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Γ-CONVEXITY

Dedicated to the memory of Professor Delin REN

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Abstract Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$ (always $d \geq 2$). A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$-convex, if for any pair of distinct points $x, y \in M$, there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$. We obtain the Γ-convexity, when $\mathcal{F}$ consists of Γ-paths. A Γ-*path* is the union of both shorter sides of an isosceles right triangle. In this paper we first characterize some Γ-convex sets, bounded or unbounded, including triangles, regular polygons, subsets of balls, right cylinders and cones, unbounded planar closed convex sets, etc. Then, we investigate the Γ-starshaped sets, and provide some conditions for a fan, a spherical sector and a right cylinder to be Γ-starshaped. Finally, we study the Γ-triple-convexity, which is a discrete generalization of Γ-convexity, and provide characterizations for all the 4-point sets, some 5-point sets and $\mathbb{Z}^d$ to be Γ-triple-convex.

Keywords Γ-convexity; Γ-starshaped sets; Γ-triple-convexity

MSC2020 52A01; 52A37

1 Introduction

The fourth author proposed a very general kind of convexity in 1974 at a meeting on convexity in Oberwolfach. Given a family $\mathcal{F}$ of sets in a certain space $\mathfrak{X}$, a set $M \subset \mathfrak{X}$ is called

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$\mathcal{F}$-convex, if for any pair of distinct points $x, y \in M$, there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$. Usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are just some examples of $\mathcal{F}$-convexity for suitably chosen families $\mathcal{F}$.

As special cases of polygonal connectedness, Bruckner and Bruckner [3], and also Magazanik and Perles [5] investigated L_n sets, where $\mathcal{F}$ consists of all polygonal paths with at most n edges in the plane; Magazanik and Perles [4] studied staircase convex sets, where $\mathcal{F}$ is the family of all polygonal paths in the plane with each edge parallel to one of the coordinate axes, and all the parallel edges having the same direction.

In 1980, Blind, Valette and the fourth author [1] first investigated rectangular convexity, the case when $\mathcal{F}$ is the family of all non-degenerated rectangles, which was also studied by Böröczky Jr [2]. In 2014, the fourth author [9] studied the right convexity, the case with $\mathcal{F}$ consisting of all (2-dimensional) right triangles. Later, the last two authors [7, 8] investigated the rt-convexity, which is a discrete generalization of the right convexity. Recently, Wang and the last two authors [6], generalizing the right convexity once again, introduced and investigated the poidge-convexity, the case with $\mathcal{F}$ consisting of the unions $\{x\} \cup \sigma$ (which are called poidges), where x is a point, σ a line-segment, and $\operatorname{conv}(\{x\} \cup \sigma)$ a non-degenerate right triangle.

Now consider an isosceles right triangle. The union of its shorter sides will be called a Γ-path. Two points of M enjoy the Γ-property, if they belong to a Γ-path included in M. A set M will be called Γ-convex, if any two of its points enjoy the Γ-property. In this paper, we investigate the Γ-convexity. One motivation for this research is the lasting intention of extending the use of $\mathcal{F}$-convexity to a substantial number of families $\mathcal{F}$. Another motivation resides in the obvious possible interest in Engineering to identify spaces fully accommodating Γ-shaped solids.

For distinct points $x, y \in \mathbb{R}^d$, where always $d \geq 2$, let xy denote the (closed) line-segment from x to y, $\overline{xy}$ the line including xy, $[xy\rangle$ the half-line from x through y, and H_{xy} the hyperplane through the midpoint of xy orthogonal to $\overline{xy}$.

For any compact $M \subset \mathbb{R}^d$, denote by C_M the smallest hypersphere containing M in its convex hull.

If L, L' are affine subspaces of $\mathbb{R}^d$, $L \parallel L'$ means that they are parallel and $L \perp L'$ that they are orthogonal. By saying that one set is orthogonal to another, we mean that the affine hulls are orthogonal.

For $M \subset \mathbb{R}^d$, we denote by $\complement M$ its complement, by $\operatorname{conv} M$ its convex hull, by $\overline{M}$ its affine hull and by $\operatorname{int} M, \operatorname{bd} M, \operatorname{cl} M$ its relative interior, boundary and closure, which means in the topology of $\overline{M}$. Also, let $\pi_M(x)$ be the orthogonal projection of x onto $\overline{M}$. Put $a_1 a_2 \cdots a_n = \operatorname{conv}\{a_1, a_2, \cdots, a_n\}$, for $a_1, \cdots, a_n \in \mathbb{R}^d$. The distance $d(x, M)$ from x to M equals $\inf_{y \in M} \|x - y\|$.

The s-dimensional Hausdorff measure will be denoted by μ_s.

Let $\mathbb{B}_d$ be the closed unit solid ball with centre $\mathbf{0}$ in $\mathbb{R}^d$, and $\mathbb{S}_{d-1} = \operatorname{bd} \mathbb{B}_d$.

In Section 2 and Section 3 we characterize some Γ-convex sets, bounded or unbounded. Some key results are as follows.

Theorem 2.3 A triangle is Γ-convex if and only if it is an isosceles right triangle.

Theorem 2.4 A regular n-gon ($n \geq 3$) is Γ-convex if and only if $n \equiv 0 \pmod 4$.

Theorem 2.9 For $d \geq 3$, the right cylinder Z_h of height h is Γ-convex if and only if

$\|h\| \leq 2$.

Theorem 2.11 For $d \geq 3$, the right cone $P = \operatorname{conv}(\mathbb{B}_{d-1} \cup \{p\})$ with $\pi_{\mathbb{B}_{d-1}}(p) = \mathbf{0}$ is Γ-convex if and only if $d(p, \mathbb{B}_{d-1}) \leq 1$.

Theorem 3.2 An unbounded planar closed convex set M with a recession cone of angle α is Γ-convex if and only if $\alpha \geq \pi/4$.

Theorem 3.5 The complement of a set included in a countable union of $(d-2)$-dimensional affine subspaces of $\mathbb{R}^d$ is Γ-convex.

In Section 4 we investigate the Γ-starshaped sets.

