Poidge-convexity

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Abstract

We investigate in this paper the poidge-convexity, which is a generalization of right convexity, introduced by one of the present authors in 2014. Although not every convex body is poidge-convex, there are many families of compact sets, some of them very different from convex bodies, which are poidge-convex. We present here several such families.

Keywords: poidge-convexity; starshaped sets; topological discs; polyhedral surfaces.

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1 Introduction

At the 1974 meeting about convexity in Oberwolfach, the third author proposed the investigation of the following very general kind of convexity. Let $\mathcal{F}$ be a family of sets in a space $\mathcal{X}$. A set $M \subset \mathcal{X}$ is called $\mathcal{F}$-convex if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$. In this paper, $\mathcal{X}$ will be $\mathbb{R}^n$; we always assume $n \geq 2$. Usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are just some examples of $\mathcal{F}$-convexity (for suitably chosen families $\mathcal{F}$).

Blind, Valette and the third author [1], and also Böröczky Jr. [2], investigated the rectangular convexity, the case when $\mathcal{F}$ is the family of all non-degenerate rectangles.

Bruckner and Bruckner [7], and also Magazanik and Perles [11] investigated $L_n$ sets, which are $\mathcal{F}$-convex sets, $\mathcal{F}$ consisting of all polygonal paths with at most $n$ edges in the plane. Magazanik and Perles [10] and Breen [3, 4, 5, 6] dealt with staircase connectedness, which is also a kind of $\mathcal{F}$-convexity, $\mathcal{F}$ being the family of all staircases. The third author studied the right convexity [18] (the case with $\mathcal{F}$ consisting of all right triangles), and the last two authors, generalizing it, investigated the $rt$-convexity, i.e. the right triple convexity [12, 13].

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A set $M \subset \mathbb{R}^n$ is called *rt-convex* if it is $\mathcal{F}$-convex, $\mathcal{F}$ being the family of all triples $\{x, y, z\} \subset \mathbb{R}^n$ with $\angle xyz = \pi/2$. If, for $x, y \in M$, there exists $z \in M$ such that the triangle $xyz$ is right, we say that the pair $x, y$ has the rt-property.

A set $M$ will be called *poidge-convex* if it is $\mathcal{F}$-convex, $\mathcal{F}$ being the family of all unions $\{x\} \cup \sigma$, called *poidges*, where $x$ is a point, $\sigma$ a line-segment, and $\text{conv}(\{x\} \cup \sigma)$ a right triangle (see Figure 1). If, for points $u, v \in M$, there is a poidge containing them included in $M$, we say that the pair $u, v$ has the poidge-property.

![Figure 1: Poidges](image)

For convex sets, right convexity, rt-convexity and poidge-convexity are equivalent. Obviously, in general, not every rt-convex set is poidge-convex, but every rightly convex set is both rt-convex and poidge-convex. Less obvious, but true, is that not every poidge-convex set is rt-convex either.

The poidge-convexity also generalizes the thin rectangular convexity (for short *tr-convexity*, where $\mathcal{F}$ is the family of all boundaries of non-degenerated rectangles), and a fortiori the rectangular convexity itself.

In this paper, we start the investigation of poidge-convexity. Our main goal will be to identify large classes of poidge-convex sets. On the one hand, not all convex bodies are poidge-convex: take, for example, the convex hull of an ellipse different from a circle. On the other hand, sets looking quite different from being convex, like meandering topological discs or single-point-kernel starshaped sets, can be found among the poidge-convex sets.

## 2 Notation

For distinct $x, y \in \mathbb{R}^n$, let $xy$ denote the line-segment from $x$ to $y$, and $\overline{xy}$ the line through $x, y$.

For any compact set $M \subset \mathbb{R}^n$, let $S_M$ be the smallest hypersphere containing $M$ in its convex hull; also, $\overline{M}$ means the affine hull of $M$, $\text{int} M$ the relative interior of $M$ (i.e., in the topology of $\overline{M}$) and $\text{bd} M$ the relative boundary of $M$.

By $\text{dim } M$ we denote the Hausdorff dimension of $M$. Also, denote by $\mathcal{P}_A(x)$ the orthogonal projection of $x$ onto the affine subspace $A$. The distance from a point $x$ to a compact set $M$ will be denoted by $\rho(x, M) = \min\{\|x - y\| : y \in M\}$. 

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Furthermore, for distinct \( x, y \in \mathbb{R}^n \), we denote by \( H_{xy} \) the hyperplane through \( x \) orthogonal to \( xy \), by \( H^+_{xy} \) the closed half-space containing \( y \) determined by \( H_{xy} \), and by \( H^-_{xy} \) the other closed half-space determined by \( H_{xy} \).

We denote by \( [xy) \) the half-line starting at \( x \) and passing through \( y \). Also, we set \( xky = \text{conv} ([kx) \cup [ky]) \), if \( k \notin xy \).

The compact ball of centre \( \omega \) and radius \( r \) is denoted by \( B_r(\omega) \). \( 0 \) is the origin of \( \mathbb{R}^n \).

In a Baire space, we say that most of its elements have property \( P \) if those not enjoying \( P \) form a set of first Baire category.

