$. Hence, it is sufficient to prove that $\Psi_U \cup \Phi_U$ is dense in $\mathcal{K}^S$. Choose $K_0 \in \mathcal{K}^S$ and a neighbourhood $\mathcal{O}$ of K_0 in $\mathcal{K}^S$. We have to find $K_3 \in (\Phi_U \cup \Psi_U) \cap \mathcal{O}$. First we choose a polytope $K_1 \in \mathcal{O}$ (by Corollary 1, all polytopes belong to $\mathcal{K}^S$). If $K_1 \in \Phi_U$ put $K_3 = K_1$ and the proof is finished; otherwise there exists a double normal of K_1 lying in $\overline{U}$. In this case, one can sightly dilate and move K_1 in order to obtain another polytope $K_2 \in \mathcal{O}$ admitting a double normal $(x, y) \in U$. For $\eta > 0$, define $x' \stackrel{\text{def}}{=} x + \eta(x - y)$, $y' \stackrel{\text{def}}{=} y + \eta(y - x)$ and $K_3 \stackrel{\text{def}}{=} \operatorname{conv}(K_2 \cup \{x', y'\})$. By Lemma 6, $(x', y') \in \mathcal{M}(K)$. If η is small enough, then K_3 still belongs to $\mathcal{O}$ and $(x', y') \in U$, whence $K_3 \in \Psi_U \cap \mathcal{O}$. □

Remark 6. In the case $d = 1$, if K is $\mathcal{C}^2$ the Hessian of ℓ_K at $b = (x, y) \in \mathcal{N}(K)$ is given by

$$\begin{pmatrix} \frac{1}{w} - \gamma_x & \frac{1}{w} \\ \frac{1}{w} & \frac{1}{w} - \gamma_y \end{pmatrix},$$

where γ_u is the curvature of ∂K at $u = x, y$ and $w = \|x - y\|$. Hence the Hessian degenerates when

$$\frac{1}{\gamma_x} + \frac{1}{\gamma_y} = w.$$

So, the index of a double normal seen as a critical point appears to be closely related to the curvature of ∂K at its feet. This contributes to the motivation for the following section. See also [16], [17], [18].

7. Curvature at feet of double normals

This section brings some light on the curvature aspect of most convex surfaces, at the endpoints of their double normals.

Consider a smooth, strictly convex body K and a point x on its boundary ∂K; the outer normal unit vector of ∂K at x is denoted by ν_x. If τ is a vector that is not collinear to ν_x, H_x^τ stands for the 2-dimensional open half-plane whose boundary line is $x + \mathbb{R}\nu_x$ and which contains $x + \tau$. For any point $z \in \partial K \setminus \{x\}$, there is exactly one circle with its centre on $x + \mathbb{R}\nu_x$ and containing both x and z. Let $r_x(z)$ be the radius of this circle. Then, if τ is a unit vector tangent to ∂K at x,

$$\rho_i^\tau(x) = \liminf_{\substack{z \to x \\ z \in H_x^\tau \cap \partial K}} r_x(z)$$

is called the *lower curvature radius* at x in direction τ. Analogously is defined the *upper curvature radius* $\rho_s^\tau(x)$. Also, $\gamma_i^\tau(x) = \rho_s^\tau(x)^{-1}$ and $\gamma_s^\tau(x) = \rho_i^\tau(x)^{-1}$ are the *lower* and *upper curvature* at x in direction τ. (See [7], p. 14.)

For distinct $x, y \in \mathbb{E}$, let $C_{xy} = \mathbb{S}(x, \|x - y\|)$ be the sphere of centre x passing through y.

Lemma 21. *For any maximizing chord c of a convex body, we have*

$$\gamma_i^\tau(x) \geq \ell(c)^{-1}$$

at each foot x of c, and in each tangent direction τ at x.

Proof. Let $c = xx^*$, and assume

$$\gamma_i^\tau(x) < \ell(c)^{-1};$$

then there exists a sequence of points $\{x_n\}_{n=1}^\infty$ in ∂K converging to x, such that

$$\|x_n - x^*\| > \ell(c).$$

But this obviously contradicts the hypothesis asking for c to be maximizing. □

Theorem 6. *For most convex bodies K and any maximizing chord c of K,*

$$\gamma_i^\tau(x) \geq \ell(c)^{-1} \quad \text{and} \quad \gamma_s^\tau(x) = \infty$$

at each foot x of c, and in each tangent direction τ at x.

Proof. By Theorem B, most convex bodies are smooth; so, one can speak of tangent directions at boundary points. By Theorem C, for most convex bodies K and any point $x \in \partial K$, we have

$$\gamma_i^\tau(x) = 0 \text{ or } \gamma_s^\tau(x) = \infty$$

in each tangent direction τ.

Since, by Lemma 21, we have

$$\gamma_i^\tau(x) \neq 0$$

for every foot x of a maximizing chord, and every tangent direction τ, the theorem follows. □

A chord c which is longest among all chords of $C \in \mathcal{K}$ is called a *metric diameter* of C. The next result strengthens Theorem 6 in the case of the metric diameter and improves Theorem 11 in [36].