Theorem 4.1 A fan is a Γ-starshaped set if its opening is at least $\pi/2$.

Theorem 4.3 For $d \geq 3$, the right cylinder Z_h is Γ-starshaped if and only if $\|h\| \leq 4$.

In Section 5 we study the Γ-triple-convexity, which is a discrete generalization of Γ-convexity, and mainly provide characterizations for all the 4-point sets, some 5-point sets and $\mathbb{Z}^d$ to be Γ-triple-convex.

2 Bounded Γ-Convex Sets

2.1 General Results for Planar Compact Sets

Lemma 2.1 Let K be a planar compact convex set. If any two points of $\operatorname{bd} K$ have the Γ-property in K, then K is Γ-convex.

Proof Consider $x, y \in K$. If $x, y \in \operatorname{bd} K$, the Γ-property is assumed. If $\{x, y\} \nsubseteq \operatorname{bd} K$, assume that $\overline{xy} \cap \operatorname{bd} K = \{x', y'\}$, and $\|x - x'\| < \|y - x'\|$. Then there is a Γ-path $ab \cup bc$ included in K containing x', y', with $\angle abc = \pi/2$. Suppose that $x' \in ab$. Let L_x denote the line parallel to ab through x, L_y the line parallel to bc through y, $L_x \cap ac = \{a'\}$, $L_y \cap ac = \{c'\}$, $L_x \cap L_y = \{b'\}$. The Γ-path $a'b' \cup b'c' \subset K$ contains x, y. □

Theorem 2.2 For planar compact sets, Γ-convexity implies simple connectedness.

Proof Let M be a planar compact set. We suppose that M is not simply connected. Since M is compact, it follows immediately from the assumption that the complement of M has an open, bounded component C. Choose a maximal disc D which is contained in $\operatorname{cl} C$. Let σ be its centre and choose $a \in \operatorname{bd} D \cap \operatorname{bd} M$. Since C is bounded, $[a, \sigma\rangle$ meets $M \setminus \{a\}$. Then, a and $b \in [a, \sigma\rangle \cap (M \setminus \{a\})$ don't have the Γ-property. □

2.2 Polygons

Theorem 2.3 A triangle is Γ-convex if and only if it is an isosceles right triangle.

Proof For the "only if" part, we assume that the triangle T is not an isosceles right triangle.

If T is an obtuse triangle or a right not isosceles triangle, we denote the vertices of the two angles less than $\pi/2$ of T by x and y. There is no Γ-path included in T containing x, y.

If T is an acute triangle, we choose a vertex of T as point x, and the point distinct from x in $\operatorname{bd} T$ on the bisector of the angle at x as point y. There is no Γ-path included in T containing x, y.

For the "if" part, let $T = abc$ with $\angle abc = \pi/2$. By Lemma 2.1, it is sufficient to show that any two points $x, y \in \operatorname{bd} T$ belong to a Γ-path in T.

Let $x \in ab$. If $y \in bc$, then x, y belong to the Γ-path $ab \cup bc$. If $y \in ac$, consider the points $x_1, x_2 \in ac$ such that $\overline{xx_1} \perp \overline{ac}$, and $\overline{xx_2} \parallel \overline{bc}$. If $y \in ax_1$, then $x, y \in ax_1 \cup x_1x$. If $y \in x_1x_2$, then $x, y \in xx_1 \cup x_1x_2$. If $y \in x_2c$, then $x, y \in yy' \cup y'a$, where $y' = \pi_{ab}(y)$.

Let both $x, y \in ac$. Consider $z \in T$ such that xyz be homothetic with T. The Γ-path $xz \cup zy \subset T$ joins x with y. □

Theorem 2.4 A regular n-gon ($n \geq 3$) is Γ-convex if and only if $n \equiv 0 \pmod 4$.

Proof For the "if" part, let R_n denote a regular n-gon of centre $\mathbf{0}$. Let x be a vertex of R_n, and y verify $\langle x, y \rangle = 0$ and $\|x\| = \|y\|$. Then $xy \cup y(-x)$ is a Γ-path. Since $n \equiv 0 \pmod 4$, y is also a vertex of R_n. This proves the Γ-property for the pair $x, -x \in \operatorname{bd} R_n$, which is a diametral pair in R_n. For any pair $x, x' \in \operatorname{bd} R_n$, xx' divides R_n into two parts R', R''; put $H_{xx'} \cap \operatorname{bd} R_n = \{u, v\}$. Assume w.l.o.g. that $\mathbf{0}, u \in \operatorname{cl} R' \subset R_n$. Take $w = (x + x')/2$. Then $\angle xux' \leq \pi/2$ and $\angle xwx' = \pi$. Hence, there exists a point $y \in uw$ such that $xy \cup yx'$ is a Γ-path. So, R_n is a Γ-convex set.

For the "only if" part, let x be a vertex of R_n and $n \not\equiv 0 \pmod 4$. Consider $y \in \operatorname{bd} R_n$ such that $\overline{xy}$ be an axis of symmetry for R_n, and put $H_{xy} \cap \operatorname{bd} R_n = \{u', v'\}$. Since $\|x - y\| > \|u' - v'\|$, the Γ-property is not available at x, y. □

2.3 Non-Convex Γ-Convex Sets

There are some examples of compact planar Γ-convex sets, which are neither convex, nor Γ-paths.