3 Starshaped sets

3.1 One-sided starshaped sets

Let \( S \) be the Baire space of all starshaped sets in \( \mathbb{R}^n \), always considered compact here. For \( M \in S \), a line-segment \( kx \subset M \) is called a ray if \( k \in \ker M \) and \( kx \) cannot be extended beyond \( x \) in \( M \).

It is known (see the corollary to Theorem 1 in [17]) that most starshaped sets have a single-point kernel. Moreover, in \( S \), the set of those starshaped sets \( M \) with \( \text{card}(M \cap S_M) = 2 \) is nowhere dense. For a proof, the reader can adapt the first part of the proof of Theorem 1 in [16], which deals with convex curves instead of starshaped sets.

More generally, we call any compact set \( M \) ordinary if \( \text{card}(M \cap S_M) \geq 3 \).

Clearly, being ordinary is a necessary condition for any compact set to be poidge-convex.

Let \( M \subset \mathbb{R}^2 \) be a starshaped set with more than one point, and let \( k \in \ker M \). The set \( M \) will be called one-sided, if there exist half-lines \( [kx) \) and \( [ky) \) such that

(i) \( M \subset xky \), or

(ii) \( [kx) \) and \( [ky) \) are opposite, i.e. \( [kx) \cup [ky) \) is a line, and \( M \) is included in one of the two half-planes with boundary \( [kx) \cup [ky) \).

Now, taking \( \angle xky \) to be minimal over all \( x, y \in \mathbb{R}^2 \setminus \{k\} \) and \( k \in \ker M \) in case (i), we call \( \omega(M) = \angle xky \) the opening of \( M \). In case (ii), put \( \omega(M) = \pi \). Thus, \( 0 \leq \omega \leq \pi \). (Note that \( M \neq \{k\} \) because \( \text{card} M > 1 \).) See Figure 2.

We call \( M \) acute, if there exists no non-degenerate triangle \( kxy \subset M \), where \( kx \) is a ray of \( M \).

**Theorem 3.1.** Let \( M \subset \mathbb{R}^2 \) be an ordinary one-sided starshaped set. If \( 0 < \omega(M) \leq \frac{\pi}{2} \), then \( M \) is poidge-convex. If \( M \) is acute and \( \frac{\pi}{2} < \omega(M) < \pi \), then \( M \) is not poidge-convex.
Proof. Take the point $k$ from the definition of the opening to be $\{0\}$. Assume $0 < \omega(M) \leq \frac{\pi}{2}$. Let $x, y \in M \setminus \{0\}$, with $\|x\| \leq \|y\|$. If $x, y$ are linearly independent, consider $z = \frac{y}{x}(x)$. Then $\{x\} \cup zy \subset M$. If $x, y$ are linearly dependent, there exists $u \in M \setminus \{0\}$, since $\omega(M) > 0$. Consider a point $v \in 0u$ close to $0$ and $w = \frac{y}{v}(v)$. Then $\{x\} \cup zw \subset M$. It remains to consider the pairs $(0, x) \in M$. Let $0y$ be the ray containing $x$. From $\text{card}(M \cap S_M) \geq 3$ it follows that $M$ has a point $z' \notin \text{int conv } S_{0y}$ different from $0$ and $y$. One possibility is that $\angle z'0y < \frac{\pi}{2}$. Then $0z' \cap S_{0y} \neq \{0, y\}$. Take $z \in 0z' \cap S_{0y} \setminus \{0, y\}$; possibly $z = z'$. Obviously, the points $0, x$ belong to the poidge $\{z\} \cup 0y \subset M$. The other possibility is that $\angle z'0y = \frac{\pi}{2}$. In this case, $0, x$ belong to the poidge $\{z'\} \cup 0y \subset M$.

Now assume $0 < \omega(M) < \pi$. There exist two rays $0x, 0y$ of $M$ such that $\frac{\pi}{2} < \angle x0y < \pi$. We claim that $x, y$ do not enjoy the poidge-property. Since every line-segment in $M$ starting at $x$ or $y$ must be collinear with $0$, otherwise $M$ would not be acute, the only way for $x, y$ to have the poidge-property would be to belong to a poidge with its line-segment $\sigma$ along one of the rays $0x, 0y$. But since $\angle x0y > \frac{\pi}{2}$, $\sigma$ would contain $0$ in its relative interior. This contradicts the inequality $\omega(M) < \pi$. \hfill \qed

It is easily seen that, for $\omega(M) = 0$, $M$ is not poidge-convex and, for $\omega(M) = \pi$, both poidge-convexity and non-poidge-convexity are possible.

We consider the family $\mathcal{S}^\kappa$ of all one-sided $M \in \mathcal{S}$ with bounded opening $\omega(M) \leq \kappa$. This family is closed in $\mathcal{S}$, and is therefore itself complete and consequently a Baire space.

**Theorem 3.2.** For $0 < \kappa \leq \pi/2$, most sets in $\mathcal{S}^\kappa$ are poidge-convex.

For $\pi/2 < \kappa < \pi$, most sets in $\mathcal{S}^\kappa$ are not poidge-convex.