Theorem 7. *Most convex bodies admit a single metric diameter c,*

$$\gamma_i^\tau(x) = \ell(c)^{-1} \quad \textit{and} \quad \gamma_s^\tau(x) = \infty$$

at each foot x of c, and in each tangent direction τ at x.

Proof. A direction or a line-segment or a hyperplane will be called *horizontal*, respectively *vertical*, if it is parallel, respectively orthogonal, to a fixed hyperplane.

By Theorem 11 in [36], most convex bodies have a single metric diameter. As the set of all convex bodies having a horizontal diameter is obviously nowhere dense, the space $\mathcal{K}'$ of all convex bodies with a single non-horizontal diameter is residual in $\mathcal{K}$, and we apply Lemma 1 to obtain generic results in $\mathcal{K}$, working in $\mathcal{K}'$.

Let xx^* be the metric diameter of $C \in \mathcal{K}'$, such that x is above and x^* below any horizontal hyperplane cutting xx^*, and let the direction τ be orthogonal to $\overline{xx^*}$. Take the points $x_n^* \in xx^*$ such that $\|x^* - x_n^*\| = 1/n$ ($n \in \mathbb{N}$, $n > \lceil 1/d(x, x^*) \rceil$), and consider the half-plane Π with xx^* on its relative boundary and $x + \tau \in \Pi$.

Let $A_n(\tau) \subset \Pi$ be the arc starting at x, of length $1/n$, of the circle $C_{x_n^* x}$ of centre x_n^* passing through x. The radius is $\operatorname{diam}(C) - 1/n$.

Let us say that $C \in \mathcal{K}'$ has the (n)-property if for its metric diameter xx^* and for some direction τ orthogonal to xx^*, $A_n(\tau)$ does not meet $\mathring{C}$.

We prove that the set $\mathcal{K}'_n$ of those $C \in \mathcal{K}'$ which enjoy the (n)-property is nowhere dense in $\mathcal{K}'$.

First, it is easily seen that each $\mathcal{K}'_n$ is closed in $\mathcal{K}'$. Then, let $C \in \mathcal{K}'$. Approximate it by a polytope P having as metric diameter xx^*. Choose $\varepsilon > 0$ very small (compared with $1/n$). Consider the $(d-1)$-sphere S with $\langle S \rangle$ orthogonal to xx^*, having its centre on xx^*, lying between $C_{x_n^* x}$ and $C_{x^* x}$, and satisfying $\operatorname{diam}(S) = \varepsilon$. Let the polytope P'' approximate $\operatorname{conv}(S)$ in $\langle S \rangle$, with $d_{PH}(P'', S)$ much smaller than ε.

Then, $P' = \operatorname{conv}(P \cup P'')$ has not the (n)-property, whence $\mathcal{K}'_n$ is nowhere dense. In conclusion, most $C \in \mathcal{K}'$ have the (n)-property for no natural number n. This means that, for every tangent direction τ at x,

$$\rho_s^\tau(x) > \mathrm{diam}(C) - 1/n$$

for infinitely many n's, yielding $\rho_s^\tau(x) = \mathrm{diam}(C)$.

Analogously, $\rho_s^\tau(x^*) = \mathrm{diam}(C)$.

By Theorem 1 in [28], for most $C \in \mathcal{K}$, at every point $z \in \partial C$ and for every direction τ at z, $\rho_i^\tau(z) = 0$ or $\rho_s^\tau(z) = \infty$. It follows that, at the endpoints x, x^* of the unique metric diameter and for any direction τ, $\rho_i^\tau(x) = \rho_i^\tau(x^*) = 0$. □

The above theorems describe the curvature at the feet of maximizing chords. However, as shown by Proposition 2, maximizing chords are rare among double normals. Concerning typical double normals we have the following result.

Theorem 8. *For most $K \in \mathcal{K}$ and most $x \in \mathcal{F}(K)$, in any tangent direction τ, $\gamma_s^\tau(x) = \infty$.*

Proof. Rephrasing the second point of Theorem C, we get that for most $K \in \mathcal{K}$ the set

$$\mathcal{I} = \{x \in \partial K | \gamma_s^\tau(x) = \infty \text{ in any tangent direction } \tau\}$$

contains a dense G_δ set in ∂K. Indeed, this set *is* a G_δ set, as a thorough examination of the proof in the original paper [29] would show. For the reader's convenience we reprove this fact. Assume that K is of class $\mathcal{C}^1$ and strictly convex. Let T_x be the set of unit vectors τ normal to ν_x.

$$\begin{aligned}\partial K \setminus \mathcal{I} &= \left\{ x \in \partial K \middle| \exists \tau \in T_x,\ \liminf_{\substack{z \to x \\ z \in \partial K \cap H_x^\tau}} r_x(z) > 0 \right\} \\ &= \bigcup_{n \in \mathbb{N}} \left\{ x \in \partial K \middle| \exists \tau \in T_x, \forall z \in \partial K \cap H_x^\tau \cap \bar{\mathbb{B}}\left(x, \frac{1}{n}\right) : r_x(z) \geq \frac{1}{n} \right\} \\ &\overset{\text{def}}{=} \bigcup_{n \in \mathbb{N}} F_n.\end{aligned}$$