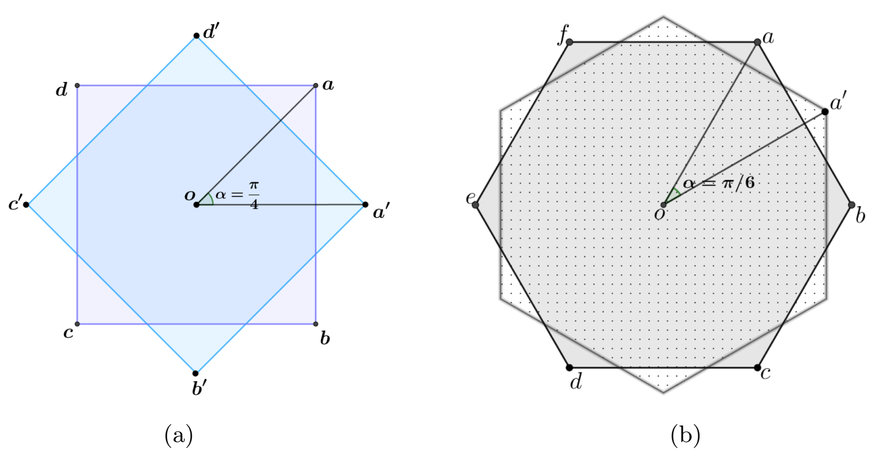


Figure 1 Examples of compact planar Γ-convex sets which are neither convex nor Γ-paths

Let S be a square. Rotate it with the angle $\pi/4$ about its centre. We get a new square S'. Then, $S \cup S'$ is a non-convex Γ-convex compact planar set different from a Γ-path, see Figure 1(a).

We can consider a regular hexagon or other regular polygons instead of a square, see Figure 1(b).

We see that in all these examples, we have line-segments in their boundaries. Is this necessary? The answer is no.

There are compact, nonconvex Γ-convex set whose boundaries contain no line-segment, see Figure 2.

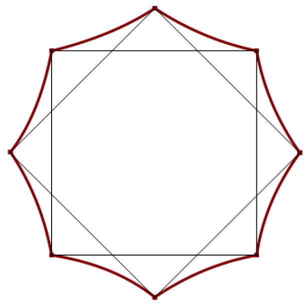


Figure 2 A nonconvex Γ-convex set whose boundary contains no line-segment.

There are also some compact Γ-convex sets in $\mathbb{R}^3$, which are neither convex, nor Γ-paths.

Consider $\mathbb{B}_{d-1} \subset \mathbb{R}^d$, and let $\|a\| = 1$ and $\overline{\mathbf{0}a} \perp \overline{\mathbb{B}_{d-1}}$. Then $\mathbb{B}_{d-1} \cup \mathbf{0}a$ is a Γ-convex set.

Also, several pairwise orthogonal unit balls of dimension at least 2 in $\mathbb{R}^d$ with centres at $\mathbf{0}$ and pairwise meeting only at $\mathbf{0}$, have a Γ-convex union.

2.4 Subsets of Balls

It is easy to prove the following.

Theorem 2.5 A ball in $\mathbb{R}^d$ is Γ-convex.

In this subsection we investigate the Γ-convexity of subsets of balls.

Let Ω be a Jordan (simple) arc in $\mathbb{S}_1$. Then $\cup\{\mathbf{0}x : x \in \Omega\}$ is called a fan, and its angle $\mu_1\Omega$ at $\mathbf{0}$ its opening.

Theorem 2.6 No fan is Γ-convex.

Proof Let F be a fan with opening α. Let a, b be the endpoints of $\Omega \subset F$. The length of Ω is $\angle a\mathbf{0}b = \alpha$.

If $\alpha < \pi/2$, consider the midpoint c of Ω. The pair of points $\mathbf{0}, c$ don't enjoy the Γ-property.

If $\pi/2 < \alpha \le \pi$, choose a point $c \in \Omega$ close to b. The Γ-property is not available at a, c.

If $\alpha > \pi$, choose a point $c \in \Omega$ close to $-a$, between a and $-a$ on Ω. The Γ-property is again missing at a, c.

Finally, the case $\alpha = \pi/2$. Let c be the midpoint of Ω. For $x \in \widehat{ac} \subset \Omega$, $y \in \mathbf{0}b$, let $m(x, y) = (x+y)/2$, and let $n(x, y)$ be the point of Ω equidistant from x and y. For $x = a, y = b$, $n(a, b) = c$ and

$$\|m(a, b) - n(a, b)\| < \|a - b\|/2.$$

Take y close to b, such that still

$$\|m(a, y) - n(a, y)\| < \|a - y\|/2.$$

Clearly, $\mathbf{0} \in C_{ay}$. Now, take x close to a. Then C_{xy} cuts $\mathbf{0}a$ in two points, one of which, say z, is close to $\mathbf{0}$. Thus, $\|y - z\|$ is close to $\|y\| < 1$ and $\|x - z\|$ is close to $\|x\| = 1$ whence $\|y - z\| < \|x - z\|$. Moreover, $\|m(x, y) - n(x, y)\| < \|x - y\|/2$. Hence, the square with xy as a diagonal has both vertices distinct from x, y outside F. □

In higher dimensions the situation is different. We consider the ball $\mathbb{B}_d$ $(d \ge 3)$ and the half-space $H \not\ni \mathbf{0}$ which meets $\mathbb{B}_d$. Then $S = \operatorname{conv}(\{\mathbf{0}\} \cup (\mathbb{B}_d \cap H))$ is called a spherical cone.

Theorem 2.7 The spherical cone S is Γ-convex if $d(\mathbf{0}, H) \le \sqrt{2}/2$.

Proof Let $x, y \in S$. Consider $S_{xy} = C_{xy} \cap H_{xy}$. Since $d(\mathbf{0}, H) \leq \sqrt{2}/2$, there exists a point $z \in S_{xy} \cap S$; thus, $xz \cup yz \subset S$ is a Γ-path. So S is Γ-convex. $\square$

A set is called a cap, if it is the intersection of a halfspace with a ball.

Theorem 2.8 For $d \geq 3$, every cap in $\mathbb{R}^d$ is Γ-convex.

Proof Let C be a cap. For any $x, y \in C$, let $S_{xy} = C_{xy} \cap H_{xy}$. This is a sphere of codimension 2. Notice that, for all positions of x, y, we have $S_{xy} \cap C \neq \emptyset$. Once $z \in S_{xy} \cap C$, we have $xz \cup zy \subset C$. Thus, C is Γ-convex. $\square$

For $d = 2$, the only Γ-convex cap is the disc.

2.5 Right Cylinders and Cones

Consider the ball $\mathbb{B}_{d-1} \subset \mathbb{R}^d$, the vector h orthogonal onto $\overline{\mathbb{B}_{d-1}}$, and the right cylinder $Z_h = \operatorname{conv}(\mathbb{B}_{d-1} \cup (\mathbb{B}_{d-1} + h))$.