**Proof.** For any $\kappa \in [0, \pi]$, most sets in $\mathcal{S}^\kappa$ are ordinary. The proof parallels the one for $\mathcal{S}$ or for the space of all convex curves instead of $\mathcal{S}^\kappa$. Moreover, the sets $M \in \mathcal{S}^\kappa$ with $\omega(M) = 0$
form a nowhere dense family. Hence, by the first part of Theorem 3.1, most sets in \( S^\kappa \) are poidge-convex.

For any \( \kappa < \pi \), most sets in \( S^\kappa \) are nowhere dense (the proof works like in the case of \( S \) instead of \( S^\kappa \), see the corollary to Theorem 1 in [17]), hence acute. By the second part of Theorem 3.1, most sets in \( S^\kappa \) are not poidge-convex. \( \square \)

### 3.2 Symmetric starshaped sets

Clearly, the space \( S^* \) of all (point) symmetric starshaped sets in \( \mathbb{R}^n \) is closed in \( S \), so it is itself a Baire space.

Like most starshaped sets, most symmetric starshaped sets have a single-point kernel.

If a starshaped set is symmetric, then its kernel is also symmetric, with the same centre of symmetry, say \( 0 \). Being convex, the kernel contains \( 0 \).

**Theorem 3.3.** A set \( M \in S^* \) is poidge-convex if and only if it is ordinary.

**Proof.** Suppose \( M \in S^* \) is ordinary. Take \( x, y \in M \setminus \{0\} \), assuming without loss of generality that \( \|x\| \leq \|y\| \). Suppose first that \( x, y \) are linearly independent. Taking \( z = \frac{1}{y}(x) \), we have \( \{x, y\} \subset \{x\} \cup zy \subset M \). If \( x, y \) are linearly dependent, then \( x \in y(\neg y) \). If \( S_{y(\neg y)} \cap M = \{y, -y\} \), then \( M \) is not ordinary. Hence, \( S_{y(\neg y)} \cap M \) contains some point \( z \notin \{y, -y\} \), and \( x, y \) belong to the poidge \( \{z\} \cup y(\neg y) \).

Suppose now that \( M \) is not ordinary, whence it has a single diameter \( x(-x) \). There is no point \( z \in S_{x(-x)} \cap M \setminus \{x, -x\} \). Moreover, the hyperplanes \( H_{x(-x)} \) and \( H_{(-x)x} \) meet \( M \) in \( x, -x \) only. So, \( x, -x \) don’t belong to any poidge in \( M \). \( \square \)

The next result has a corresponding one for convex bodies (with a similar proof), and uncovers a major discrepancy in comparison with the non-symmetric case.

**Theorem 3.4.** Most sets belonging to \( S^* \) are not ordinary.

**Proof.** Let \( S^*_m \) be the set of all \( M \in S^* \) having a pair of diameters at Pompeiu-Hausdorff distance at least \( 1/m \) from each other. We prove that \( S^*_m \) is nowhere dense in \( S^* \).

Let \( O \subset S^* \) be open and \( M \in O \). Choose a diameter of \( M \). This is also a diameter of \( S_M \). Extend it in both directions by \( \varepsilon > 0 \) to a line-segment \( \Delta_\varepsilon \). Then \( M_\varepsilon = M \cup \Delta_\varepsilon \) belongs to \( S^* \) and has the unique diameter \( \Delta_\varepsilon \). If \( \varepsilon \) is small enough, \( M_\varepsilon \in O \), too. There exists an open neighbourhood \( N \) of \( M_\varepsilon \) in \( O \) such that every set belonging to \( N \) has all its diameters at distance less than \( 1/2m \) from \( \Delta_\varepsilon \). This shows that they are not in \( S^*_m \). Thus, \( S^*_m \) is nowhere dense, and the set \( \cup_{m=1}^\infty S^*_m \) of all \( M \in S^* \) having more than one diameter is of first Baire category. \( \square \)

**Corollary 3.5.** Most sets belonging to \( S^* \) are not poidge-convex.
3.3 Starshaped sets close to convex

We introduce a measure of non-convexity for non-convex starshaped sets, which might be of independent interest. Line-segments $xy$ joining $x, y \in \text{bd } M$ are called chords of the set $M \subset \mathbb{R}^n$. Let $k$ belong to the kernel $\ker M$ of $M \in \mathcal{S}$. Let $\Gamma(M)$ be the set of chords $xy$ of $M$ not included in $M$. Let

$$\xi(M) = \sup_{xy \in \Gamma(M)} \inf_{k \in \ker M} \angle xky.$$  

See Figure 3. If the sequence $\{M_m\}_{m=1}^\infty$ of starshaped sets converges, and $\xi(M_m) \to 0$, then the limit set is convex. Thus, $\xi$ indicates how close to being convex a starshaped set is. We shall see that starshaped sets “closer” to convex ones, i.e. with smaller value of $\xi$, are more likely to be poidge-convex.