Now, we prove that F_n is closed for n large enough. We assume that $\frac{1}{n} < \min\{d(x, y) | x, y \in \partial K, \nu_x \in T_y\}$. Let $x_p \in F_n$ converge to $x \in \partial K$. By the definition of F_n, there exists $\tau_p \in T_{x_p}$ such that for any $z \in \partial K \cap H_{x_p}^{\tau_p} \cap \bar{\mathbb{B}}\left(x_p, \frac{1}{n}\right)$, $r_{x_p}(z) \geq \frac{1}{n}$. Passing if necessary to a subsequence, one may assume that τ_p is converging to a unit vector τ. Since K is $\mathcal{C}^1$, $\tau \in T_x$. Choose $z \in \bar{\mathbb{B}}\left(x, \frac{1}{n}\right) \cap \partial K \cap H_x^\tau$. It is easy to see that, for p large enough, there exists a unique point z_p in $H_{x_p}^{\tau_p} \cap \mathbb{S}(x_p, \|z - x\|) \cap \partial K$, and moreover, z_p converges to z. By the choice of τ_p and by the definition of F_n, $r_{x_p}(z_p) \geq \frac{1}{n}$, thus, since r is continuous with respect to x and z, $r_x(z) \geq \frac{1}{n}$. This holds for any $z \in \partial K \cap H_x^\tau \cap \bar{\mathbb{B}}\left(x, \frac{1}{n}\right)$, so $x \in F_n$.

It follows that $\mathcal{I} \cap \mathcal{F}(K)$ is a G_δ set in $\mathcal{F}(K)$, which contains, by Theorem 6, all feet of maximizing chords of K. Now, by Proposition 3, the set of those feet is dense in $\mathcal{F}(K)$, whence the conclusion. □

Remark 7. We still ignore, for typical convex bodies, the behaviour of the lower curvature at the feet of (most) double normals. The existence of double normals with finite upper curvature at a foot is also unknown; however, these curvatures cannot be finite at both feet of the same double normal (see Theorem 4.1 in [1]).

Let us consider a typical convex body among those that admit a given line-segment as double normal.

Theorem 9. *For most convex bodies admitting the double normal c,*

$$\gamma_i^\tau(x) = 0 \quad \textit{and} \quad \gamma_s^\tau(x) = \infty$$

at each foot x of c, and in each tangent direction τ at x.

Theorem 9 shows that the curvature behaviour at the endpoints of c coincides with the curvature behaviour at most points. (See [29] for the latter result in $\mathcal{K}$; the result is also valid in the space $\mathcal{K}''$ defined below, and the proof parallels that for $\mathcal{K}$.)

Proof. Let $\mathcal{K}''$ be the Baire space of all convex bodies admitting c as a double normal. We may assume that $c = xx^*$ is vertical, with x above x^*.

Following the same steps as in the proofs of Klee [14] or Gruber [11], one can show that most $C \in \mathcal{K}''$ are smooth (boundary of class $\mathcal{C}^1$). This justifies the use of "tangent directions" at x.

Let the direction τ be orthogonal to $\overline{xx^*}$. Consider the points $x_n \in \overline{xx^*}$, $x'_n \in xx^*$, such that $x \notin x^*x_n$ and $\|x - x_n\| = \|x - x'_n\|^{-1} = n$. Take the half-plane Π with xx^* on its boundary and $x + \tau \in \Pi$.

Let $A_n(\tau) \subset \Pi$, $A'_n(\tau) \subset \Pi$ be the arcs starting in x, of length $1/n$, of the circle $C_{x_n x}$, respectively $C_{x'_n x}$. The radii are n and $1/n$, respectively.

We now say that $C \in \mathcal{K}''$ has the (n)-property if, for some horizontal direction τ, $A_n(\tau) \cap \mathring{C}\mathrm{int}C = \emptyset$ or $A'_n(\tau) \subset C$.

We prove that the set $\mathcal{K}''_n$ of those $C \in \mathcal{K}''$ which enjoy the (n)-property is nowhere dense in $\mathcal{K}''$.

Again, it is easily checked that each $\mathcal{K}''_n$ is closed in $\mathcal{K}''$. Approximate $C \in \mathcal{K}''$ by a polytope P with vertices x, x^* such that ∂P has no horizontal direction at x or x^*. We now use the polytope P' constructed in the proof of Theorem 7. This polytope has not the (n)-property, whence $\mathcal{K}''_n$ is nowhere dense. Hence, most $C \in \mathcal{K}''$ have the (n)-property for no natural number n. Thus, for every tangent direction τ at x,

$$\rho_s^\tau(x) > n \ \text{ and } \ \rho_i^\tau(x) < 1/n$$

for infinitely many n's, i.e. $\rho_s^\tau(x) = \infty$ and $\rho_i^\tau(x) = 0$.

Analogously, $\rho_s^\tau(x^*) = \infty$ and $\rho_i^\tau(x^*) = 0$. □

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