Theorem 2.9 For $d \geq 3$, the right cylinder Z_h is Γ-convex if and only if $\|h\| \leq 2$.

Proof To prove the "if" part, let $x, y \in Z_h$. We consider the "worst" case, when $x \in \operatorname{bd}\mathbb{B}_{d-1}$ and $y = h - x$. Obviously, $h/2 \in xy$. Clearly, $\|(h/2) - s\| = \|x - y\|/2 = \sqrt{4 + \|h\|^2}/2$, for all $s \in \operatorname{bd}(\mathbb{B}_{d-1} + h)$. If s moves from y to $x + h$ on $\operatorname{bd}(\mathbb{B}_{d-1} + h)$, $\angle y(h/2)s$ varies from 0 to $\alpha = \angle y(h/2)(x + h)$, and $\alpha \geq \pi/2$ because $\|h\| \leq 2$. Hence, for some position s^* of s, $\angle y(h/2)s^* = \pi/2$. Consequently, $xs^* \cup s^*y$ is a Γ-path in Z_h.

Now we prove the "only if" part. If $\|h\| > 2$, we consider some diameter ab of Z_h. Obviously, the Γ-property is not available at a, b. $\square$

Consider the ball $\mathbb{B}_{d-1}$, the point $p \notin \overline{\mathbb{B}_{d-1}}$, and the cone $P = \operatorname{conv}(\mathbb{B}_{d-1} \cup \{p\})$. This cone is right, if $\pi_{\mathbb{B}_{d-1}}(p) \in \mathbb{B}_{d-1}$.

Theorem 2.10 For $d \geq 3$, the right cone $P = \operatorname{conv}(\mathbb{B}_{d-1} \cup \{p\})$ is Γ-convex, if $d(p, \mathbb{B}_{d-1}) \leq 1$.

Proof Let $x, y \in P$. Put $p' = \pi_{\mathbb{B}_{d-1}}(p)$. We consider the case $x = p$, $y = -p'$, $p' \in \operatorname{bd}\mathbb{B}_{d-1}$. Denote by m the midpoint of $p(-p')$; thus, $\pi_{\mathbb{B}_{d-1}}(m) = \mathbf{0}$. Clearly, $\|m - z\| = \|p + p'\|/2$, for all $z \in \operatorname{bd}\mathbb{B}_{d-1}$. If z moves from $-p'$ to p' on $\operatorname{bd}\mathbb{B}_{d-1}$, $\angle(-p')mz$ varies from 0 to $\alpha = \angle(-p')mp'$, and $\alpha > \pi/2$ because $\|m\| = d(p, \mathbb{B}_{d-1})/2 < 1 = \|p'\|$. Hence, for some position z^* of z, $\angle(-p')mz^* = \angle pz^*(-p') = \pi/2$. Consequently, $pz^* \cup z^*(-p')$ is a Γ-path in P.

This case can be adapted for all $y \in \mathbb{B}_{d-1} \backslash A$, where $A = \{z \in \mathbb{B}_{d-1} : \|z - p'\| < \|p - p'\|\}$, by considering half of the cone $\operatorname{conv}((\mathbb{B}_{d-1} \cap C_{yp'}) \cup \{p\})$.

If $x = p$, $y \in \operatorname{bd}\mathbb{B}_{d-1}$ such that $\|p' - y\| < \|p - p'\|$, then $\|x - y\| < \|y + p'\|$. Denote by m'' the midpoint of xy; $mm'' \perp p'y$, so $mm'' \perp xy$. Since $\|m'' - m\| = \|y + p'\|/2 > \|x - y\|/2$, there is a point $z \in mm''$ such that $\|m'' - z\| = \|x - y\|/2$. Then $xz \cup zy$ is a Γ-path in P. $\square$

Theorem 2.11 For $d \geq 3$, the right cone $P = \operatorname{conv}(\mathbb{B}_{d-1} \cup \{p\})$ with $\pi_{\mathbb{B}_{d-1}}(p) = \mathbf{0}$, is Γ-convex if and only if $d(p, \mathbb{B}_{d-1}) \leq 1$.

Proof The "if" part follows from Theorem 2.10.

For the "only if" part, if $d(p, \mathbb{B}_{d-1}) > 1$, let $S_{\mathbf{0}p} = C_{\mathbf{0}p} \cap H_{\mathbf{0}p}$. Since the radius of $S_{\mathbf{0}p}$ is greater than 1/2, $S_{\mathbf{0}p} \cap P = \emptyset$. And since $d(p, \mathbb{B}_{d-1})$ is greater than the radius of $\mathbb{B}_{d-1}$, there is no Γ-path included in P containing $\mathbf{0}, p$. So, P is not Γ-convex. $\square$

Remark If a cone is not right, it is not Γ-convex. Indeed, let P' be a cone that is not a right one; choose the apex of P' as x, and the farthest point y from x on the boundary of the base of P'. The pair of points x and y don't enjoy the Γ-property.

3 Unbounded Γ-Convex Sets

Theorem 3.1 The non-empty intersection of two half-spaces in $\mathbb{R}^d$ $(d \geq 3)$ is Γ-convex.

Proof Let B be the intersection of the two half-spaces H_x, H_y. Consider $x, y \in B$. The only non-trivial situation appears when $\text{bd } H_x \cap \text{bd } H_y \neq \emptyset$, $x \in \text{bd } H_x \setminus \text{bd } H_y$, $y \in \text{bd } H_y \setminus \text{bd } H_x$. Take $z \in H' \cap \text{bd } H_x \cap \text{bd } H_y$, where H' is the hyperplane through x orthogonal to $\overline{xy}$. We have $D = \text{conv } ([y(y + x - z)\rangle \cup [z, x\rangle) \subset B$. Since $E = \text{cl } (D \setminus xyz)$ includes the half-disc $E \cap C_{xy}$, x, y have the Γ-property. □

When $d = 2$, a slab bounded by two parallel lines is unbounded, but not Γ-convex.