![Figure 3: The function $\xi$.](image)

**Theorem 3.6.** If $M \subset \mathbb{R}^2$, $M \in \mathcal{S}$, $M$ is ordinary, $\dim \ker M > 1$, and $\xi(M) < \frac{\pi}{2}$, then $M$ is poidge-convex.

**Proof.** Let $x, y \in M$.

**Case 1.** $xy \subset M$.

Being ordinary, $M$ must have a point $z \notin (\text{int conv } S_{xy}) \cup \{x, y\}$.

If $z \in S_{xy}$, then $\{z\} \cup xy$ is a suitable poidge. If $z \in H_{xy} \cup H_{yx}$, then again $\{z\} \cup xy$ is a suitable poidge. So, let $z$ lie in one of the four components of $\mathbb{R}^2 \setminus (H_{xy} \cup H_{yx} \cup \text{conv } S_{xy})$, namely $H_{xy}^-, H_{yx}^-, E_1, E_2$.

Take $k \in (\ker M) \setminus (xy \cup yz \cup yz)$. This is possible, because $\dim \ker M > 1$.

If $k \in (H_{xy} \cup H_{yx} \cup S_{xy})$, we are done.

If $k \in E_1 \cup E_2$, then $kx \cap S_{xy} \neq \emptyset$, and we are done again.
If \( k \in H_{xy}^- \), then \( ky \) meets \( H_{xy} \) and the intersection point is not \( x \). For \( k \in H_{yx}^- \), we have an analogous situation. So, assume \( k \) lies in \( \text{int conv} \ S_{xy} \).

Now, we consider again the positions that \( z \) may take.

If \( z \in H_{xy}^- \), then \( kz \cap H_{xy} \neq \emptyset \) and the intersection point is not \( x \).

If \( z \in H_{yx}^- \), then \( kz \cap H_{yx} \neq \emptyset \) and the intersection point is not \( y \).

Finally, if \( z \in E_1 \cup E_2 \), then \( kz \cap S_{xy} \neq \emptyset \).

Thus, in all cases a poidge in \( M \) including \( xy \) can be found.

**Case 2.** \( xy \not\subset M \).

Extend \( xy \) until we obtain a chord \( x'y' \in \Gamma(M) \). Since \( \inf_{k \in \ker M} \angle x'ky' \leq \xi(M) < \pi/2 \), we find a point \( k^* \in \ker M \) satisfying \( \angle x'y'y' < \pi/2 \), which implies \( \angle xk^*y < \pi/2 \).

Assume without loss of generality that \( \|k^* - x\| \leq \|k^* - y\| \). Now, if \( z = p_{k^*y}(x) \), then \( \{x\} \cup zy \) is a suitable poidge.

\[\square\]

4 Topological discs

4.1 Smooth topological discs

For a set \( M \subset \mathbb{R}^2 \) with differentiable boundary, a chord \( xy \) is called a double normal of \( M \) if it is orthogonal to both tangent lines of \( \text{bd} \ M \) at \( x \) and \( y \) (see Figure 4).

![Figure 4: A double normal xy.](image)

Let \( \mathcal{D}_m \) be the space of all planar topological discs with a \( C^m \)-boundary.

**Theorem 4.1.** A topological disc \( D \in \mathcal{D}_2 \) is poidge-convex if for any double normal \( xy \) of \( \text{bd} \ D \), the curvature of \( \text{bd} \ D \) at \( x \) or \( y \) is less than \( \frac{2}{\|x-y\|} \).

**Proof.** Clearly, \( x, y \) have the poidge-convex property if one of them is interior to \( D \). So, let \( x, y \in \text{bd} \ D \). If \( xy \) is not a double normal, then the tangent line at \( x \) or \( y \) is not orthogonal to \( xy \). Suppose the tangent line \( T \) at \( x \) is not orthogonal to \( xy \). Then there exists a small line-segment \( xz \subset D \) orthogonal to \( xy \). Thus, \( \{y\} \cup xz \) is a suitable poidge.
If \( xy \) is a double normal, by hypothesis, at \( x \) or \( y \), say at \( x \), the curvature radius satisfies \( \rho(x) > \frac{||x-y||}{2} \). This implies that \( S_{xy} \) has a small arc starting at \( x \), inside of \( D \). If that arc is \( \widehat{xz} \), then the poidge \( \{y\} \cup xz \) is as required. (Notice that the case that \( D \) is not locally convex at \( x \) is not excluded. In that case, the curvature condition is even superfluous.)

**Remarks.** A topological disc must have at least one double normal: take its diameter!

The condition, which works for \( rt \)-convexity, that \( D \) has at least two diameters, or that \( D \) is ordinary (i.e. \( \text{card} \ (D \cap S_D) \geq 3 \)), is not sufficient for the poidge-convexity.

We offer now an extension of Theorem 4.1 to topological discs with boundaries of class \( C^1 \).

Suppose \( D \subset \mathbb{R}^2 \) is a topological disc, locally convex at some boundary point \( x \). Then, if \( \text{bd} \ D \) is differentiable at \( x \), a lower and an upper curvature, \( \gamma_l(x) \) and \( \gamma_u(x) \), at \( x \) can be defined, as in [8] (page 14). The preceding theorem can be (in the same way) proven in the following more general form.

**Theorem 4.2.** A topological disc \( D \in \mathcal{D}_1 \) is poidge-convex if, for any double normal \( xy \) of \( \text{bd} \ D \), at one of its endpoints, say at \( x \), either \( D \) is locally convex and \( \gamma_u(x) < \frac{2}{||x-y||} \), or \( \mathbb{R}^2 \setminus \text{int} \ D \) is locally convex.

### 4.2 Unions of convex sets

Here, we consider unions of compact convex sets in \( \mathbb{R}^n \), possibly of distinct dimensions.

**Theorem 4.3.** Every ordinary connected union of two ordinary compact convex sets is poidge-convex.

**Proof.** Let \( M = A \cup B \), with \( A, B \) compact convex sets, satisfying the hypothesis. Let \( x, y \in M \). If both \( x, y \) lie in \( A \), or both in \( B \), then we have a right triangle \( xyz \) in \( A \) or in \( B \) because they are rightly convex, by Theorem 3 in [18]. So, assume without loss of generality that \( x \in A \setminus B \) and \( y \in B \setminus A \). If inside \( S_{xy} \) there is no point of \( M \), then \( A \cap B = \emptyset \) and \( M \) is not connected. If \( M \subset \text{conv} S_{xy} \), then, \( M \) being ordinary, there exists a point \( z \in S_{xy} \setminus \{x, y\} \). If \( M \not\subset \text{conv} S_{xy} \), but \( S_{xy} \cap M = \{x, y\} \), then either \( A \subset \text{conv} S_{xy} \) and \( B \cap \text{conv} S_{xy} = \{y\} \), or vice-versa. In both cases \( M \) becomes disconnected, or one of the sets \( A, B \) is not ordinary, which is false.

Hence, in any (possible) case, there exists \( z \in S_{xy} \setminus \{x, y\} \). Now, if \( z \in A \) then \( \{y\} \cup xz \) is a suitable poidge, and if \( z \in B \), a suitable poidge is \( \{x\} \cup yz \). □

For \( m \geq 3 \), an ordinary connected union of \( m \) ordinary compact convex sets may not be poidge-convex, see Figure 5.
5 Not simply connected sets

5.1 Convex bodies with spherical holes

It is easily seen that there are poidge-convex sets which are not simply connected. So, for example, the set $B_1(0) \setminus \text{int } B_{\varepsilon}(0)$ in $\mathbb{R}^2$, for any $\varepsilon \in ]0, 1[$.

Theorem 5.1. Let $K \subset \mathbb{R}^2$ be an ordinary convex body, and assume that $0$ is the centre of $S_K$. The set $K \setminus \text{int } B_\alpha(0)$ is poidge-convex, if $\alpha$ satisfies the following conditions. For any double normal $ab$ of $bd\ K$,

(i) if $ab$ is not a chord of $S_K$, then $\alpha \leq \rho(0, ab)$, and

(ii) if $ab$ is a chord of $S_K$ and $c \in S_K \cap K \setminus \{a, b\}$, then $\alpha \leq \max\{\rho(0, bc), \rho(0, ca)\}$.

Proof. Let $L = K \setminus \text{int } B_\alpha(0)$, and $x, y \in L$. We need to investigate only the case when $xy$ is a double normal of $bd\ K$, which we now assume.

Case 1. $xy$ is not a chord of $S_K$.

In this case, $xy \subset L$, because $xy \cap \text{int } B_\alpha(0) = \emptyset$, by condition (i). Since $K$ is ordinary, $S_{xy} \cap K \neq \{x, y\}$. Choose $w \in S_{xy} \cap K \setminus \{x, y\}$. So, $\{w\} \cup xy$ is a poidge in $L$.

Case 2. $xy$ is a chord of $S_K$.

Since $K$ is ordinary, $S_K \cap K \setminus \{x, y\}$ contains a point $z$. Now, we have $yz \cap \text{int } B_\alpha(0) = \emptyset$ or $zx \cap \text{int } B_\alpha(0) = \emptyset$, by condition (ii) of the hypothesis. Assume without loss of generality that $yz \cap \text{int } B_\alpha(0) = \emptyset$.

Since $\angle yzx = \pi/2$, we have the poidge $\{x\} \cup yz \subset L$. \hfill \Box

Is always the existence of such a number $\alpha$ guaranteed?
Theorem 5.2. If $K \subset \mathbb{R}^2$ is an ordinary convex polygon, in which every double normal passing through the centre 0 of $S_K$ is a chord of $S_K$, then a number $\alpha$ satisfying the conditions in Theorem 5.1 does exist.

Proof. Let $\mathcal{D}_1$ be the family of those double normals of $K$ which do not contain 0, and $\mathcal{D}_2$ the complementary family of double normals, which must be chords of $S_K$.

Since no double normal in $\mathcal{D}_1$ contains 0, $\beta = \min\{\rho(0, N) : N \in \mathcal{D}_1\} > 0$.