Theorem 3.2 An unbounded planar closed convex set M with a recession cone of angle α is Γ-convex if and only if $\alpha \geq \pi/4$.

Proof Let V be the recession cone of M with angle α at the apex $\mathbf{0}$. There are two rays R_a, R_b such that $\text{bd } V = R_a \cup R_b$ and the angle between them is α.

The "only if" part is obvious.

For the "if" part, let $x, y \in M$.

Consider the cones $V_x = V + x$ and $V_y = V + y$.

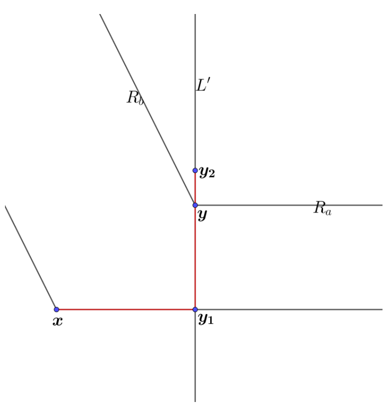


Figure 3 $V_x \supset V_y$.

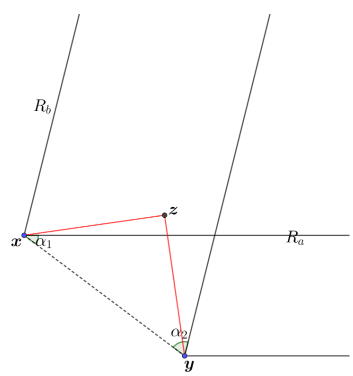


Figure 4 $V_x \not\supset V_y$ and $V_y \not\supset V_x$.

First, suppose $V_x \supset V_y$. Assume w.l.o.g. that y is not closer to $R_b + x$ than to $R_a + x$. Let $L' \ni y$ be a line orthogonal to $R_a + x$, and $\{y_1\} = L' \cap (R_a + x)$, see Figure 3. Consider the point $y_2 \in L'$ not separated from y by $R_a + x$, such that $\angle y_1 x y_2 = \pi/4$. Then $xy_1 \cup y_1 y_2 \subset V_x$ is a Γ-path containing x, y.

Suppose $V_x \not\supset V_y$ and $V_y \not\supset V_x$, see Figure 4. Let α_1 be the angle between $[xy\rangle$ and $R_a + x$. The angle between $[xy\rangle$ and $R_b + x$ is larger than $\alpha \geq \pi/4$. Analogously, the angle between $[yx\rangle$ and $R_a + y$ is larger than $\pi/4$. Therefore, for some $z \in \text{conv}\,(R_b + x \cup R_a + y) \subset M$, $\angle yxz = \angle xyz = \pi/4$, and $xz \cup zy$ is a suitable Γ-path. □

Theorem 3.3 Let M be an unbounded planar closed convex set with a recession cone of angle α. Then $\text{cl}\,\complement M$ is Γ-convex if $\alpha \geq \pi/2$.

Proof It suffices to show that, for any pair of points $x, y \in \operatorname{bd} M$, there exists a Γ-path in $\operatorname{cl} \complement M$ containing x, y. Consider a point z such that $\overline{xz}$ be a supporting line of M at x and $\overline{yz}$ a supporting line of M at y. Clearly, $\angle xzy \geq \alpha$. Consider the isosceles right triangle xyu, such that $\overline{xy}$ does not separate u from z.

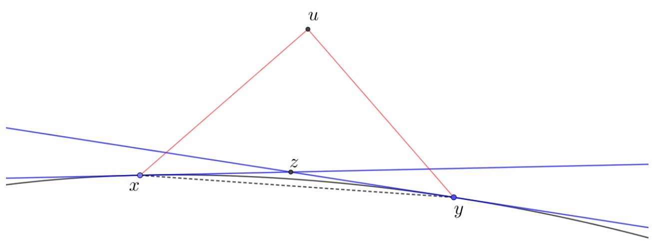


Figure 5 $z \in xyu$.

If $z \in xyu$, then $xu \cup uy$ is a suitable Γ-path (see Figure 5).

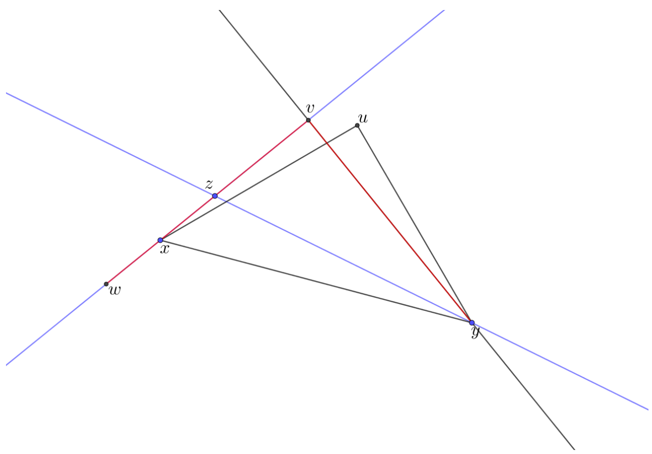


Figure 6 $z \notin xyu$.

If $z \notin xyu$, assume w.l.o.g. that $\angle zxy > \pi/4$ (see Figure 6). Then $\angle xyz < \pi/4$, and $\|x - z\| < \|y - z\|$. Put $v = \pi_{\overline{xz}}(y)$. We have $\|x - v\| < \|y - v\|$. Take the point w such that $x \in wv$ and $\|w - v\| = \|y - v\|$. Now, $wv \cup vy$ is a suitable Γ-path. □

A set $M \subset \mathbb{R}^d$ is called acyclic, if it includes no sphere of dimension $\dim \overline{M} - 1$.

Theorem 3.4 If $M \subset \mathbb{R}^d$ is an acyclic set included in a hyperplane, then $\complement M$ is Γ-convex.

Proof Let $x, y \notin M$. The existence of a Γ-path from x to y is obvious if x and y are not separated by $\overline{M}$. Suppose now they are separated by $\overline{M}$. It follows that $\overline{M}$ is a hyperplane. Then $S_{xy} = C_{xy} \cap \overline{M}$ is a sphere of dimension $d-2$. Since M is an acyclic set, there exists a point $z \in S_{xy} \backslash M$. Then, $\angle xzy = \pi/2$. This implies the existence of a Γ-path including $\{x, y\}$, contained in $\complement M$. □

Theorem 3.5 The complement of a set included in a countable union of $(d-2)$-dimensional affine subspaces of $\mathbb{R}^d$ is Γ-convex.

Proof Let $M \subset \mathbb{R}^d$ be a set as described in the statement. Thus, $M \subset \bigcup_{n=1}^{\infty} A_n$, where each A_n is an affine subspace of $\mathbb{R}^d$ of codimension 2.

Choose $x, y \in \complement M$. Let

$$\sigma_n = \left\{ \frac{z-x}{\|z-x\|} : z \in A_n \right\} \subset \mathbb{S}_{d-1}.$$

Since the $(d-1)$-dimensional measure $\mu_{d-1}(\sigma_n)$ of each σ_n is 0, we also have $\sum_{n=1}^{\infty} \mu_{d-1}(\sigma_n) = 0$. Consider the half-line L_τ of direction τ starting at x. It follows that $\mu_{d-1}(\cup_{\tau\in S}(L_\tau \cap \mathbb{S}_{d-1})) = 0$, where $S = \cup_{n=1}^{\infty}\sigma_n$. Hence, $\mu_{d-1}(\cup_{\tau\in S}(\overline{L_\tau} \cap \mathbb{S}_{d-1})) = 0$. This yields $\mu_{d-1}(\cup_{\tau\in S}(\overline{L_\tau} \cap C_{xy})) = 0$. Hence, for almost all $z \in C_{xy}$, $\overline{xz} \cap M = \emptyset$. Analogously, $\overline{yz} \cap M = \emptyset$ for almost all $z \in C_{xy}$. Hence, we find $z \in C_{xy}$ such that $\overline{xz} \cup \overline{yz} \subset \complement M$. Since $\overline{xz} \perp \overline{yz}$, $\complement M$ is Γ-convex. □

4 Γ-Starshaped Sets

If there exists a point $k \in M$ such that for any $x \in M$, k, x enjoy the Γ-property in M, then M is a Γ-starshaped set relative to the point k. The set of points in M which can play the role of k form the kernel of M and is denoted by $ker_\Gamma(M)$.

Theorem 4.1 A fan is a Γ-starshaped set if its opening is at least $\pi/2$.

Proof Let F be a fan with opening α. Let a, b be the endpoints of $\Omega \subset F$ (from the definition of a fan). The length of Ω is $\angle a\mathbf{0}b = \alpha$. Consider the arbitrary point $y \in F\backslash\{\mathbf{0}\}$; then, $\angle a\mathbf{0}y \geq \pi/4$ or $\angle b\mathbf{0}y \geq \pi/4$ since $\alpha \geq \pi/2$. There exists a Γ-path in F with $\mathbf{0}$ and y as endpoints. So, F is a Γ-starshaped set. □

Consider $a, b, c \in \mathbb{S}_2$. Then $C = \mathbb{B}_3 \cap \mathrm{conv}\,([\mathbf{0}a\rangle \cup [\mathbf{0}b\rangle \cup [\mathbf{0}c\rangle)$ is called the spherical sector determined by a, b, c.

Theorem 4.2 If $a, b, c \in \mathbb{S}_2$, $\angle a\mathbf{0}c, \angle b\mathbf{0}c \geq \pi/2$, then the spherical sector determined by a, b, c is Γ-starshaped.

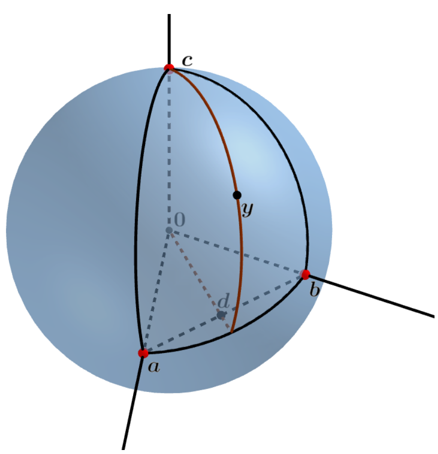


Figure 7 A spherical sector determined by a, b, c with $\angle a\mathbf{0}c, \angle b\mathbf{0}c \geq \pi/2$.

Proof Consider $d \in ab$, i.e. $d = \alpha a + \beta b$, where $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$, see Figure 7. Since $\angle a\mathbf{0}c, \angle b\mathbf{0}c \geq \pi/2$, $\langle a, c\rangle \leq 0$ and $\langle b, c\rangle \leq 0$. Then $\langle d, c\rangle = \langle \alpha a + \beta b, c\rangle = \alpha\,\langle a, c\rangle + \beta\,\langle b, c\rangle \leq 0$, whence $\angle d\mathbf{0}c \geq \pi/2$. Let C be the given spherical sector and $y \in C \setminus \{\mathbf{0}\}$. Then $\overline{c\mathbf{0}y} \cap C$ is a fan with opening at least $\pi/2$. According to the proof of Theorem 4.1, $\mathbf{0}, y$ enjoy the Γ-property.

□

Theorem 4.3 For $d \geq 3$, the right cylinder Z_h is Γ-starshaped if and only if $\|h\| \leq 4$.

Proof The "only if" part is obvious.

For the "if" part, let $x \in Z_h$. By applying Theorem 2.9 in the cylinders $\operatorname{conv}(\mathbb{B}_{d-1} \cup (\mathbb{B}_{d-1} + h/2))$ and $\operatorname{conv}((\mathbb{B}_{d-1} + h/2) \cup (\mathbb{B}_{d-1} + h))$, we see that x and $h/2$ enjoy the Γ-property in Z_h. □

Let $R_n = \operatorname{conv}\{x_1, x_2, \cdots, x_n\} \subset \mathbb{R}^2$ $(n \geq 3)$ be a regular n-gon of centre $\mathbf{0}$, and D_n be the intersection of the n discs with the n vertices of R_n as centres, and the diameter of R_n as radii.