The set of double normals passing through 0 is finite. For each double normal $N = ab \in \mathcal{D}_2$, consider

$$\gamma_N = \max\{\max\{\rho(0, ax), \rho(0, bx)\} : x \in S_K \cap \text{bd} K \setminus \{a, b\} \}$$

and $\gamma = \min\{\gamma_N : N \in \mathcal{D}_2\} > 0$.

By choosing $\alpha = \min\{\beta, \gamma\}$, both conditions of Theorem 5.1 are satisfied, condition (i) for the double normals of $\mathcal{D}_1$, and condition (ii) for the double normals of $\mathcal{D}_2$. \hfill \square

Corollary 5.3. If $K \subset \mathbb{R}^2$ is an ordinary convex polygon, in which no double normal passes through the centre 0 of $S_K$, then a number $\alpha$ satisfying the conditions in Theorem 5.1 does exist.

Theorem 5.2 presents only sufficient conditions for a convex polygon to acquire an appropriate number $\alpha$. All regular polygons are poidge-convex, although they do not satisfy the conditions of Theorem 5.2.

5.2 Tetrahedral surfaces

Starting with this section, we investigate the poidge-convexity of convex surfaces in $\mathbb{R}^n$, for $n = 3$ or for larger $n$.

The following lemma is straightforward.

Lemma 5.4. Every non-obtuse triangle is poidge-convex.

Lemma 5.5. If $P$ is a $(n-1)$-dimensional polytope in $\mathbb{R}^n$, $x \in \mathbb{R}^n$, and $y \in P \setminus (V(P) \cup \{x\})$, then there exists a poidge in $P \cup \{x\}$ containing both $x$ and $y$.

Proof. Indeed, $H_{yx} \cap P$ includes a non-degenerate line-segment $\sigma$ with an endpoint in $y$. Thus, $\{x\} \cup \sigma$ is a suitable poidge. \hfill \square

Theorem 5.6. Suppose $abcd$ is a tetrahedron in $\mathbb{R}^3$ with a non-obtuse facet (i.e. 2-dimensional face) $abc$. Then $\text{bd} (abcd)$ is poidge-convex if and only if

$$d \in \left((H_{ba}^+ \cup H_{ca}^+) \cap (H_{ab}^+ \cup H_{cb}^+) \cap (H_{ac}^+ \cup H_{bc}^+) \right).$$
The boundary of a tetrahedron with all facets obtuse is not poidge-convex.

Proof. Suppose abc and d are as stipulated in the first part of the theorem. We prove that bd (abcd) is poidge-convex.

Let x, y ∈ bd (abcd) be any two distinct points.

Case 1. There is a face F of abcd such that x, y ∈ F.

If F is a non-obtuse face, then there exists a poidge containing x, y and contained in F, by Lemma 5.4. If F is an obtuse face, then we obviously have to consider only the case that x, y are vertices of the longest edge of bd F. If that edge is included in bd (abc), then we find a poidge containing x and y in the face abc, by Lemma 5.4. So, suppose the obtuse angle of F is not at d. Let d′ be the orthogonal projection of d on the plane abc.

Subcase 1.1. d′ ∈ Hba ∩ Hca ∩ Hcb ∩ Haab ∩ Hac ∩ Hbc. (See Figure 6.)

All the angles dac, dca, dcb, dabc, dba, dab are non-obtuse. Then the obtuse face F must have an obtuse angle at d, and we obtain a contradiction.

Subcase 1.2. d′ ∈ (Hca ∩ Hac) \ (Hba ∩ Hbc). (See Figure 7.)

All angles dac, dca, dcb, dab are non-obtuse. Clearly, F ≠ cad. If F = bcd, it must have its obtuse angle at b. If d′′ is the orthogonal projection of d onto abc, then d′′ ∈ abc and x, y belong to the poidge cd ∪ {d}. The case F = abd is analogous.

The cases d′ ∈ (Hbc ∩ Hcb) \ (Hac ∩ Hab) and d′ ∈ (Hba ∩ Hab) \ (Hca ∩ Hcb) are analogous to subcase 1.2.

Case 2. There are two distinct faces F1 and F2 such that x ∈ F1 and y ∈ F2.

If x ∈ int F1 or y ∈ int F2, then they have the poidge-property, by Lemma 5.5. Otherwise, x ∈ bd F1 and y ∈ bd F2, which means that we are in Case 1, unless x and y are in opposite edges. Without loss of generality, we can assume that x ∈ ad and y ∈ bc. Let w be the
orthogonal projection of $x$ onto $bc$. Then $w \in bc$, and $x, y$ will be contained in the poidge
\{x\} \cup wb or \{x\} \cup wc.

Now, let us show that, if $abc$ is as required, but $d$ not (see Figure 8), then bd $(abcd)$ is
not poidge-convex. This will be proved by showing that $d$ and one of the other three vertices
do not enjoy the poidge-property.

If $d \notin H_{ba}^+ \cup H_{ca}^+$, then the angles $\widehat{dca}$ and $\widehat{db}$ are obtuse. Then, obviously, $a$ and $d$
do not enjoy the poidge-property. The cases $d \notin H_{ba}^+ \cup H_{ca}^+$ and $d \notin H_{ab}^+ \cup H_{cb}^+$ are analogous.
The proof of the first part of the theorem is now complete.

For the second part, assume all facets of the tetrahedron $abcd$ are obtuse. Assume without loss of generality that $ab$ is a diameter of $abcd$. The pair of points $a, b$ has not the poidge-property. Indeed, $S \cap abcd = \{a, b\}$, $H_{ab} \cap abcd = \{a\}$ and $H_{ba} \cap abcd = \{b\}$. \hfill \Box

### 5.3 Boundaries of Cartesian products and cones

As we could see in the preceding section, there exist poidge-convex convex surfaces. However, many convex surfaces are not poidge-convex: for example, no strictly convex body has a poidge-convex boundary. Therefore, most convex surfaces are not poidge-convex (see [9]). We present now some classes of convex surfaces which are poidge-convex.

We consider the Cartesian product in $\mathbb{R}^n$ of a $k$-dimensional compact convex set $K$ and an $(n-k)$-dimensional compact convex set $L$, hence with $\overline{K}$ orthogonal to $\overline{L}$.

#### Theorem 5.7. Every Cartesian product of compact convex sets of positive dimensions has a poidge-convex boundary.

Proof. We consider $\overline{K}$ to be spanned by the first $k$ axes, and $\overline{L}$ by the last $n-k$ axes. Let $S = \text{bd} (K \times L)$.

Let $x, y \in S$. We have $x = u \times t_u$ and $y = v \times t_v$, where $u, v \in K$ and $t_u, t_v \in L$.

Consider the chord $t'_u t'_v = \overline{t_u t_v} \cap L$ of $L$.

**Case 1.** $x, y \in \{u\} \times L$, where $u \in \text{bd} K$.

Take $u' \in K \setminus \{u\}$ arbitrarily. Then the poidge $\{u' \times t'_u\} \cup (\{u\} \times t'_u t'_v)$ contains $x, y$ and lies in $S$.

**Case 2.** $x \in \{u\} \times L$, $y \in \{v\} \times L$, where $u, v \in \text{bd} K$ are distinct.

If $t'_u, t_u, t_v, t'_v$ lie in this order on $\overline{t_u t_v}$, we have $\overline{t_u t_v} \subset t_u t'_v$. Then $\{x\} \cup (\{v\} \times t_u t'_v)$ is a suitable poidge in $S$.

**Case 3.** $x, y \in K \times \{t\}$, where $t \in \text{bd} L$.

For this case we have the poidge $(u \times t') \cup x y \subset S$, where $t' \in (\text{bd} L) \setminus \{t\}$.

**Case 4.** $x \in K \times \{t_u\}$, $y \in K \times \{t_v\}$, where $t_u, t_v \in \text{bd} L$ are distinct.

If $u \neq v$, a good poidge is $\{y\} \cup x (v \times t_u)$. Otherwise, $\{y\} \cup x z$ is a good one, $z$ being an arbitrary point of $K \times \{t_u\}$ different from $x$.

**Case 5.** $x \in K \times \{t_u\}$, $y \in (\text{bd} K) \times L$, where $t_u \in \text{bd} L$.

Now, if we are not in Case 1 or 4, a suitable poidge is $\{x\} \cup (\{v\} \times I)$, where $I$ is an arbitrary line-segment in $L$ starting at $t_u$. \hfill \Box

Let $K$ be an $(n-1)$-dimensional convex body in $\mathbb{R}^n$, and $v \in \mathbb{R}^n \setminus K$. Put $v' = \overline{p_{\overline{K}}(v)}$. We call the cone $C = \text{conv} (\{v\} \cup K)$ right, if $v' \in K$ and, for any pair of points $a, b \in \text{bd} K$, $\angle avb \leq \pi/2$. 

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Figure 9: A conical surface.

The boundary of a right cone will be called here a conical surface. See Figure 9.

**Theorem 5.8.** Every conical surface is poidge-convex.

**Proof.** Let $S = \text{bd } C$, where $C = \text{conv } \{v\} \cup K$ is a right cone. We observe the preceding notation.

Let $x, y \in S$. We assume without loss of generality that $\|v - x\| \leq \|v - y\|$.  

**Case 1.** $x, y \in K$.

If $x, y$ are not both boundary points of $K$, then they clearly have the poidge-property (in $K$). Suppose now $x, y \in \text{bd } K$. Since $\angle xvy \leq \pi/2$ and $\|v - x\| \leq \|v - y\|$, $z = \text{p}_v_y(x)$ belongs to $vy$. Thus, $\{x\} \cup yz$ is a suitable poidge.

**Case 2.** The points $v, x, y$ are collinear.

The point $x$ is closer from $v$ than $y$, possibly $x = v$.

Let $\{y\} = \pi_y \cap K$.

If $y' = y$, then we consider the poidge $\{x\} \cup yz$, where $z = \text{p}_v_y(x)$.

If $y' \neq y$, then take $u \in (\text{bd } K) \setminus \{y'\}$ so close to $y'$ that $\|v - u\| > \|v - y\|$. Then $\angle vuy < \pi/2$, and the projection $z = \text{p}_{vuy}(x)$ belongs to $vu$. So, we obtain the poidge $\{z\} \cup xy \subset S$.