Theorem 4.4 For every $n \geq 3$, D_n is Γ-starshaped.

Proof It is obvious that $\mathbf{0}$ is the centre of D_n and D_n is convex. For any point $x \in \operatorname{bd} D_n$, we prove that $\mathbf{0}$ and x enjoy the Γ-property. Let $H_{\mathbf{0}x} \cap \operatorname{bd} D_n = \{z_1, z_2\}$; then, $\|z_1\| = \|x - z_1\|$ and $\angle \mathbf{0}xz_1 > \pi/4$. There is a point $z \in \operatorname{conv}\{\mathbf{0}, z_1, x\} \subset D_n$ such that $\mathbf{0}z \cup zx$ is a suitable Γ-path. □

Theorem 4.5 D_n is Γ-convex if and only if $n \equiv 0 \pmod 4$.

Proof The proof is similar to the proof of Theorem 2.4. □

5 Γ-Triple-Convexity

Let $M \subset \mathbb{R}^d$. A Γ-triple is the vertex set of an isosceles right triangle. A pair of points in M is said to enjoy the Γt-property in M, if there exists a third point in M, such that the three points form a Γ-triple. A set $M \subset \mathbb{R}^d$ is called Γ-triple convex, or Γt-convex if any pair of points in M enjoys the Γt-property.

Theorem 5.1 A set of four points is Γt-convex, if and only if it is one of the following:

(1) The set of the four vertices of a square.

(2) The set of the vertices and the circumcenter of an isosceles right triangle.

(3) A set similar to $\{\mathbf{0}, (1,0,0), (0,1,0), (0,0,1)\} \subset \mathbb{R}^3$.

Proof The "if" part is obvious.

For the "only if" part: Let $M = \{a, b, c, d\}$ be a Γt-convex set. Assume that $\{a, b, c\}$ is a Γ-triple, and $\angle abc = \pi/2$. Since M is Γt-convex, at least two triples among $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$ are Γ-triples. Hence, at least one triple of $\{a, b, d\}$, $\{b, c, d\}$ is a Γ-triple. Assume w.l.o.g. that $\{a, b, d\}$ is a Γ-triple. The possible positions of d are those shown in Figure 8, plus those obtained after a rotation about $\overline{ab}$. Let S_{xy} denote the circle with centre x and radius $\|x - y\|$, orthogonal to xy.

<u>Case 1.</u> $\{a, b, d\}, \{a, c, d\}$ are Γ-triples.

Among all positions of d mentioned above, only the positions d_2 and d_5 in Figure 8 give Γ-triples. These are the alternatives (1) and (2) in the statement.

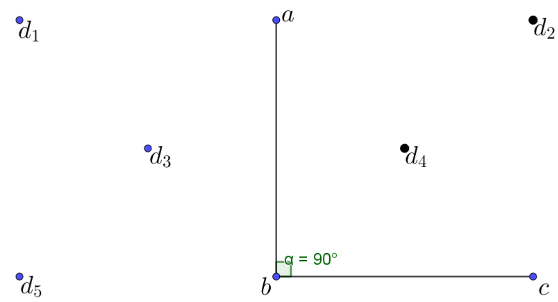


Figure 8 $\{a, b, d\}, \{b, c, d\}$ are Γ-triples.

Case 2. $\{a, b, d\}, \{b, c, d\}$ are Γ-triples.

Among all positions of d mentioned above, $\{b, c, d\}$ becomes a Γ-triple only if d is d_2 in Figure 8, or d is d_4 in Figure 8, or d takes the position of c after the mentioned rotation, of $\pi/2$. These correspond to alternatives (1), (2) and (3) of the statement, respectively. □

For the next two theorems we omit the elementary, but lengthy proofs.

Theorem 5.2 A set $\{a, b, c, d, e\} \subset \mathbb{R}^3$ of five points with a 4-point Γt-convex subset is Γt-convex, if and only if one of the following happens:

(1) $abcd$ is a square, e is the centre of $abcd$.

(2) a, b, c, d, e are the vertices of a regular octahedron minus one of them.

(3) $abcd$ is a square, ecb is an isosceles triangle such that $\|e - c\| = \|e - b\| = \|c - a\| =$ and $\overline{ec} \perp \overline{ca}$.

(4) $abcd$ is a square and ade is an equilateral triangle orthogonal to ab.

(5) abc and ace are orthogonal isosceles right triangles with the common hypotenuse ac.

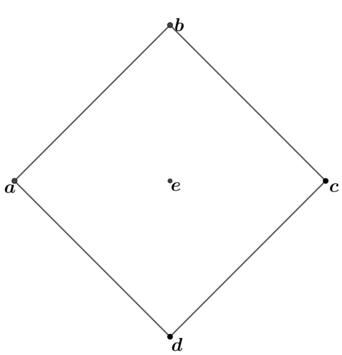


Figure 9 (1)

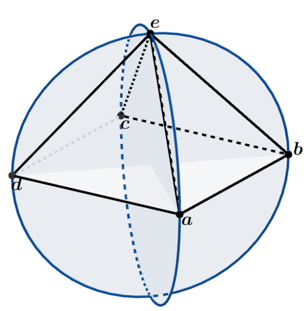


Figure 10 (2)

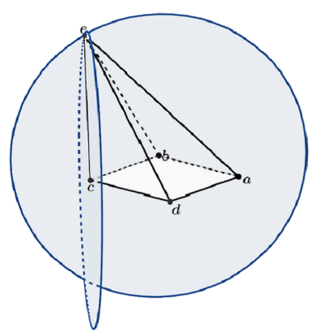


Figure 11 (3)

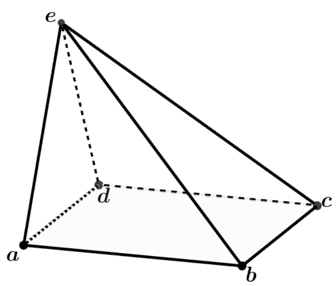


Figure 12 (4)

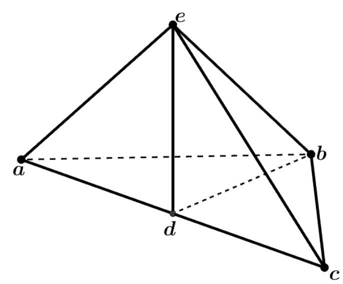


Figure 13 (5)

Theorem 5.3 A planar 5-point set is Γt-convex if and only if it is the set of four vertices and the centre of a square.