**Case 3.** $x \in vx', y \in vy'$, with $x', y' \in \text{bd } K$.

We may suppose $x \neq v, y \neq v$, and $x' \neq y'$, otherwise we are in Case 2. Since $\angle xvy \leq \pi/2$, we have $z \in vy' \setminus \{y\}$, where $z = \text{p}_{vuy}(x)$. Thus, $\{x\} \cup yz$ is a suitable poidge.

**Case 4.** $x \in K, y \notin K$.

Let $z = \text{p}_x(y)$. If $z \neq x$, then $\{y\} \cup xz$ is a good poidge. If $z = x$, take any point $z' \in K$ different from $z$. The poidge $\{y\} \cup zz'$ will do it.  

For results on right convexity in cylinders and cones, see [18].
5.4 Polyhedral surfaces

We have already seen that some tetrahedral surfaces are poidge-convex, some are not. The same is true in the larger frame of all polyhedral surfaces.

The star $S_x$ at a vertex $x$ of a polytope in $\mathbb{R}^n$ is the union of all facets (i.e. $(n-1)$-dimensional faces) having $x$ as a vertex. Two vertices of a polytope in $\mathbb{R}^n$ will be called opposite if $H_{xy}$ and $H_{yx}$ are supporting hyperplanes of the polytope.

Recall that a compact set $M \subset \mathbb{R}^n$ is called ordinary, if $\text{card}(S_M \cap M) \geq 3$. This is equivalent to the property that, for any pair of points $x, y \in M$, not all points of $M$ different from $x, y$ lie inside $S_{xy}$, i.e. $M \setminus \text{int conv } S_{xy} \neq \{x, y\}$.

Now, we say that a polytope $P \in \mathbb{R}^n$ is extraordinary, if, for any pair of opposite vertices $x, y \in P$, not all points of $S_x \cup S_y$ different from $x, y$ lie inside $S_{xy}$, i.e. $(S_x \cup S_y) \setminus \text{int conv } S_{xy} \neq \{x, y\}$.

Of course, every extraordinary polytope is ordinary, but not vice-versa.

Among the extraordinary polytopes we find all those admitting a circumscribed sphere, i.e. a sphere containing all vertices. The Platonic and the Archimedean Solids are well-known examples.

Theorem 5.9. Every extraordinary polytope in $\mathbb{R}^n$ has a poidge-convex boundary.

Proof. By Lemma 5.5, we only need to verify the poidge-property for pairs of vertices $x, y$ of the given extraordinary polytope $P$.

![Figure 10: An ordinary, not extraordinary polytope.](image)
Take $x, y \in V(P)$. Let $H_x$ be a supporting hyperplane of $P$ at $x$. Consider the hyperplane $H_y \ni y$ parallel to $H_x$.

If $H_y$ is supporting $P$, then our hypothesis implies the existence of some point $z \in (S_x \cup S_y) \setminus \text{int conv } S_{xy}$ different from $x$ and $y$. Assume without loss of generality that $z \in S_x$. Then $zx \cap S_{xy} \neq \{x\}$. If $z' \in zx \cap S_{xy} \setminus \{x\}$, then $xz' \subset S_x$, as $\{x\}$ is the kernel of the starshaped set $S_x$. Moreover, $\angle xz'y = \pi/2$. Thus, $\{y\} \cup xz'$ is a suitable poidge.

If $H_y$ is not supporting $P$, then both $H_y$ and $S_{xy}$ are locally cutting bd $P$ at $y$. Locally, the intersection $S_{xy} \cap S_y$ is a union of pieces of spheres of dimension $n - 2$. Take a point $z$ in $S_{xy} \cap S_y$ different from $y$. Then $\{x\} \cup yz$ is a suitable poidge. \qed

Figure 10 shows an ordinary polytope which is not extraordinary.

Sometimes even the 1-skeleta of polytopes are poidge-convex.

In $\mathbb{R}^3$, let $T_1$, $C_1$, $O_1$, $D_1$, $I_1$ be the boundary 1-complexes of the regular tetrahedron, cube, regular octahedron, regular dodecahedron, and regular icosahedron, respectively.

**Theorem 5.10.** $T_1$, $C_1$, $O_1$ are poidge-convex, while $D_1$, $I_1$ are not.

**Proof.** We leave to the reader the first part of the statement. Consider $D_1$. Let $abcde$ be a face of $D_1$, and $aa'$ an edge different from $ab$ and $ea$. Consider the points $x \in aa'$ and $y \in bc$. Obviously, $H_{xy} \cap aa' = \{x\}$, and $H_{yx} \cap bc = \{y\}$. Moreover, $xy \not\subset D_1$. Hence, $x, y$ have not the poidge-property.

For $I_1$, the proof is similar. \qed

6 Problems

We end the paper with two problems about the relationship between various $\mathcal{F}$-convexities.

**Problem 6.1.** Which poidge-convex sets are not rt-convex?

We mentioned (and it is easily verified) that tr-convexity implies poidge-convexity.

**Problem 6.2.** Which poidge-convex sets are not tr-convex?

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