The Γt-convexity can also be investigated in infinite discrete sets, for example in lattices.

Theorem 5.4 $\mathbb{Z}^d$ is Γt-convex if and only if d is even.

Proof For the "if" part: It suffices to prove the Γt-property for the points $v = (x_1, x_2, x_3, x_4, \cdots, x_{d-1}, x_d)$, $\mathbf{0} \in \mathbb{Z}^d$. We just have to take $w = (-x_2, x_1, -x_4, x_3, \cdots, -x_d, x_{d-1})$; then $\{v, \mathbf{0}, w\}$ is a Γ-triple, and thus $\mathbb{Z}^d$ is Γt-convex.

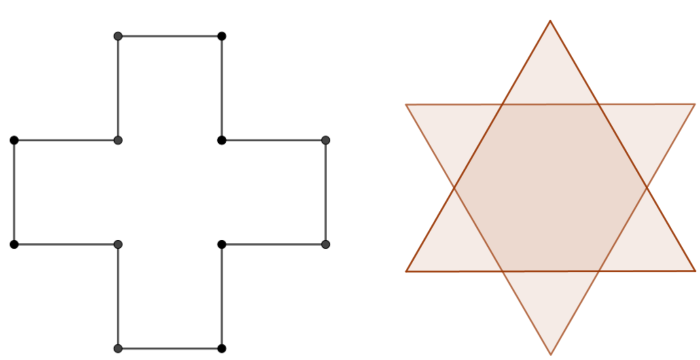
For the "only if" part: Let $v = (1, 1, \cdots, 1)$, $\mathbf{0} \in \mathbb{Z}^d$. The existence of a point $w = (x_1, x_2, \cdots, x_d) \in \mathbb{Z}^d$ such that $\langle w, v \rangle = 0$ and $\|w\| = \|v\|$ means that

$$\begin{cases} x_1 + x_2 + \cdots + x_d = 0, \\ x_1^2 + x_2^2 + \cdots + x_d^2 = d. \end{cases}$$

Since 0 and d have the same parity, d is even. □

If a set is Γt-convex, it is obviously rt-convex and it-convex. An $\mathcal{F}$-convex set is rt-convex (it-convex), if $\mathcal{F}$ is the family of all vertex-sets of right (resp. isosceles) triangles. Conversely, if a set is rt-convex and it-convex, it does not have to be Γt-convex. Indeed, a rectangle different from a square is rt-convex and it-convex, but not Γt-convex.

There are non-convex sets with Γt-convex complements. For example: the interior of the polygon shown in Figure 14 (a); a fan with opening $\alpha \in (\pi, 3\pi/2]$; the union of an equilateral triangle with the triangle symmetric with respect to its centre, see Figure 14 (b).



(a) (b)

Figure 14 Examples of non-convex sets with Γt-convex complements.

A set M in a subspace X of $\mathbb{R}^d$ is hyper-Γt-convex in X, if it is Γt-convex and, for any point $s \in X \backslash M$, the set $M \cup \{s\}$ is Γt-convex, too.

Theorem 5.5 The complement of any open convex set in $\mathbb{R}^d$ is hyper-Γt-convex in $\mathbb{R}^d$.

Proof Let $M \subset \mathbb{R}^d$ be an open convex set. Consider $x \in \mathbb{R}^d$, $y \in \complement(M \cup \{x\})$. Let $F = \{z \mid \angle xyz = \pi/2 \text{ and } \|x-y\| = \|z-y\|\}$. There is at least one point $z \in F \backslash M$. Indeed, otherwise, for two points z_1, z_2 diametrally opposite in the $(d-2)$-dimensional sphere F, $y \in z_1 z_2$. Since M is convex, $y \in M$, which contradicts $y \in \complement M$. So, there exists a point $z \in \complement M$ such that $\{x, y, z\}$ be a Γ-triple. □

Consider the subset $Z_d \subset \mathbb{Z}^d$ defined by $Z_d = \mathbb{Z}^{d-1} \times \{1, 2, 3, \cdots\}$.

Theorem 5.6 The set Z_d is hyper-Γt-convex in $\mathbb{Z}^d$, if d is even.

Proof To prove the theorem, it suffices to show that, for any $v \in Z_{k,d} = \mathbb{Z}^{d-1} \times \{k, k+1, k+2, \cdots\}$, there exists $w \in Z_{k,d}$, such that $\{\mathbf{0}, v, w\}$ be a Γ-triple.

Consider $v = (x_1, x_2, x_3, x_4, \cdots, x_{d-1}, x_d)$. If $x_{d-1} \geq 0$, we take $w^+ = (-x_2,\ x_1,\ -x_4,\ x_3,\ \cdots,\ -x_d,\ x_{d-1}) \in Z_d$; then $\{v, \mathbf{0}, w^+\}$ is a Γ-triple. If $x_{d-1} < 0$, we take $w^- = (x_2,\ -x_1,\ x_4,\ -x_3,\ \cdots,\ x_d,\ -x_{d-1}) \in Z_d$; then $\{v, \mathbf{0}, w^-\}$ is also a Γ-triple.

For $x_{d-1} \geq 0$, take $w = v + w^+ = (x_1 - x_2, x_2 + x_1, x_3 - x_4, x_4 + x_3, \cdots, x_{d-1} - x_d, x_d + x_{d-1})$. For $x_{d-1} < 0$, take $w = v + w^- = (x_1 + x_2, x_2 - x_1, x_3 + x_4, x_4 - x_3, \cdots, x_{d-1} + x_d, x_d - x_{d-1})$. In this way, $v \in Z_{k,d}$ implies $w \in Z_{k,d}$, and $\{\mathbf{0}, v, w\}$ is a Γ-triple. □

Conflict of Interest The authors declare no conflict of interest.

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